The L-space Conjecture

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February 14 – February 23, 2025

1 Overview

The L-space conjecture ([BGW], [Juh]) contends that the following conditions are equivalent for a closed connected orientable irreducible 3-manifold M.

- The fundamental group of M is left-orderable (i.e. M is *LO*).
- M admits a co-oriented taut foliation (i.e. M is *CTF*).
- M is not a Heegaard Floer L-space (i.e. M is *NLS*).

The conjecture links seemingly unrelated group theoretic, topological, and analytic aspects of 3-manifold topology, and though much supporting evidence has appeared, the heuristic connections between these aspects remain mysterious. As such, any progress towards verifying them should provide deep structural information about 3-dimensional spaces. The only implication known is that CTF implies NLS, which was proved by Ozsváth and Szabó for C^2 foliations in [OSz] and later extended to C^0 foliations by Kazez-Roberts [KR] and independently by Bowden [Bn].

When the first Betti number of M is positive, M is LO [BRW], CTF [Ga], and NLS (by definition), so the interesting case arises when M is a rational homology 3-sphere. For such manifolds it is difficult to know whether any of the conditions holds or not, though there are many families for which we know the status of at least one of them and this motivates developing techniques to determine the status of the others.

Much evidence supporting the L-space conjecture has been produced. For instance, it is known to hold for non-hyperbolic geometric 3-manifolds (i.e. Seifert fibre spaces and Sol manifolds), graph manifolds, and many infinite families of hyperbolic manifolds. What remains to be verified are the cases of general toroidal 3-manifolds and general hyperbolic 3-manifolds. The slope detection and gluing method, which was used to verify the conjecture for graph manifolds ([BC1], [HRRW]), provides a promising line of attack for proving it for toroidal 3-manifolds, though non-trivial challenges remain to complete the argument. The focus of this RIT programme was dealing with these challenges.

2 Recent Developments and Open Problems

The three participants in the RIT programme have produced four papers [BGH1, BGH2, BGH3, BGH4] in the area over the last few years. These papers focus on

- the L-space conjecture for toroidal 3-manifolds [BGH1, BGH3];
- using pseudo-Anosov flows to study the property LO for hyperbolic manifolds [BGH2, BGH3];
- the L-space conjecture for cyclic branched covers of 3-manifolds [BGH1, BGH2, BGH4].

Define a *knot manifold* to be a compact connected, orientable irreducible 3-manifold whose boundary is an incompressible torus.

A relative form of the L-space conjecture has emerged which is intimately connected with the L-space conjecture for toroidal 3-manifolds. Here, certain slopes on the boundary of a knot manifold are singled out (i.e. *detected*) using Heegaard Floer homology (NLS-detection), left-orders (LO-detection), or foliations (CTF-detection). The *relative L-space conjecture* is that the sets of NLS-detected slopes, LO-detected slopes, and CTF-detected slopes coincide.

Let $*$ denote one of LO, CTF, and NLS. It is known that gluing knot manifolds together along their boundaries in such a way that a $*$-detected slope on one side is matched with a $*$-detected slope on the other yields a manifold which is $*$ ([HRW], [BC2], [BGH1]). These gluing theorems connect the theory of $*$-detected slopes to the L-space conjecture for toroidal 3-manifolds. Indeed, if the relative L-space conjecture holds, then the L-space conjecture for toroidal 3-manifolds would follow if the converses to the gluing theorems held. This converse is known to be true in the NLS case [HRW]. Here are some open problems on CTF- and LO-detection whose NLS analogues are known to be true.

1. Show that the converses of the LO and CTF gluing theorems hold.
2. Show that the set of $*$-detected slopes on a knot manifold is connected with rational endpoints when $*$ is LO or CTF.
3. Show that up to taking a mirror image, the set of LO-detected slopes, resp. CTF-detected slopes, on the boundary of the exterior of a non-trivial knot K of genus g in the 3-sphere is either the set of all slopes or the set of slopes in the interval $[-\infty, 2g(K) - 1]$.
4. Show that any slope of distance 1 from the longitude of an integer homology solid torus is CTF-detected.
5. In a special case of problem (3), show that the meridional slope of a non-trivial knot in the 3-sphere is CTF-detected.

Even partial progress on these problems will have important consequences in 3-manifold topology.

3 Focus of the RIT programme

The focus of this Research in Team programme was to make progress on proving the converse of the CTF gluing theorem. We mapped out a detailed strategy for accomplishing this in which a key component involves showing that the set of CTF-detected slopes on a knot manifold is a closed in the space of all slopes. Most of our efforts went into investigating the latter problem.

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