Worst-Case Per-User Error Bound for Asynchronous Unsourced Multiple Access

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Collaborators and Acks

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Disclaimer

- About two years ago, the information theory and physical layer security group has started to work on unsourced multiple access.
- The topic is still outside of my comfort zone.
- Please excuse if references or recent results are missing or not properly cited.
- The work presented is submitted to IEEE ISIT 2024 and it can be found on arxiv[1].

[1] Jyun-Sian Wu et al. *Worst-Case Per-User Error Bound for Asynchronous Unsourced Multiple Access*. 2024. arXiv: 2401.14265 [cs.IT].

Introduction, Motivation, and State of the Art

Introduction

- Internet-of-things (IoT), sensor networks, and ultra-reliable low latency massive machine-type communications have attracted attention for 6G communications and beyond.
- The main challenges of the codebook designs for these systems are:
  1. Short-blocklength codewords
  2. A large number of devices that an access point has to serve.
  3. Sporadic and asynchronous activity
- Classical information theory uses the multiple-access channel (MAC) to analyze these systems. The classical MAC considers individual codebooks for all devices.
- Increasing number of devices prohibits using individual codebooks practically.

Motivation

- A perspective on massive random access with a proposal to unify existing models is provided[2].
- In the unsource multiple access channel (UMAC), all transmitters share an identical codebook, and the amount of data transmitted at each transmitter is the same.
- Decoder needs to estimate the transmitted messages.
- Several variations and different model assumptions exists.

[2] Yury Polyanskiy. "A perspective on massive random-access." In: *2017 IEEE International Symposium on Information Theory (ISIT)*. 2017, pp. 2523–2527. DOI: `10.1109/ISIT.2017.8006984`.

State of the Art I

- The first-order capacity is studied[3] when the numbers of users are some functions of the blocklength, and users apply individual codebooks for identification and an identical codebook for transmitting information.
- The second-order asymptotic achievable rates of the grant-free random access system, where users access the channel without any prior request, are analyzed[4],[5].
- Rateless coding as well as feedback is applied.

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[3] Xu Chen, Tsung-Yi Chen, and Dongning Guo. "Capacity of Gaussian Many-Access Channels." In: *IEEE Transactions on Information Theory* 63.6 (2017), pp. 3516–3539. DOI: `10.1109/TIT.2017.2668391`.

[4] Recep Can Yavas, Victoria Kostina, and Michelle Effros. "Random Access Channel Coding in the Finite Blocklength Regime." In: *IEEE Transactions on Information Theory* 67.4 (2021), pp. 2115–2140. DOI: `10.1109/TIT.2020.3047630`.

[5] Recep Can Yavas, Victoria Kostina, and Michelle Effros. "Gaussian Multiple and Random Access Channels: Finite-Blocklength Analysis." In: *IEEE Transactions on Information Theory* 67.11 (2021), pp. 6983–7009. DOI: `10.1109/TIT.2021.3111676`.

State of the Art II

- The energy efficiency of synchronous UMAC with per-user error probability (PUPE) constraint is studied in[6].
- T-fold ALOHA and a low-complexity coding scheme for the grant-free Gaussian random access channel is proposed[7]. The scheme exploits a compute-and-forward approach.
- Furthermore, the minimum Eb/N0 of the T-fold ALOHA and the low-complexity coding scheme is analyzed.

[6] Yury Polyanskiy. "A perspective on massive random-access." In: *2017 IEEE International Symposium on Information Theory (ISIT)*. 2017, pp. 2523–2527. DOI: 10.1109/ISIT.2017.8006984.

[7] Or Ordentlich and Yury Polyanskiy. "Low complexity schemes for the random access Gaussian channel." In: *2017 IEEE International Symposium on Information Theory (ISIT)*. 2017, pp. 2528–2532. DOI: 10.1109/ISIT.2017.8006985.

State of the Art in Asynchronous UMAC I

- Asynchronous systems are worth investigating due to the difficulty of synchronising a large number of devices.
- For asynchronous classical MAC, the capacity is the same as the synchronous MAC[8], assuming the ratio of delay to blocklength asymptotically vanishes.
- Authors in[9] apply a sparse orthogonal frequency-division multiple access (OFDMA) scheme and compressed sensing-based algorithms to reliably identify arbitrarily asynchronous devices and decode messages.

[8] T. Cover, R. McEliece, and E. Posner. "Asynchronous multiple-access channel capacity." In: *IEEE Transactions on Information Theory* 27.4 (1981), pp. 409–413. DOI: 10.1109/TIT.1981.1056382.

[9] Xu Chen et al. "Asynchronous Massive Access and Neighbor Discovery Using OFDMA." In: *IEEE Transactions on Information Theory* 69.4 (2023), pp. 2364–2384. DOI: 10.1109/TIT.2022.3224951.

State of the Art in Asynchronous UMAC II

- For asynchronous UMAC (AUMAC),[10][11] utilize the T-fold ALOHA and the orthogonal frequency-division multiplexing (OFDM), transforming the time-asynchronous problem to a frequency-shift problem.
- The capacity region of the non-fading asynchronous MAC is the same as the usual synchronous MAC.
- The maximum delay must be smaller than the length of the cyclic prefix.

[10] Kirill Andreev et al. "Low Complexity Energy Efficient Random Access Scheme for the Asynchronous Fading MAC." In: *2019 IEEE 90th Vehicular Technology Conference (VTC2019-Fall)*. 2019, pp. 1–5. DOI: `10.1109/VTCFall.2019.8891549`.

[11] Suhas S Kowshik et al. "Short-Packet Low-Power Coded Access for Massive MAC." In: *2019 53rd Asilomar Conference on Signals, Systems, and Computers*. 2019, pp. 827–832. DOI: `10.1109/IEEECONF44664.2019.9048748`.

Contributions, System Model, Main Results

Contributions I

- We consider the AUMAC system with a bounded delay, i.e., maximum delay $\mathrm{D_m} \in \mathbb{Z}^+ \cup 0$, and $\frac{\mathrm{D_m}}{n}$ is a constant.
- Transmitters send a fixed payload size with an identical finite-length n codebook.
- The delays of active users $d_1, \ldots, d_{\mathrm{K_a}}$ are smaller than $\mathrm{D_m}$.
- In our considered model, the messages have to be decoded within n channel uses.
- Receivers decoding without completely receiving codewords are investigated in broadcast channels[12] [13].

[12] Pin-Hsun Lin et al. "Second Order Rate Regions of Gaussian Broadcast Channels under Heterogeneous Blocklength Constraints." In: *IEEE Transactions on Communications* (2023), pp. 1–1. DOI: 10.1109/TCOMM.2023.3329219.

[13] Marcel Mross, Pin-Hsun Lin, and Eduard A. Jorswieck. "Second-Order Performance of Early Decoding with Shell Codes in Gaussian Broadcast Channels." In: *IEEE International Symposium on Information Theory, ISIT 2023, Taipei, Taiwan, June 25-30, 2023*. IEEE, 2023, pp. 2123–2128. DOI: 10.1109/ISIT54713.2023.10206483. URL: https://doi.org/10.1109/ISIT54713.2023.10206483.

Contributions II

- We analyze the PUPE of AUMAC with decoding from incompletely received codewords with finite blocklength n.
- To provide a more precise analysis than the typically used Berry-Esseen theorem (BET) in finite blocklength[14] we apply the saddlepoint approximation[15].
- The permutation-invariant property is invalid due to the asynchronicity. Each k out of K_a combination of the wrong decoded messages has different tail probability.
- Analysis requires the sum of $2^{\mathsf{K}_\mathsf{a}} - 1$ different tail probabilities. Hence, we derive a uniform upper bound of PUPE for our considered AUMAC.
- By studying the worst-case delay pattern, it turns out that receiving more symbols is better than receiving less interference.
- Numerical results compare achievable Eb/N0 for the proposed AUMAC to synchronous UMAC.

[14] Alfonso Martinez and Albert Guillén i Fàbregas. "Saddlepoint approximation of random-coding bounds." In: *2011 Information Theory and Applications Workshop*. 2011, pp. 1–6. DOI: `10.1109/ITA.2011.5743590`.

[15] Jens Ledet Jensen. *Saddlepoint approximations*. English. Oxford Science Publications. Clarendon Press Oxford, 1995. ISBN: 0-19-852295-9.

System Model i

- We consider an AUMAC, which has additive white Gaussian noise (AWGN), one receiver, and multiple transmitters, where the number of active transmitters is denoted by a positive integer K_a.
- All transmitters utilize the same codebook with the same maximal power constraint, P', to transmit the same (and fixed) size of payloads, i.e., $\log M$ nats, to the receiver.
- The codewords are independent and identically distributed (i.i.d.) generated from a Gaussian distribution with mean zero and variance P, where $P < P'$ due to the power backoff.
- The power backoff reduces the probability that the maximal power constraint violations occur.

System Model ii

Definition

We define the asynchronicity in terms of the vector of time shifts (delay) as

$$D^{\mathsf{K_a}} := [d_1,\ d_2,\ \ldots,\ d_{\mathsf{K_a}}] \in \{\mathbb{Z}_0^+\}^{\mathsf{K_a}},$$

where $0 = d_1,\ d_i \leq \mathsf{D_m}$ and $d_i \leq d_\ell,\ \forall \ell > i$ for all $i \in [\mathsf{K_a}]$. The i-th entry, d_i, represents the delay of the i-th received codeword relative to the first received codeword, and $\mathsf{D_m}$ denotes the delay constraint. We define $\alpha := \frac{\mathsf{D_m}}{n} \in [0, 1)$, which is constant w.r.t. the blocklength n, and $\bar{\alpha} = 1 - \alpha$.

System Model iii

- We assume that the receiver has perfect knowledge of the asynchronicity and jointly detects the transmitted messages by maximum information density decoding.
- Asynchronous communication systems may result from asynchronous clocks between transmitters and receivers, different idle times among transmitters, or channel delays.

Remark

We consider that every transmitter transmits with the same codebook, and the receiver is not interested in identifying the senders of the received codewords. Therefore, d_i *indicates the delay of the* i*-th received codeword but does not indicate the identification of the transmitter.*

System Model iv

- In the asynchronous model, the number of transmitted codeword symbols of each channel use can be different. For a given delay $D^{\mathsf{K_a}}$ and the set of erroneously decoded messages $\mathcal{S} \subseteq [\mathsf{K_a}]$, we define a vector

$$a^n(\mathcal{S},\ D^{\mathsf{K_a}}) := [a_1(\mathcal{S},\ D^{\mathsf{K_a}}), a_2(\mathcal{S},\ D^{\mathsf{K_a}}), \ldots, a_n(\mathcal{S},\ D^{\mathsf{K_a}})], \tag{1}$$

  where $a_i(\mathcal{S}, D^{\mathsf{K_a}}) \leq a_\ell(\mathcal{S}, D^{\mathsf{K_a}})$, $\forall \ell > i$, $i \in [n]$ and $a_i(\mathcal{S}, D^{\mathsf{K_a}}) \in \mathbb{Z}_0^+$, $\forall i \in [n]$. For a given $D^{\mathsf{K_a}}$ and a given $i \in [n]$, the i-th entry of $a^n(\mathcal{S}, D^{\mathsf{K_a}})$, i.e., $a_i(\mathcal{S}, D^{\mathsf{K_a}})$, indicates the number of simultaneously received symbols, which belong to $\mathcal{S}$, at the i-th channel use.
- To simplify notations, we use $a^n := [a_1, a_2, \ldots, a_n]$ to represent $a^n(\mathcal{S}, D^{\mathsf{K_a}})$. For example, considering a $\mathsf{K_a}$-active-user AUMAC with $D^{\mathsf{K_a}} = [0, 1, 3, 5, \ldots, 5]$ in Figure, for the set $\mathcal{S} = \{1, 2\}$, $a^n = [a_1 = 1, a_{[n]\setminus[1]} = 2]$; for the set $\mathcal{S} = \{2, 3, 4\}$, $a^n = [a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 2, a_5 = 2, a_{[n]\setminus[5]} = 3]$. Note that for a given $\mathcal{S}$ and $D^{\mathsf{K_a}}$, $a_{[n]\setminus[\alpha n]} = |\mathcal{S}|$.

System Model v

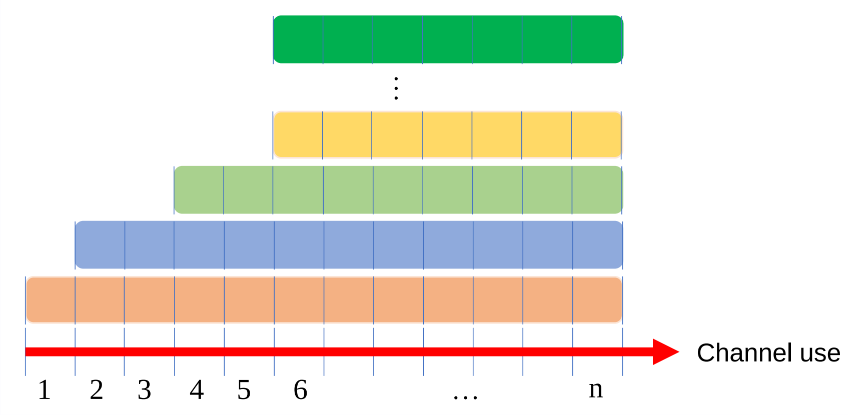

Figure 1: A K_a-active-user AUMAC with $D^{K_a} = [0, 1, 3, 5, ..., 5]$.

System Model vi

- For any $\ell \in [n]$, we define a shift function $\tau_{d_i}(X_i^n, \ell) := X_{i,\ell-d_i}$, where $X_{i,\ell-d_i}$ is the $(\ell - d_i)$-th element of X_i^n, and if $\ell - d_i \notin [n]$, $X_{i,\ell-d_i} = 0, \ \forall i \in [\mathsf{K_a}]$.
- The received symbol at the receiver at time $\ell \in [n]$ is

$$Y_\ell = \sum_{i=1}^{\mathsf{K_a}} \tau_{d_i}(X_i^n, \ell) + Z_\ell, \tag{2}$$

where the channel input $X_i^n \in \chi^n \subset \mathbb{R}^n$, where $\chi^n := \{x^n : x^n \in \mathbb{R}^n, \ \|x^n\|^2 \le n\mathsf{P}'\}$ is the channel input satisfying the maximal power constraint and $Z_\ell \sim \mathcal{N}(0,1)$ is an i.i.d. AWGN, $\forall \ell \in [n]$.

Definition AUMAC Code

Definition

An $(n, \mathsf{M}, \epsilon, \mathsf{K_a}, \alpha, D^{\mathsf{K_a}})-$code, $\mathcal{C}_1$, for an AUMAC described by $P_{Y|X_{[\mathsf{K_a}]}}$, consists of

- one message set $\mathcal{M} = \{1, 2, ..., \mathsf{M}\}$,
- one encoder $f : \mathcal{M} \to \chi^n$,
- one decoder $g : \mathbb{R}^n \to \binom{[\mathcal{M}]}{\mathsf{K_a}}$, where $\binom{[\mathcal{M}]}{\mathsf{K_a}}$ is a set containing $\mathsf{K_a}$ distinct elements from the set $\mathcal{M}$,

and the delay $D^{\mathsf{K_a}}$ fulfills the delay constraint αn in Def. 1, the PUPE satisfies

$$\mathsf{P}_{\mathsf{PUPE}|D^{\mathsf{K_a}}} := \frac{1}{\mathsf{K_a}} \sum_{i=1}^{\mathsf{K_a}} \Pr(\tilde{\mathcal{E}}_i | D^{\mathsf{K_a}}) \leq \epsilon, \tag{3}$$

where $\tilde{\mathcal{E}}_i := \{ \cup_{\ell \neq i} \{M_i = M_\ell\} \cup \{M_i \notin g(Y^n)\} \cup \{\|f(M_i)\|^2 > n\mathsf{P}'\}\}$, $i \in [\mathsf{K_a}]$, and $M_i \sim \mathrm{Unif}(\mathcal{M})$ is the i-th transmitted message.

Main Results i

Theorem: PUPE bound

Fix $0 < \mathsf{P} < \mathsf{P}'$. There exists an $(n, \mathsf{M}, \epsilon, \mathsf{K_a}, \alpha, D^{\mathsf{K_a}})$−code for an AUMAC such that the PUPE can be upper bounded by the following:

$$\sum_{\mathcal{S}\subseteq[\mathsf{K_a}]} \frac{|\mathcal{S}| g_1(a^n, t_0(a^n))}{\mathsf{K_a}\sqrt{2\pi}} \Big[g_2(a^n, t_0(a^n)) + \xi(a^n, t_0(a^n)) \Big] + p_0 \leq \epsilon, \tag{4}$$

if there exists a $t_0(a^n) \in (0,1)$ such that $E_t^{(1)}(a^n, t_0(a^n)) = |\mathcal{S}| \log \mathsf{M}$, where

$$g_1(a^n, t) := \exp(t|\mathcal{S}|\log \mathsf{M} - E(a^n, t)), \tag{5}$$

$$g_2(a^n, t) := \left(t(1-t)\sqrt{-E_t^{(2)}(a^n, t)} \right)^{-1}. \tag{6}$$

$$E(a^n,t) := \frac{1}{2}\sum_{i=1}^{n}\left(t\log(1+a_i\mathrm{P})+\log\left(1-\frac{a_i\mathrm{P}t^2}{1+a_i\mathrm{P}}\right)\right), \tag{7}$$

$$\xi(a^n,t) := \frac{1}{2\pi j}\int_{t-j\infty}^{t+j\infty}\exp\left(-\frac{E_t^{(2)}(a^n,t)}{2}(\rho-t)^2\right) \cdot \frac{1}{\rho(1-\rho)}\sum_{m=1}^{\infty}\frac{\bar{\xi}(a^n,t,\rho)^m}{m!}d\rho, \tag{8}$$

$$\bar{\xi}(a^n,t,\rho) := -\sum_{i=3}^{\infty}E_t^{(i)}(a^n,t)\frac{(\rho-t)^i}{i!}, \tag{9}$$

and $p_0 := \frac{\mathrm{K_a}(\mathrm{K_a}-1)}{2\mathrm{M}} + \sum_{i=1}^{\mathrm{K_a}}\Pr(\|X_i^n\|^2 > n\mathrm{P}')$.

Definition

An $(n, \mathsf{M}, \epsilon, \mathsf{K_a}, \alpha)-$code, $\mathcal{C}_2$, for an AUMAC described by $P_{Y|X_{[\mathsf{K_a}]}}$ consists of one message set $\mathcal{M}$, one encoder f, and one decoder g defined by

$$g(Y^n) = \underset{X^n_{[\mathsf{K_a}]} \in \mathcal{C}_2}{\arg\max} \sum_{\ell=1}^{n} i\left(\left\{\tau_{d_m}(X^n_m, \ell)\right\}_{m \in [\mathsf{K_a}]}; Y_\ell\right), \tag{10}$$

such that for the power constraint P' and any $D^{\mathsf{K_a}}$ satisfying the maximum delay constraint, the PUPE satisfies

$$\mathsf{P}_{\mathsf{PUPE}} := \max_{D^{\mathsf{K_a}}:\ d_{\mathsf{K_a}} \leq \alpha n} \sum_{i=1}^{\mathsf{K_a}} \frac{1}{\mathsf{K_a}} \Pr(\tilde{\mathcal{E}}_i | D^{\mathsf{K_a}}) \leq \epsilon, \tag{11}$$

where $\tilde{\mathcal{E}}_i$ is defined as before.

Main Results iv

Theorem: universal PUPE upper bound

Fix $0 < \mathrm{P} < \mathrm{P}'$. There exists an $(n, \mathrm{M}, \epsilon, \mathrm{K_a}, \alpha)-$code for AUMAC, such that the PUPE can be upper bounded by the following:

$$\frac{1}{\mathrm{K_a}\sqrt{2\pi}} \sum_{|\mathcal{S}|=1}^{\mathrm{K_a}} \left(\binom{\mathrm{K_a}-1}{|\mathcal{S}|} \frac{|\mathcal{S}| g_1(a_0^{n*}, t_0(a_0^{n*}))}{T_0^* \sqrt{-E_t^{(2)}(a_0^{n*}, \underline{t}_0)}} + \binom{\mathrm{K_a}-1}{|\mathcal{S}|-1} \frac{|\mathcal{S}| g_1(a_1^{n*}, t_0(a_1^{n*}))}{T_1^* \sqrt{-E_t^{(2)}(a_1^{n*}, \underline{t}_1)}} \right) + p_0 + O\left(\frac{\exp(-n)}{\sqrt{n}}\right) \leq \epsilon \quad (12)$$

if $t_0(a_\iota^{n*}) \in \mathcal{A} \cap \mathcal{B}$, $\bar{t}_\iota \in \mathcal{A} \cap \bar{\mathcal{B}}$, $\underline{t}_\iota \in \mathcal{A} \cap \underline{\mathcal{B}}$, and $\underline{t}_\iota \leq t_0(a^n) \leq \bar{t}_\iota$, where $a_\iota^{n*} = [\iota^{\alpha n}, |\mathcal{S}|^{n-\alpha n}]$, $T_\iota^* := \min\{\underline{t}_\iota - \underline{t}_\iota^2,\ \bar{t}_\iota - \bar{t}_\iota^2\}$, $\iota := 1(1 \in \mathcal{S})$, $\mathcal{A} := \{t : t \in (0,1)\}$,

$$\mathcal{B} := \left\{ t : E_t^{(1)}(a_\iota^{n*}, t) = |\mathcal{S}| \log \mathrm{M} \right\}, \tag{13}$$

$$\bar{\mathcal{B}} := \left\{ t : \sum_{i=1}^{n} \frac{a_{\iota,i}^* \mathrm{P} t}{1 + a_{\iota,i}^* \mathrm{P}(1 - t^2)} = \frac{n}{2} \log(1 + |\mathcal{S}|\mathrm{P}) - |\mathcal{S}| \log \mathrm{M} \right\}, \tag{14}$$

$$\underline{\mathcal{B}} := \left\{ t : \frac{|\mathcal{S}| n \mathrm{P} t}{1 + |\mathcal{S}|\mathrm{P}(1 - t^2)} = \sum_{i=1}^{n} \frac{\log(1 + a_{\iota,i}^* \mathrm{P})}{2} - |\mathcal{S}| \log \mathrm{M} \right\}, \tag{15}$$

and $a_{\iota,i}^*$ is the i-th element of a_ι^{n*}.

Remark

The upper bound decreases as a_i increases for any $D^{\mathsf{K_a}}$ and $\mathcal{S}$ because $\frac{\partial}{\partial a_i} g_1(a^n, t) g_2(a^n, t) \leq 0$ for all $t \in (0, 1)$ and $i \in [\alpha n]$. In fact, having more overlap in the transmission leads to more interference. A larger number of overlapping symbols has one positive and one negative effect on the receiver: it leads to more received energy but, meanwhile, more interference. By our analysis, we found that the positive effect is dominant.

Numerical Assessments and Discussions

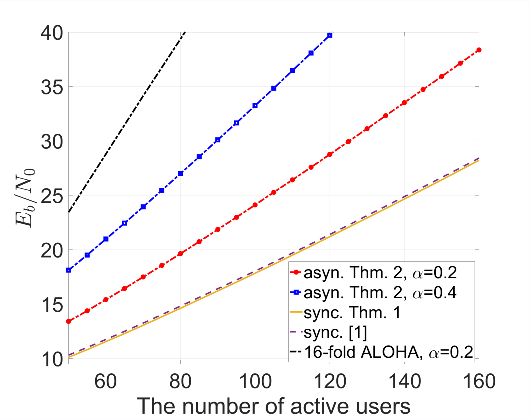

Figure 2: $\frac{E_b}{N_0}$ of AUMAC compared to synchronous UMAC for different numbers of active users.

Numerical Simulations ii

- We numerically optimize P from the second result with $\alpha = 0.2$ and $\alpha = 0.4$ and compare the $\frac{E_b}{N_0}$ of AUMAC and that of synchronous UMAC.
- Numerical results show that the AUMAC that has larger α causes the transmitters to consume more energy to transmit in the worst case of delay.
- Observing the curves of $\alpha = 0.2$, $\alpha = 0.4$ and $\alpha = 0$ (synchronous), we can conclude that for the AUMAC with larger α, which means fewer interference for the first αn channel uses, the PUPE increases.
- It is because the receiver decodes the messages based on fewer transmitted codewords symbols, which is equivalently based on less received energy.
- This effect is illuminated in Remark 2. Thus, codebooks of our considered model require more energy to achieve the same PUPE constraint.

Conclusions and Future Works

Conclusions

- We analyze the FBL performance of the asynchronous UMAC system with bounded and non-vanishing delay constraints αn.
- The derivations using saddlepoint approximation provide FBL performance bounds.
- We also investigate a uniform upper bound of the PUPE, which highly simplifies the analysis to multiply the uniform upper bound with the corresponding binomial coefficient instead of calculating tail probabilities of all error events.
- The numerical results show the trade-off between $\frac{E_b}{N_0}$ and delay constraint αn.
- Although asynchronous transmissions have less interference, reducing the error probability of the first few codewords, it increases PUPE as the receiver decodes shorter codewords, which is analytically and numerically shown.
- Compared to the synchronous case, the achievable energy-per-bit $\frac{E_b}{N_0}$ for the asynchronous case shows that the required $\frac{E_b}{N_0}$ increases as the receiver decodes shorter codewords, even though interference reduces.

Future Works

- Relax strict decoding constraint after n symbols
- Sliding-window decoding: decode whenever a codeword has been completely received. Then perform SIC.
- Relax assumptions on knowledge of number of active users and their delays.
- Estimate active number of users in a time slot and then try to decode if a codeword is complete. (What happens if one new codeword begins when another just ended?)
- Extend to multi-carrier and/or multi-antenna systems
- Extension from MAC to IC

Backup: Proof of Theorem 1 i

- Theorem 1 is derived by the maximal information density decoder with the random coding union (RCU) bound[16] to express the per-user probability of error (PUPE) as a sum of tail probabilities.
- We define $\tilde{\mathcal{E}}_\ell := \{ \cup_{i \neq \ell} \{M_\ell = M_i\} \cup \{M_\ell \notin g(Y^n)\} \cup \{\|f(M_\ell)\|^2 > n\mathrm{P}'\}\}$, which represents the ℓ-th user's error event of the PUPE, $\mathcal{E}_\ell := \{\{M_\ell \neq M_i,\ \forall i \neq \ell\} \cap \{\|f(M_i)\|^2 \leq n\mathrm{P}',\ \forall i \in [\mathrm{K_a}]\}\}$, $\ell \in [\mathrm{K_a}]$, which represents the event that the other transmitted messages are distinct to the i-th transmitted message and transmitted codewords fulfill the power constraint, and $p_0 := \frac{\mathrm{K_a}(\mathrm{K_a}-1)}{2\mathrm{M}} + \sum_{i=1}^{\mathrm{K_a}} \Pr(\|X_i^n\|^2 > n\mathrm{P}')$ is the upper bound of the probability that collisions or power constraint violations occur.

Backup: Proof of Theorem 1 ii

- The PUPE of an $(n, \mathsf{M}, \epsilon, \mathsf{K_a}, \alpha, D^{\mathsf{K_a}})$-code can be upper bounded by the union bound as follows:

$$\mathrm{P}_{\mathrm{PUPE}|D^{\mathsf{K_a}}} := \sum_{\ell=1}^{\mathsf{K_a}} \frac{1}{\mathsf{K_a}} \Pr\left(\tilde{\mathcal{E}}_\ell | D^{\mathsf{K_a}}\right) \tag{16}$$

$$\leq p_0 + \sum_{\ell=1}^{\mathsf{K_a}} \frac{1}{\mathsf{K_a}} \Pr\left(M_\ell \notin g(Y^n) | D^{\mathsf{K_a}}, \mathcal{E}_\ell\right). \tag{17}$$

- To simplify the notation, we omit the condition $D^{\mathsf{K_a}}$ in the following derivation.

- For any subset $\mathcal{S} \subseteq [\mathsf{K_a}]$, we define

$$\tilde{\gamma}(\bar{X}^n_{\mathcal{S}}, X^n_{[\mathsf{K_a}]\setminus\mathcal{S}}) := \sum_{\ell=1}^{n} i(\{\tau_{d_m}(\bar{X}^n_m, \ell)\}_{m\in\mathcal{S}}, \{\tau_{d_m}(X^n_m, \ell)\}_{m\in[\mathsf{K_a}]\setminus\mathcal{S}}; Y_\ell)$$

and

$$\gamma(\bar{X}^n_{\mathcal{S}}) := \sum_{\ell=1}^{n} i(\{\tau_{d_m}(\bar{X}^n_m, \ell)\}_{m\in\mathcal{S}}; Y_\ell | \{\tau_{d_m}(X^n_m, \ell)\}_{m\in[\mathsf{K_a}]\setminus\mathcal{S}}).$$

Backup: Proof of Theorem 1 iv

- We define a set

$$\Sigma(\ell) := \{\mathcal{S} : \mathcal{S} \subseteq [\mathsf{K_a}], \ell \in \mathcal{S}\}, \tag{18}$$

which contains all possible subsets $\mathcal{S}$ of the error event $\{M_\ell \notin g(Y^n)\}$. Substitute the definition of the maximal information density decoder into $\Pr(M_\ell \notin g(Y^n)|\mathcal{E}_\ell)$,

Backup: Proof of Theorem 1 v

we have

$$
\begin{aligned}
&\Pr(M_\ell \notin g(Y^n)|\mathcal{E}_\ell) \\
&= \Pr\left(\bigcup_{\substack{\mathcal{S}\in\Sigma(\ell),\\ \bar{X}^n_{\mathcal{S}}\neq X^n_{\mathcal{S}}}} \left\{ \tilde{\gamma}(\bar{X}^n_{\mathcal{S}}, X^n_{[\mathrm{K_a}]\setminus\mathcal{S}}) > \tilde{\gamma}(X^n_{[\mathrm{K_a}]}) \right\} \middle| \mathcal{E}_\ell \right) \qquad (19) \\
&= \Pr\left(\bigcup_{\substack{\mathcal{S}\in\Sigma(\ell),\\ \bar{X}^n_{\mathcal{S}}\neq X^n_{\mathcal{S}}}} \left\{ \gamma(\bar{X}^n_{\mathcal{S}}) > \gamma(X^n_{\mathcal{S}}) \right\} \middle| \mathcal{E}_\ell \right) \qquad (20)
\end{aligned}
$$

Backup: Proof of Theorem 1 vi

$$= \mathbb{E}\left[\Pr\left(\bigcup_{\substack{\mathcal{S}\in\Sigma(\ell),\\ \bar{X}^n_{\mathcal{S}}\neq X^n_{\mathcal{S}}}} \{\gamma(\bar{X}^n_{\mathcal{S}}) > \gamma(X^n_{\mathcal{S}})\} \,\middle|\, X^n_{[\mathsf{K_a}]}, Y^n, \mathcal{E}_\ell\right)\right] \tag{21}$$

$$\leq \mathbb{E}\Bigg[\min\Bigg\{1, \sum_{\mathcal{S}\in\Sigma(\ell)} \binom{\mathsf{M}-\mathsf{K_a}}{|\mathcal{S}|}|\mathcal{S}|! \cdot \Pr\left(\gamma(\bar{X}^n_{\mathcal{S}}) > \gamma(X^n_{\mathcal{S}}) \,\middle|\, X^n_{[\mathsf{K_a}]}, Y^n, \mathcal{E}_\ell\right)\Bigg] \tag{22}$$

$$\leq \mathbb{E}\left[\min\left\{1, \sum_{\mathcal{S}\in\Sigma(\ell)} \mathsf{M}^{|\mathcal{S}|}\exp(-\gamma(X^n_{\mathcal{S}}))\right\}\right] \tag{23}$$

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$$\leq \sum_{\mathcal{S}\in\Sigma(\ell)} \mathrm{E}\left[\min\left\{1, \mathsf{M}^{|\mathcal{S}|}\exp(-\gamma(X_{\mathcal{S}}^{n}))\right\}\right] \tag{24}$$

$$\leq \sum_{\mathcal{S}\in\Sigma(\ell)} \Pr\left(\mathsf{M}^{|\mathcal{S}|}\exp(-\gamma(X_{\mathcal{S}}^{n})) \geq U\right) \tag{25}$$

$$= \sum_{\mathcal{S}\in\Sigma(\ell)} \Pr\left(\log\left(\mathsf{M}^{|\mathcal{S}|}\exp(-\gamma(X_{\mathcal{S}}^{n}))\right) - \log(U) \geq 0\right) \tag{26}$$

$$= \sum_{\mathcal{S}\in\Sigma(\ell)} \Pr(W_{\mathcal{S}} \geq 0), \tag{27}$$

where (19) is due to the definition of the maximum information density decoder, (20) is due to the chain rule of information density. The random coding scheme and union bound are used in (21) and (22), respectively. Note that the asynchronous model does not have the permutation-invariant property. Therefore, the number of permutations of the erroneously decoded messages, $|\mathcal{S}|!$, is summed up. The inequality (23) follows from the fact that $\binom{\mathsf{M}-\mathsf{K_a}}{|\mathcal{S}|} \cdot |\mathcal{S}|! \leq \mathsf{M}^{|\mathcal{S}|}$ and

$$\Pr(\gamma(\bar{X}^n_{\mathcal{S}}) > \gamma(X^n_{\mathcal{S}})) \leq \exp(-\gamma(X^n_{\mathcal{S}})),$$

where $\bar{X}^n_{\mathcal{S}}$ is an independent copy of $X^n_{\mathcal{S}}$[17]. The inequality (24) follows from $\min\{1, \beta_1 + \beta_2\} \leq \min\{1, \beta_1\} + \min\{1, \beta_2\}$ for $\beta_1, \beta_2 \in \mathbb{R}$. The inequality (25) follows from $\mathbb{E}[\min\{1, V\}] = \Pr(V \geq U)$ [12, eq.(77)] for a non-negative random variable V, where $U \sim \mathrm{Unif}(0, 1)$ is independent of V. The equality (27) follows from defining

$$W_{\mathcal{S}} := \log\left(\mathsf{M}^{|\mathcal{S}|} \exp(-\gamma(X^n_{\mathcal{S}}))\right) - \log(U).$$

- We apply the CGF, the Taylor expansion, and the inverse Laplace transform to derive $\Pr(W_{\mathcal{S}} \geq 0)$.

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- We denote by $\psi_{W_{\mathcal{S}}}(t) = \log(\mathbb{E}[\exp(tW_{\mathcal{S}})])$ the CGF of the random variable $W_{\mathcal{S}}$ with parameter t.

$$\begin{aligned}
&\psi_{W_{\mathcal{S}}}(t) \\
&= \log\left(\mathbb{E}\left[\exp\left(t\log\left(\mathsf{M}^{|\mathcal{S}|}\exp(-\gamma(X_{\mathcal{S}}^n))\right) - t\log(U)\right)\right]\right) && (28)\\
&= t|\mathcal{S}|\log\mathsf{M} - \log(1-t) + \log(\mathbb{E}[\exp(-t\cdot\gamma(X_{\mathcal{S}}^n))]) && (29)\\
&= t|\mathcal{S}|\log\mathsf{M} - \log(1-t) - E(a^n, t) && (30)\\
&= \tilde{\psi}_{W_{\mathcal{S}}}(t) - \log(1-t), && (31)
\end{aligned}$$

where $t \in (0,1)$, $\tilde{\psi}_{W_{\mathcal{S}}}(t) := t|\mathcal{S}|\log \mathsf{M} - E(a^n, t)$, and (30) is due to the following definition in Theorem 1,

$$E(a^n, t) := -\log(\mathbb{E}[\exp(-t\gamma(X_{\mathcal{S}}^n))])$$
$$= \frac{1}{2}\sum_{i=1}^{n}\left(t\log(1+a_i\mathsf{P}) + \log\left(1 - \frac{a_i\mathsf{P}t^2}{1+a_i\mathsf{P}}\right)\right),$$

where

$$\exp(-t\cdot\gamma(X_{\mathcal{S}}^n)) = \prod_{\ell=1}^{n}\left(\frac{dP_{Y|X_{[\mathsf{K_a}]}}\left(Y_\ell\middle|\left\{\tau_{d_m}(X_m^n,\ell)\right\}_{m\in[\mathsf{K_a}]}\right)}{dP_Y(Y_\ell)}\right)^t.$$

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- For $t \in (0,1)$, the CGF converges, which is proved as follows. Since the CGF is the summation of the logarithm of the following n terms,

$$\mathbb{E}[\exp(t \cdot i(\{\tau_{d_m}(X_m^n, \ell)\}_{m\in\mathcal{S}}; Y_\ell | \{\tau_{d_m}(X_m^n, \ell)\}_{m\in[\mathsf{K_a}]\setminus\mathcal{S}}))], \tag{32}$$

  $\ell = 1, 2, \ldots, n$, for a CGF to converge, a sufficient condition is that (32) converges in term of t for all $\ell \in [n]$. We apply the Gaussian integral to derive (32). The corresponding range of convergence for any $\ell \in [n]$ is $t \in \left(-\frac{1+a_\ell \mathsf{P}}{a_\ell \mathsf{P}}, \sqrt{\frac{1+a_\ell \mathsf{P}}{a_\ell \mathsf{P}}}\right)$. When $t < 0$, it is possible that $|\mathcal{S}| \log \mathsf{M} > \sum_{\ell=1}^{n} \frac{1}{2} \log(1 + a_\ell \mathsf{P})$, which means that the corresponding error probability approaches 1.
- For all $\ell \in [n]$ and $a_\ell \in \mathbb{Z}^+$, $\sqrt{\frac{1+a_\ell \mathsf{P}}{a_\ell \mathsf{P}}} \geq 1$. If $a_\ell = 0$, $\ell \in [n]$, it represents that no codeword symbol is transmitted at the ℓ-th channel use. Thus, the information density is 0 and the corresponding (32) must converge.

- Therefore, in both theorems, we choose $t = t_0(a^n) \in (0, 1)$, which fulfills

$$|\mathcal{S}| \log \mathrm{M} = E_t^{(1)}(a^n, t_0(a^n)), \tag{33}$$

to guarantee the convergence.

- The PDF of $W_{\mathcal{S}}$ is obtained by the inverse Laplace transform:

$$f_{W_{\mathcal{S}}}(w) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \exp(\psi_{W_{\mathcal{S}}}(t) - tw) dt, \tag{34}$$

where $c \in (0, 1)$.

- The probability, $\Pr(W_{\mathcal{S}} \geq 0)$, is obtained by changing the order of integration, i.e.,

$$\Pr(W_{\mathcal{S}} \geq 0) = \frac{1}{2\pi j} \int_{0}^{\infty} \left\{ \int_{c-j\infty}^{c+j\infty} \exp(\psi_{W_{\mathcal{S}}}(t) - tw) dt \right\} dw \tag{35}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \exp(\psi_{W_{\mathcal{S}}}(t)) \frac{dt}{t} \tag{36}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \exp(\tilde{\psi}_{W_{\mathcal{S}}}(t)) \frac{dt}{t(1-t)}. \tag{37}$$

- The last equality follows from (31).

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- By applying the Taylor expansion to $\tilde{\psi}_{W_{\mathcal{S}}}(t)$ at the point $t = t_0(a^n)$, which fulfills (33), we have

$$\begin{aligned}\tilde{\psi}_{W_{\mathcal{S}}}(t) =& t_0(a^n)|\mathcal{S}|\log \mathrm{M} - E(a^n, t_0(a^n)) \\ &+ \left[|\mathcal{S}|\log \mathrm{M} - E_t^{(1)}(a^n, t_0(a^n))\right](t - t_0(a^n)) \\ &- E_t^{(2)}(a^n, t_0(a^n))\frac{(t - t_0(a^n))^2}{2} + \bar{\xi}(a^n, t_0(a^n), t), \end{aligned} \tag{38}$$

where $\left[|\mathcal{S}|\log \mathrm{M} - E_t^{(1)}(a^n, t_0(a^n))\right](t - t_0(a^n)) = 0$ due to (33),

$$\bar{\xi}(a^n, t_0(a^n), t) := \sum_{i=3}^{\infty} -E_t^{(i)}(a^n, t_0(a^n))\frac{(t - t_0(a^n))^i}{i!}$$

is the sum of higher order terms of Taylor expansion, and $t_0(a^n)$ satisfies (33).

- Substitute (38) and $c = t_0(a^n)$ into (37), we have

$$\frac{1}{2\pi j}\int_{t_0(a^n)-j\infty}^{t_0(a^n)+j\infty}\exp(\tilde{\psi}_{W_S}(t))\frac{dt}{t(1-t)}$$
$$=\frac{\eta}{j}\int_{t_0(a^n)-j\infty}^{t_0(a^n)+j\infty}\exp\left(\beta\frac{(t-t_0(a^n))^2}{2}+\bar{\xi}(a^n,t_0(a^n),t)\right)\frac{dt}{t(1-t)} \tag{39}$$

$$=\eta\left\{\frac{1}{j}\int_{t_0(a^n)-j\infty}^{t_0(a^n)+j\infty}\exp\left(\beta\frac{(t-t_0(a^n))^2}{2}\right)\frac{dt}{t(1-t)} + 2\pi\xi(a^n,t_0(a^n))\right\} \tag{40}$$

$$
\begin{aligned}
&= \eta \Big\{ \int_{-\infty}^{\infty} \exp\Big(-\beta \frac{\rho^2}{2} \Big) \frac{d\rho}{t_0(a^n) + j\rho} \\
&\quad + \int_{-\infty}^{\infty} \exp\Big(-\beta \frac{\rho^2}{2} \Big) \frac{d\rho}{1 - t_0(a^n) - j\rho} + 2\pi \xi(a^n, t_0(a^n)) \Big\},
\end{aligned} \tag{41}
$$

where $\eta := \frac{g_1(a^n, t_0(a^n))}{2\pi}$, $\beta := -E_t^{(2)}(a^n, t_0(a^n))$, $\rho := \frac{t - t_0(a^n)}{j}$, and

$$
g_1(a^n, t) := \exp(t|\mathcal{S}| \log \mathrm{M} - E(a^n, t)).
$$

The equality (40) follows from $e^x = 1 + \sum_{i=1}^{\infty} \frac{x^i}{i!}$ and by letting $x = \bar{\xi}(a^n, t_0(a^n), t)$,

$$
\begin{aligned}
\xi(a^n, t_0(a^n)) &:= \frac{1}{2\pi j} \int_{t_0(a^n) - j\infty}^{t_0(a^n) + j\infty} \exp\Big(\frac{\beta}{2} (t - t_0(a^n))^2 \Big) \\
&\quad \cdot \frac{1}{t(1-t)} \sum_{m=1}^{\infty} \frac{\bar{\xi}(a^n, t_0(a^n), t)^m}{m!} dt.
\end{aligned} \tag{42}
$$

- By multiplying $\frac{t_0(a^n)-j\rho}{t_0(a^n)-j\rho}$ to the first integral in (41), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp\left(-\beta\frac{\rho^2}{2}\right)\frac{d\rho}{t_0(a^n)+j\rho} \\
&= \int_{-\infty}^{\infty} \exp\left(-\beta\frac{\rho^2}{2}\right)\frac{t_0(a^n)d\rho}{t_0(a^n)^2+\rho^2} \\
&\qquad - \int_{-\infty}^{\infty} \exp\left(-\beta\frac{\rho^2}{2}\right)\frac{j\rho d\rho}{t_0(a^n)^2+\rho^2} \qquad (43)
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \exp\left(-\beta\frac{\rho^2}{2}\right)\frac{t_0(a^n)d\rho}{t_0(a^n)^2+\rho^2} \tag{44}$$

$$= 2\pi \exp\left(\frac{t_0(a^n)^2\beta}{2}\right) Q\left(t_0(a^n)\sqrt{\beta}\right) \tag{45}$$

$$\leq \frac{\sqrt{2\pi}}{t_0(a^n)}\frac{1}{\sqrt{\beta}} \tag{46}$$

$$= \frac{\sqrt{2\pi}}{t_0(a^n)}\frac{1}{\sqrt{-E_t^{(2)}(a^n, t_0(a^n))}}, \tag{47}$$

where the second integral in (43) is the integral of an odd function, which equals 0. By applying the Voigt function **[Finn:1965:TBF]** to the integral in (44), we have (45). The inequality (47) follows from the upper bound of the Gaussian Q-function,

$Q(x) \leq \frac{1}{x\sqrt{2\pi}} \exp(-\frac{x^2}{2})$. The last equality follows from the definition: $\beta := -E_t^{(2)}(a^n, t_0(a^n))$.

- By multiplying $\frac{1-t_0(a^n)+j\rho}{1-t_0(a^n)+j\rho}$ with the same steps used in deriving (47), the second integral in (41) is bounded by

$$\int_{-\infty}^{\infty} \exp\left(-\beta\frac{\rho^2}{2}\right)\frac{d\rho}{1-t_0(a^n)-j\rho} \leq \frac{\sqrt{2\pi}}{1-t_0(a^n)}\frac{1}{\sqrt{-E_t^{(2)}(a^n, t_0(a^n))}}. \tag{48}$$

- Consequently, we can upper bound the sum of the two integrations in (41) as follows:

$$\begin{aligned}
&\eta\Bigg\{\int_{-\infty}^{\infty}\exp\Big(-\beta\frac{\rho^2}{2}\Big)\frac{d\rho}{t_0(a^n)+j\rho} \\
&\quad+\int_{-\infty}^{\infty}\exp\Big(-\beta\frac{\rho^2}{2}\Big)\frac{d\rho}{1-t_0(a^n)-j\rho}+2\pi\xi(a^n,t_0(a^n))\Bigg\} \\
&\leq \frac{g_1(a^n,t_0(a^n))}{\sqrt{2\pi}(1-t_0(a^n))t_0(a^n)}\frac{1}{\sqrt{-E_t^{(2)}(a^n,t_0(a^n))}} \\
&\qquad\qquad+g_1(a^n,t_0(a^n))\xi(a^n,t_0(a^n)).
\end{aligned} \tag{49}$$

By combining (17), (27), (37), (41), and (49), the PUPE of the AUMAC system for a given $D^{\mathsf{K_a}}$ is

$$\sum_{\ell=1}^{\mathsf{K_a}} \frac{1}{\mathsf{K_a}} \Pr(M_\ell \notin g(Y^n) | D^{\mathsf{K_a}}, \mathcal{E}_\ell) + p_0$$
$$\leq \sum_{\ell=1}^{\mathsf{K_a}} \frac{1}{\mathsf{K_a}} \sum_{\mathcal{S} \in \Sigma(\ell)} \left\{ \frac{g_1(a^n, t_0(a^n))}{(1 - t_0(a^n)) t_0(a^n)} \frac{1}{\sqrt{-2\pi E_t^{(2)}(a^n, t_0(a^n))}} \right.$$
$$\left. + g_1(a^n, t_0(a^n)) \xi(a^n, t_0(a^n)) \right\} + p_0 \tag{50}$$

$$= \sum_{\mathcal{S}\subseteq[\mathsf{K_a}]} \frac{|\mathcal{S}|}{\mathsf{K_a}} \Bigg\{ \frac{g_1(a^n, t_0(a^n))}{(1 - t_0(a^n))t_0(a^n)} \frac{1}{\sqrt{-2\pi E_t^{(2)}(a^n, t_0(a^n))}} + g_1(a^n, t_0(a^n))\xi(a^n, t_0(a^n)) \Bigg\} + p_0, \tag{51}$$

where $\Sigma(\ell)$ is defined in (18).

[16] Yury Polyanskiy, H. Vincent Poor, and Sergio Verdu. "Channel Coding Rate in the Finite Blocklength Regime." In: *IEEE Trans. Inf. Theory* 56.5 (2010), pp. 2307–2359. DOI: 10.1109/TIT.2010.2043769.

[17] Yury Polyanskiy and Yihong Wu. *Information Theory: From Coding to Learning*. English. Cambridge University Press, 2023. ISBN: 0-19-852295-9, Corollary 18.4.

Appendix: Proof of Theorem 2 i

- In the following, in addition to Theorem ??, we derive a uniform upper bound of the PUPE of an $(n, \mathsf{M}, \epsilon, \mathsf{K_a}, \alpha)$-code as indicated in Theorem 2. In particular, we will find the worst-case asynchronicity, which implies finding the worst-case of a^n and $t_0(a^n)$ in Theorem ??.
- To simplify the derivation, we denote $\iota := (1 \in \mathcal{S})$ and all possible a^n's w.r.t. ι by the set $\mathcal{F}_{k,\iota} := \{a^n : \iota, |\mathcal{S}| = k\}$, where a^n is defined in (1) as a function of $\mathcal{S}$ and $D^{\mathsf{K_a}}$.
- We will show that for all $t \in (0, 1)$, there exists an a_ι^{n*} resulting in a uniform upper bound of PUPE for all $a^n \in \mathcal{F}_{|\mathcal{S}|,\iota}$, such that the upper bound of the PUPE in (4) has the following property

$$g_1(a^n, t)g_2(a^n, t) \leq g_1(a_\iota^{n*}, t)g_2(a_\iota^{n*}, t).$$

- However, for any $a^n \in \mathcal{F}_{|\mathcal{S}|,\iota}$, the order between

$$g_1(a^n, t_0(a^n))g_2(a^n, t_0(a^n))$$

and

$$g_1(a_\iota^{n*}, t_0(a_\iota^{n*}))g_2(a_\iota^{n*}, t_0(a_\iota^{n*}))$$

is not fixed, since the sign of $\frac{\partial}{\partial t} g_1(a^n, t)g_2(a^n, t)$ is not the same for all $t \in (0, 1)$.

- Therefore, for fixed a_ι^{n*}, we will show that the choices of T_0^*, T_1^*, $\underline{t}_0$, and $\underline{t}_1$ uniformly upper bound the PUPE regardless $D^{\mathsf{K_a}}$.

Appendix: Proof of Theorem 2 iii

- We start from (4) restated as follows

$$\sum_{\mathcal{S}\subseteq[\mathsf{K_a}]} \frac{|\mathcal{S}|}{\mathsf{K_a}\sqrt{2\pi}} g_1(a^n, t_0(a^n))g_2(a^n, t_0(a^n)), \tag{52}$$

  while omitting the term p_0 and also the approximation error term $\xi(a^n, t_0(a^n))$ since we do not bound these terms.
- To proceed, we use the following lemma.

Lemma

Let $g_1(a^n, t) = \exp(f_1(a^n, t))$ and $g_2(a^n, t) = (f_2(a^n, t))^{-\frac{1}{2}}$, where $a^n \in \{\mathbb{Z}_0^+\}^n$, $t \in (0, 1)$, $f_1(a^n, t) \in \mathbb{R}$ and $f_2(a^n, t) > 0$. Then $g_1(a^n, t)g_2(a^n, t)$ is a non-increasing function w.r.t. a_i, $\forall i \in [n]$ if $f_{1,a_i}^{(1)}(a_i, t) \leq 0$ and $f_{2,a_i}^{(1)}(a_i, t) \geq 0$.

The proof of Lemma 1 is relegated to the next Appendix.

Appendix: Proof of Theorem 2 iv

- Then, we apply Lemma 4 by defining $f_1(a^n, t) := t|\mathcal{S}|\log \mathrm{M} - E(a^n, t)$ and $f_2(a^n, t) := -(t - t^2)^2 E_t^{(2)}(a^n, t)$. The first derivatives of $f_1(a^n, t)$ and $f_2(a^n, t)$ w.r.t. a_i are expressed as follows, respectively

$$f_{1,a_i}^{(1)}(a_i, t) = \frac{\mathrm{P}(t^2 - t) + a_i \mathrm{P}^2(t^3 - t)}{2(1 + a_i\mathrm{P})(1 + a_i\mathrm{P} - a_i\mathrm{P}t^2)} \tag{53}$$

and

$$f_{2,a_i}^{(1)}(a_i, t) = \frac{\mathrm{P}(1 - t)^2 t^2 (1 + a_i\mathrm{P} + 3a_i\mathrm{P}t^2)}{(1 + a_i\mathrm{P} - a_i\mathrm{P}t^2)^3}. \tag{54}$$

- For $t \in (0, 1)$, it is clear that $f_{1,a_i}^{(1)}(a_i, t) \leq 0$ and $f_{2,a_i}^{(1)}(a_i, t) \geq 0$. We then conclude that $g_1(a^n, t) g_2(a^n, t)$ is a non-increasing function w.r.t. a_i, $i \in [\alpha n]$ according to Lemma 4.

- It implies that the PUPE of any given $\mathcal{S}$ decreases with increasing $a_i,\ i \in [\alpha n]$. Namely, reducing $a_i,\ i \in [\alpha n]$ will upper bound the error probability.
- Therefore, to upper bound the PUPE, we can consider the following case, where the number of transmitted symbols that belong to $\mathcal{S}$ at the first αn channel use, $a_{[\alpha n]}$, are reduced to the minimum, which is $a^*_{\iota,[\alpha n]} = \iota$. Namely $a^{n*}_{\iota} = [\iota^{\alpha n}, |\mathcal{S}|^{n-\alpha n}]$.
- Consequently, for all $a^n \in \mathcal{F}_{|\mathcal{S}|,\iota}$ and a given $t = t_0(a^n)$, we have

$$g_1(a^n, t_0(a^n))g_2(a^n, t_0(a^n)) \leq g_1(a^{n*}_{\iota}, t_0(a^n))g_2(a^{n*}_{\iota}, t_0(a^n)). \tag{55}$$

- We have shown that the error probability is non-increasing w.r.t. a_i. However, the sign of $\frac{\partial}{\partial t} g_1(a^n, t)g_2(a^n, t)$ w.r.t. t changes for $t \in (0, 1)$.

- To solve it, we can show that given a_ι^{n*}, if $t_0(a_\iota^{n*}) \in \mathcal{A} \cap \mathcal{B}$, $\bar{t}_\iota \in \mathcal{A} \cap \bar{\mathcal{B}}$, $\underline{t}_\iota \in \mathcal{A} \cap \underline{\mathcal{B}}$, and $\underline{t}_\iota \le t_0(a^n) \le \bar{t}_\iota$, then there exist a uniform upper bound of the error probability for all $D^{\mathsf{K_a}}$ satisfying delay constraint αn, where

$$\mathcal{A} := \{t : t \in (0,1)\}, \tag{56}$$

$$\mathcal{B} := \left\{t : E_t^{(1)}(a_\iota^{n*}, t) = |\mathcal{S}| \log \mathsf{M}\right\}, \tag{57}$$

$$\bar{\mathcal{B}} := \left\{t : \sum_{i=1}^{n} \frac{a_{\iota,i}^* \mathsf{P} t}{1 + a_{\iota,i}^* \mathsf{P}(1 - t^2)} = \frac{n}{2} \log(1 + |\mathcal{S}|\mathsf{P}) - |\mathcal{S}| \log \mathsf{M}\right\}, \tag{58}$$

$$\underline{\mathcal{B}} := \left\{t : \frac{|\mathcal{S}| n \mathsf{P} t}{1 + |\mathcal{S}|\mathsf{P} - |\mathcal{S}|\mathsf{P} t^2} = \sum_{i=1}^{n} \frac{1}{2} \log(1 + a_{\iota,i}^* \mathsf{P}) - |\mathcal{S}| \log \mathsf{M}\right\}, \tag{59}$$

and $a_{\iota,i}^*$ is the i-th element of a_ι^{n*}.

- To proceed, we find upper bounds of $g_1(a_\iota^{n*}, t_0(a^n))$ and $g_2(a_\iota^{n*}, t_0(a^n))$ as u_1 and u_2. Then we upper bound $g_1(a_\iota^{n*}, t_0(a^n))g_2(a_\iota^{n*}, t_0(a^n))$ by $u_1 u_2$.
- Since the second partial derivative w.r.t. t,

$$f_{1,t}^{(2)}(a^n, t) = \sum_{i=1}^{n} \frac{(a_i \mathrm{P}) + (a_i \mathrm{P})^2 + (a_i \mathrm{P})^2 t^2}{(1 + a_i \mathrm{P} - a_i \mathrm{P} t^2)^2}, \tag{60}$$

is positive for $t \in (0, 1)$, $f_1(a^n, t)$ is a convex function regarding t. Moreover, $f_{1,t}^{(1)}(a^n, t_0(a^n)) = 0$ by (57). Namely, $f_1(a^n, t)$ achieves minimum at $t = t_0(a^n)$.

- Therefore, for any $a^n \in \mathcal{F}_{|\mathcal{S}|,\iota}$, we have

$$g_1(a^n, t_0(a^n)) \leq g_1(a^n, t_0(a_\iota^{n*})) \leq g_1(a_\iota^{n*}, t_0(a_\iota^{n*})), \tag{61}$$

where the first inequality is because $g_1(a^n, t)$ achieves the minimum at $t = t_0(a^n)$. If $a^n = a_\iota^{n*}$, the equalities hold. The second inequality follows from the fact that $g_1(a^n, t)$ is a non-decreasing function for a given $t \in (0, 1)$ w.r.t. a_i, $\forall i \in [\alpha n]$, since

$$g_{1,a_i}^{(1)}(a_i, t) = \exp(f_1(a^n, t)) \cdot f_{1,a_i}^{(1)}(a_i, t) \leq 0,$$

for $t \in (0, 1)$, where $f_{1,a_i}^{(1)}(a_i, t)$ is given in (53).

Appendix: Proof of Theorem 2 ix

- We define $f_2(a^n, t) := (f_3(t))^2 f_4(a^n, t)$, where $f_3(t) := t - t^2$ and $f_4(a^n, t) := -E_t^{(2)}(a^n, t)$. Since the first partial derivative w.r.t. t of $f_{4,t}(a^n, t)$ is as follows

$$f_{4,t}^{(1)}(a^n, t) = \sum_{i=1}^{n} 2(a_i \mathrm{P})^2 t \frac{3 + 3a_i \mathrm{P} + a_i \mathrm{P} t^2}{(1 + a_i \mathrm{P} - a_i \mathrm{P} t^2)^3}, \tag{62}$$

which is positive for $t \in (0, 1)$, $f_4(a^n, t)$ is a non-decreasing function of t. Then, we have

$$f_4(a_\iota^{n*}, t_0(a_\iota^{n*})) \geq f_4(a_\iota^{n*}, \underline{t}_\iota). \tag{63}$$

Appendix: Proof of Theorem 2 x

- By the condition, $\underline{t}_\iota \leq t_0(a^n) \leq \bar{t}_\iota$, there exists a $\lambda \in [0,1]$ such that $t_0(a^n) = \lambda \underline{t}_\iota + \bar{\lambda} \bar{t}_\iota$, where $\bar{\lambda} = 1 - \lambda$. Since $f_3(t)$ is concave, it satisfies

$$\begin{aligned} f_3(t_0(a^n)) &\geq \lambda f_3(\underline{t}_\iota) + \bar{\lambda} f_3(\bar{t}_\iota) && (64) \\ &\geq \lambda \min\{f_3(\bar{t}_\iota), f_3(\underline{t}_\iota)\} + \bar{\lambda} \min\{f_3(\bar{t}_\iota), f_3(\underline{t}_\iota)\} && (65) \\ &= \min\{f_3(\bar{t}_\iota), f_3(\underline{t}_\iota)\} =: T_\iota^*. && (66) \end{aligned}$$

- Consequently, for all $a^n \in \mathcal{F}_{|\mathcal{S}|,\iota}$, we have

$$g_1(a_\iota^{n*}, t_0(a^n)) \leq g_1(a_\iota^{n*}, t_0(a_\iota^{n*})), \tag{67}$$

stated in (61), and

$$g_2(a_\iota^{n*}, t_0(a^n)) = \frac{1}{f_3(t_0(a^n))\sqrt{f_4(a_\iota^{n*}, t_0(a^n))}} \tag{68}$$

$$\leq \frac{1}{T_\iota^* \sqrt{f_4(a_\iota^{n*}, t_0(a^n))}} \tag{69}$$

$$\leq \frac{1}{T_\iota^* \sqrt{f_4(a_\iota^{n*}, \underline{t}_\iota)}}, \tag{70}$$

where (68) is by definition, (69) follows from (66) and (70) follows from (63).

- Consequently, we have

$$
\begin{aligned}
& g_1(a^n, t_0(a^n))g_2(a^n, t_0(a^n)) \\
& \leq g_1(a_\iota^{n*}, t_0(a^n))g_2(a_\iota^{n*}, t_0(a^n)) && (71) \\
& \leq g_1(a_\iota^{n*}, t_0(a_\iota^{n*}))g_2(a_\iota^{n*}, t_0(a^n)) && (72) \\
& \leq \frac{g_1(a_\iota^{n*}, t_0(a_\iota^{n*}))}{T_\iota^* \sqrt{-E_t^{(2)}(a_\iota^{n*}, \underline{t}_\iota)}}, && (73)
\end{aligned}
$$

which completes the proof of Theorem 2.

Appendix: Proof of Lemma i

Let $g_1(a^n, t) = \exp(f_1(a^n, t))$ and $g_2(a^n, t) = (f_2(a^n, t))^{-\frac{1}{2}}$, where $a^n \in \{\mathbb{Z}_0^+\}^n$, $t \in (0, 1)$ and $f_1(a^n, t) \in \mathbb{R}$, and $f_2(a^n, t) > 0$. Then the first partial derivative of $g_1(a^n, t)g_2(a^n, t)$ w.r.t. a_i is

$$\begin{aligned}
&\frac{\partial}{\partial a_i} g_1(a^n, t) g_2(a^n, t) \\
&= g_2(a^n, t) \frac{\partial}{\partial a_i} g_1(a^n, t) + g_1(a^n, t) \frac{\partial}{\partial a_i} g_2(a^n, t) \qquad (74) \\
&= g_2(a^n, t) \exp(f_1(a^n, t)) \frac{\partial}{\partial a_i} f_1(a^n, t) \\
&\qquad - \frac{1}{2} g_1(a^n, t)(f_2(a^n, t))^{-\frac{3}{2}} \frac{\partial}{\partial a_i} f_2(a^n, t). \qquad (75)
\end{aligned}$$

Therefore, $g_1(a^n, t)g_2(a^n, t)$ is a non-increasing function w.r.t. a_i, if $\frac{\partial}{\partial a_i} f_1(a^n, t) \leq 0$ and $\frac{\partial}{\partial a_i} f_2(a^n, t) \geq 0$.

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