BIRS Conference Report
Advances in Hierarchical Hyperbolicity (24w5254)
May 2024

Organizers: Carolyn Abbott, Jason Behrstock, Jacob Russell

1 Conference Focus and Goals

Gromov's definition of a hyperbolic group gave a robust notion of negative curvature for finitely generated groups, revolutionizing the study of group theory and its connections to low-dimensional topology. Many important groups from topology (e.g., mapping class groups, 3-manifold groups, Artin groups) have features reminiscent of negative curvature, but fail to be Gromov-hyperbolic as they contain a mixture of flat and negatively curved phenomena. Understanding the geometry of these "non-positively curved" groups has been a central theme of geometric group theory for the last 30 years.

A recent but powerful and prolific notion of non-positive curvature is hierarchical hyperbolicity. Hierarchical hyperbolicity provides an axiomatic framework that unifies the geometry of the mapping class group, 3-manifold groups, many cubulated groups, and many Artin groups. This framework results in powerful machinery for understanding the geometry of a wide variety of important groups and spaces simultaneously. It has led to advances in understanding a number of important properties of these spaces, including their asymptotic dimension, the structure of their quasi-flats, and other aspects of their metric geometry. This unified framework has also allowed for the development of techniques that would have been inaccessible in any of the individual settings.

This conference brought together leading experts and junior researchers from around the world who are working in the new area of hierarchical hyperbolicity to discuss the flurry of promising new results in the area and directions for the future.

In addition to research lectures, we also included both survey lectures and discussion periods. The surveys focused on some of the major subtopics in the area. The discussion periods each had an appointed moderator and served to allow interactive follow-up conversations to the lectures, as well as time for rooms full of experts and aspiring-experts to bounce ideas around and have discussions to help lead junior and senior researchers toward new directions of study.

2 Program

Monday	**May 27, 2024**
09:00–10:00	Alessandro Sisto: *Combinatorial HHSs**
10:40–11:40	Matthew Durham: *Cubical approximations and asymptotic CAT(0) metrics**
14:30–15:00	Abdul Zalloum: *Injective metric spaces*
15:30–16:30	Harry Petyt: *Wallspaces and fine structures for HHSs*
16:30–17:30	*Group Discussion 1b: Cubulation of Hulls*
16:30–17:30	*Group Discussion 1a: Combinatorial HHS*

Tuesday	**May 28, 2024**
09:00–10:00	Christopher Leininger: *Atoroidal surface bundles*
10:30–11:30	Spencer Dowdall: *Lattice Veech groups and geometric finiteness in mapping class groups**
13:30–14:00	Stefanie Zbinden: *Using strong contraction to obtain hyperbolicity*
14:00–14:30	Eliot Bongiovanni: *Extensions of Finitely Generated Veech Groups*
14:30–15:00	Brian Udall: *Parabolically geometrically finite subgroups of mapping class groups*
15:30–16:30	Giorgio Mangioni: *Combinatorial data from quasi-isometries, and quasi-isometric rigidity of (random quotients of) mapping class groups.*
16:30–17:30	*Group Discussion 2a: Geometric Finiteness*
16:30–17:30	*Group Discussion 2b: QI Rigidity*

Wednesday	**May 29, 2024**
09:00–10:00	Mara Cumplido: *Hyperbolicity in Artin groups**
10:15–10:45	Thomas Ng: *Uniform exponential growth in HHGs*
10:45–11:30	*Discussion: Artin Groups*

Thursday	**May 30, 2024**
09:00–10:00	Mark Hagen: *Universal real cubings**
10:30–11:30	Alex Wright: *Spheres in the curve graph and linear connectivity of the Gromov boundary*
14:00–15:00	Wenyuan Yang: *Uniform exponential growth for groups with proper product actions on hyperbolic spaces*
15:30–16:30	Mahan Mj: *Hyperbolic commensurations and infinite hierarchies*
16:30–17:30	*Discussion: New Examples and Obstructions*

Five of the talks, marked with asterisks above, were primarily expository, aimed at introducing researchers to key questions and techniques in a major area of research. The remaining talks focused on new developments.

We aimed to make each day of the conference somewhat thematic. The first day was primarily about combinatorical HHSs and cubulations of hulls. The second day about geometrical finiteness and quasi-isometric rigidity. The third day was about Artin groups. The last day was about different new directions that people are developing using the HHS machinery. We ended the conference with a discussion which resulted in a problem list which we will make public and distribute on our websites.

3 Presentations

Alessandro Sisto
Title: Combinatorial HHSs
Abstract: I will discuss a criterion of combinatorial flavor to check that a given space is an HHS, in particular giving a "user's guide" on how to apply it in practice. This criterion is quite useful to construct new examples of HHSs, and it is often a lot simpler than checking the axioms directly.

Matthew Durham
Title: Cubical approximations and asymptotic CAT(0) metrics
Abstract: Behrstock-Hagen-Sisto's cubical model theorem says HHSes are locally modeled by CAT(0) cube complexes analogously to how manifolds are locally modeled by Euclidean space. In the first part of this talk, I'll first discuss some of the remarkable applications of this fundamental structural result, and then sketch an alternative construction which realizes each such cubical model as a weakly convex subset of a product of simplicial trees naturally arising from the ambient HHS structure. In the second part, I'll discuss joint work with Minsky and Sisto, in which we show that these local cubical approximations are coarsely coherent, analogous to having a manifold with well-behaved transition maps. Our main application builds an asymptotically CAT(0) metric for most HHSes, from which we derive a number of consequences.

Abdul Zalloum
Title: Injective metric spaces
Abstract: I will introduce the notion of an injective metric space and discuss certain injective spaces on which HHGs admit nice actions. When the HHG is the mapping class group of a finite type surface, I will show that geodesics in these injective spaces define re-parameterized quasi-geodesics in curve graph of the underlying surface.

Harry Petyt
Title: Wallspaces and fine structures for HHSs
Abstract: Sageev's construction says that, under a finiteness condition, you can make a

cube complex dual to a collection of walls by counting how many walls separate each pair of points. With Abdul Zalloum, we observed that if you restrict which families of walls you count, then you can relax the finiteness condition. For an HHS X, this lets you use natural walls (those used in the cubulation of hulls) to build a dual median algebra quasiisometric to X, which has improved fine properties.

Christopher Leininger
Title: Atoroidal surface bundles
Abstract: I will discuss joint work with Autumn Kent in which we construct the first known examples of compact atoroidal surface bundles over surfaces for which the base and fiber genus are both at least 2.

Spencer Dowdall
Title: Lattice Veech groups and geometric finiteness in mapping class groups
Abstract: Going beyond the setting of convex cocompactness, there is an effort to develop a theory of geometric finiteness for subgroups of mapping class groups that captures a broader range of behaviors and relates these to the structure of Teichmller space, the action on the curve complex and the geometry of surface group extensions as viewed, for example, via hierarchical hyperbolicity. I will survey some progress in this direction and highlight various examples that fit into this picture. We will focus on the case of lattice Veech groups, which are perhaps the prototypical candidates for geometric finiteness. I will describe the geometric structure of these groups and how it gives rise to hierarchical hyperbolicity and quasi-isometric rigidity for the associated surface group extensions. Joint work with Matt Durham, Chris Leininger, and Alex Sisto.

Stefanie Zbinden
Title: Using strong contraction to obtain hyperbolicity
Abstract: For almost 10 years, it has been known that if a group contains a strongly contracting element, then it is acylindrically hyperbolic. Moreover, one can use the Projection Complex of Bestvina, Bromberg and Fujiwara to construct a hyperbolic space where said element acts WPD. For a long time, the following question remained unanswered: if Morse is equivalent to strongly contracting, does there exist a space where all generalized loxodromics act WPD? In this talk, I will introduce the contraction space, a space which answers this question positively.

Eliot Bongiovanni
Title: Extensions of finitely generated Veech groups
Abstract: Given a closed surface S, a subgroup G of the mapping class group of S has an associated extension group Γ, which is the fundamental group of an S–bundle with monodromy an isomorphism to G. A general problem is to infer features of Γ from G. I take G to be a finitely generated Veech group and show that Γ is hierarchically hyperbolic. This is a generalization of results from Dowdall, Durham, Leininger, and Sisto regarding lattice

Veech groups. The focus of this talk is constructing a hyperbolic space $\hat{E}$ on which Γ acts nicely (isometrically and cocompactly). This examples contributes to the growing evidence of a good notion of "geometric finiteness" for subgroups of mapping class groups.

Brian Udall
Title: Parabolically geometrically finite subgroups of mapping class groups
Abstract: We will discuss the parabolically geometrically finite (PGF) subgroups of mapping class groups, which is a class of groups generalizing the definition of convex cocompact groups via the curve complex. This class contains all finitely generated Veech groups, as well as free products of multitwist groups on sufficiently far apart multicurves. We will discuss two results about these groups. First, they are undistorted as subgroups of the mapping class group. Second, we give a combination theorem which allows one to build many more examples of PGF groups. With whatever time is remaining afterwards, we will discuss open problems.

Giorgio Mangioni
Title: Combinatorial data from quasi-isometries, and quasi-isometric rigidity of (random quotients of) mapping class groups
Abstract: A result of Behrstock, Hagen, and Sisto states that, under suitable conditions, a self-quasi-isometry of a hierarchically hyperbolic space induces an automorphism of the "hinge graph", which roughly encodes the intersection patterns of standard quasiflats of the maximum dimension. If one is able to identify who this graph is, and most importantly the group of its simplicial automorphisms, then one can aim at some classification of quasi-isometries. In this talk we will first review the construction of the hinge graph and its automorphism. The machinery can then be applied to prove that random quotients of mapping class groups of surfaces are quasi-isometrically rigid, meaning that if such a quotient and a finitely generated group are quasi-isometric then they are weakly commensurable. The key point is that, in this case, the hinge graph is related to the corresponding quotient of the curve graph, whose automorphism group is the quotient group itself (this is the analogue of a result of Ivanov-Korkmaz for mapping class groups). If time permits, we will also show how quasi-isometric rigidity can be used to deduce other "rigidity" results, such as the fact that all automorphisms of such quotients are inner.

María Cumplido
Title: Hyperbolicity in Artin groups
Abstract: This talk aims to explain the state of the art of hyperbolicity in Artin groups and related complexes. We will see what the latest advances have been in hyperbolicity in the sense of Gromov, acylindrical hyperbolicity, and hierarchical hyperbolicity, as well as the main techniques that have been used.

Thomas Ng
Title: Uniform exponential growth in HHGs

Abstract: Exponential growth is well-known to be preserved under quasi-isometry. While, changing finite generating sets may yield exponential growth rate that are not uniformly bounded over all finite generating sets, uniform exponential growth has been proven for RAAGs using algebraic tools, and for mapping class groups using actions on homology or curve complexes. One of the challenges in determining uniform exponential growth in acylindrically hyperbolic group is controlling elliptic subgroups. I will discuss joint work with Abbott and Spriano that constrains dynamic of elliptic subgroups on lower level domains using the hierarchy structure. These tools give a unified geometric approach for showing uniform exponential growth in HHGs and have seen various other applications in hierarchically hyperbolic groups.

Mark Hagen
Title: Universal real cubings
Abstract: There has been a great deal of work on median spaces and groups acting on them; these spaces are natural and important, being a common generalisation of real trees and CAT(0) cube complexes. Another natural source of median algebras/median metric spaces comes from large-scale geometry, namely work by Bowditch and Zeidler showing that asymptotic cones of finite-rank coarse median spaces are canonically bilipschitz to median spaces. In particular, studying asymptotic cones of HHGs means studying median spaces. But in exactly the same way that passing from a cube complex to an HHS structure requires a "descending chain condition" on certain convex subcomplexes — namely the existence of a factor system — there is a subclass of connected, finite-rank median spaces that admit a factor system in an appropriate continuous sense; these we called "real cubings" and have various advantageous properties compared to general median spaces. These relate back to hierarchical hyperbolicity via our first result: asymptotic cones of HHSes are bilipschitz to real cubings. This extends an earlier result of Behrstock-Drutu-Sapir on mapping class groups. The next step is the construction, inspired by related work of Osin-Sapir and Sisto for tree-graded spaces, is to construct so-called universal real cubings determined by local data. An interesting feature of universal real cubings is that they are not only spaces, but groups. I will explain the sense in which they are "continuous RAAGs". I will then briefly outline how these pieces fit into a proof that many hierarchically hyperbolic groups have unique asymptotic cones (up to bilipschitz equivalence). This is all joint work with Montserrat Casals-Ruiz and Ilya Kazachkov.

Alex Wright
Title: Spheres in the curve graph and linear connectivity of the Gromov boundary
Abstract: For a vertex c and an integer radius r, the sphere $S_r(c)$ is the induced graph on the set of vertices of distance r from c. We will show that spheres in the curve graph are typically connected, and discuss connectivity properties of the Gromov boundary. We will also explain the motivation and context for this work.

Wenyuan Yang
Title: Uniform exponenetial growth for groups with proper product actions on hyperbolic

spaces
Abstract: Uniform exponential growth of finitely generated groups has been a classical problem in geometric group theory. In recent years, there is an increasing interest in understanding the product set growth, which could be thought of as a stronger version of uniform exponential growth. In this talk, I will discuss these two problems, provided that the finitely generated group under consideration acts properly on a finite product of hyperbolic spaces. Under the assumption of coarsely dense orbits or shadowing property on factors, we prove that any finitely generated non-virtually abelian subgroup has uniform exponential growth. These assumptions are full-filled in many hierarchically hyperbolic groups, including mapping class groups, specially cubulated groups and BMW groups. Via a weakly acylindrical action on each factor, we are able to classify subgroups with product set growth in any group acting discretely on a simply connected manifold with pinched negative curvature, and in groups acting acylindrically on trees, and in 3-manifold groups. This is based on a joint work with Renxing Wan (PKU).

Mahan Mj
Title: Hyperbolic commensurations and infinite hierarchies
Abstract: We shall start with a recent theorem (joint with Nir Lazarovich and Alex Margolis) that states the following: Let $H < G$ be an infinite commensurated hyperbolic subgroup of infinite index in a hyperbolic group. Then H is virtually a free product of surface and free groups. This leads us to the study of the abstract commensurator group $\mathrm{Comm}(H)$ of surface groups H. The rest of the talk will be a survey of what is known, going back to the work of Sullivan, Biswas, Nag, Penner, Saric and others. It turns out that $\mathrm{Comm}(H)$ naturally acts on spaces that admit a "hierarchically hyperbolic structure with an infinite hierarchy."

4 Problems

One of the outcomes of the project about which we (and others as well) are very excited is the creation of an open problem list to highlight interesting problems and to help this area grow even futher. We held a problem session, during which the participants all worked together to generated a list of open problems in the area. This has since been edited further resulting in the following list of problems, which we will be circulating publicly as a separate document (in which we thank BIRS, of course!).

Many questions raised at the session (or added afterwards) are unattributed when they were well known in the community and we couldn't identify a particular original source. Attributions are by full name on the first appearance and by surname on subsequent ones. We grouped questions and comments by broader topic, so the order in these notes differs from the chronological order of events during the session in some places.

- Generalizations of HHSs
  - Direct limits of HHGs and infinite hierarchies (Mahan Mj)

  - What are the connections with $\mathbb{R}$-cubings?
  - Which, if any, tools/ideas from HHSs could be applied to big mapping class groups?
  - One possible generalization of an HHS could be to keep the assumption that nesting chains are finite length, but drop the requirement that this length be uniform across all nesting chains.

- Which free-by-cyclic groups are HHGs?
  - In general, polynomially growing free-by-cyclic groups aren't HHGs. Even linearly growing ones aren't in general and the problem with these persists in higher strata. There are additional hypotheses which ensure they are HHGs. This is an an obstruction to being HHG for many free-by-cyclic groups where the automorphism has a polynomially growing component (possibly with some assumptions). Question: give a complete set of conditions that characterize when they are HHGs (Harry Petyt)[1].
    * The following result explains why it suffices to consider the polynomially growing ones:
      If G is (f.g. free)-by-cyclic, it is hyperbolic relative to free-by-cyclic subgroups whose monodromy is polynomially growing (including the possibility of a degenerate relatively hyperbolic structure, i.e., that G itself has polynomial growth monodromy (see Theorem 3.5 in `https://arxiv.org/pdf/1901.06760#page=31.13`).

- Hierarchical hyperbolicity of Coxeter groups
  - We know that right-angled Coxeter groups are HHGs, and we know a few other cases as well.
  - Is a Coxeter group always virtually an HHG?
  - Is it an HHG iff it is cocompactly cubulated? (Mark Hagen)
  - Are all Coxeter groups HHSs, even if they are not all HHGs?

- Similarities between $\mathrm{Out}(\mathbb{F}_n)$ and an HHS
  - The Kolchin subgroups act elliptically on all known hyperbolic complexes for $\mathrm{Out}(\mathbb{F}_n)$. Does coning off the Kolchin subgroups result in an HHS? Note that this is weaker than being a relative HHS because there's no known way to project onto these subgroups.
  - Does there exist an action of (a finite-index subgroup of) $\mathrm{Out}(\mathbb{F}_n)$ on a hyperbolic space where a polynomially growing element acts non-elliptically? Loxodromically? (Bestvina)
    * Remark: if the answer is no, then there is no hope of an HHS-like structure.

[1] There is now a paper by Zach Munro and Petyt with a precise statement of the obstruction alluded to here.

    * Remark: $\mathrm{Out}(\mathbb{F}_n)$ acts on a product of free splitting complexes (which are hyperbolic spaces) and everything except the Kolchin subgroups acts there.
    * Remark: Elements in the exponential stratum act loxodromically here as shown in "Subfactor projections" by Mladen Bestvina and Mark Feighn.

- Quotients of HHGs
  - Let $g_1, \cdots, g_k \in MCG(\Sigma)$. Does there exist a such that for all multiples a_i of a, $MCG(\Sigma)/\langle\langle g_1^{a_1}, \cdots, g_k^{a_k}\rangle\rangle$ is an HHG? (Mangioni–Sisto)
    * True if all g_i pseudo-Anosov or if g_i all Dehn twists and k is sufficiently large.
    * Work in progress by Mangioni–Sisto will show this holds for the pure mapping class group for $\Sigma = S_{0,5}$, for full mapping class group or more general surfaces the problem appears more delicate and has surprising connections to bounded cohomology.
    * The simplest unknown case is when $k = 1$ and g is a Dehn twist.
  - Does the above hold if $MCG(\Sigma)$ is replaced with a RAAG? (Thomas Ng)
    * Work of Wise (cubical small-cancellation) might give a way to cubulate this sort of quotient, and it seems plausible that **if** the quotient is cocompactly cubulated then the cube complex will have a factor system, so the quotient will be HHG. (Hagen)
  - Are all hierarchically quasiconvex subgroups of mapping class groups separable?
    * Evidence: it is true for multicurve stabilisers by Leininger-McReynolds and for certain convex-cocompact subgroups (Hagen-Sisto, using Manning-Mj-Sageev).
    * Motivation/analogy: Haglund–Wise proved that convex-cocompact subgroups of fundamental groups of special cube complexes are separable.

- Recognizing HHGs from asymptotic cones
  - (Possibly with some other reasonable conditions) is being virtually an HHG characterized by having all asymptotic cones be $\mathbb{R}$-cubings? (Montserrat Casals-Ruiz)
  - It is not known whether Sam Shepherd's example of a cubulation which does not admit a factor system is an HHG or not, but Hagen conjectured in the problem session that it is not. One way to see it would be to exhibit an asymptotic cone whose candidate $\mathbb{R}$–cubing structure fails the finite complexity condition. (Hagen)
  - More generally, can one construct an HHG structure for a finitely generated group using its asymptotic cones? (Casals-Ruiz)
  - Can $\mathbb{R}$-cubings distinguish the asymptotic cones of the pentagon RAAG and a rank 2 mapping class group? (Behrstock)

- Coarse medians and quasi-isometries with cube complexes
  - Does being strongly coarse median plus something else (e.g., an analogue of admitting a factor system) imply being an HHS?

  - Do there exist hierarchical structures on the mapping class group with different coarse median structures? (Hagen and Abdul Zalloum)
    * Note that having the same median implies having the same hierarchy paths.
    * Do mapping class groups have more than one coarse median structure? [2]
  - Are all HHSs quasi-isometric to CAT(0) cube complexes?
  - Are there CAT(0) cube complexes that are not HHSs?
    * Hagen conjectures that perhaps Shepherd's group is an example (i.e., the cubulation without a factor system does not admit an HHS structure, even some other structure unrelated to the cubical structure; see also above question on R–cubings).
  - Does being quasi-isometric to a CAT(0) cube complex imply anything stronger than cubical approximation? (Hagen)
  - Is there some natural deformation space of hierarchical structures? (Behrstock)
  - Is there some natural deformation space of coarse median structures?
    * For coarse median cubical structures, a paper by Fioravanti–Levcovtiz–Sageev provides a nice set-up for this question and some rigidity results.
  - What properties are preserved if hyperbolic spaces are replaced by coarse median spaces of bounded rank? (Tang)

- Algorithmic properties of HHGs
  - Do HHGs have solvable isomorphism problem?
    * Zlil Sela's recent work is in this direction.
    * This will be a very hard problem.
  - Does there exist an algorithm that (semi-)decides if a group is an HHG or coarse median? (Sisto)
    * This exists for hyperbolic groups.
  - Are HHGs automatic?
    * Motiejus Valiunas has conjectured this to be false.
  - Can one always find an infinite order element of an HHG uniformly quickly? (Zalloum)
  - Is there a polynomial-exponential divergence gap? Are there (fast?) algorithms to detect the divergence from the HHS structure? (Behrstock)

- Which Artin groups are HHGs?
  - Are all Artin groups HHGs?
  - Are the (2,4,4) and (5,3,2) Artin groups HHGs? (Petyt)

[2] After the conference Mangioni posted a preprint showing that the 5–punctured sphere has more than one such structure, but the question remains open for other surfaces.

    - Conjectured to be true for both.
  - Are there other examples of 3-vertex Artin groups which are not known to be HHGs? (Petyt)
  - Are FC-type Artin groups HHGs?

- Which other familiar groups are hierarchically hyperbolic?
  - Which Out(RAAG) or Out(RACG) are HHGs? (Russell)
    - This is false for complete and false for discrete, but what about in-between?
  - Which surface group extensions of any other subgroups of mapping class groups are HHGs or HHSs? (Christopher Leininger)
    - Possible examples/counterexample for this include: Torelli groups, surface braid groups, handlebody groups with genus 2, pure braid groups, RACGs, RAAGs.
    - For some of these subgroups it is unknown if they are HHGs or HHSs themselves, e.g., higher genus Torelli groups, surface braid groups.

- Boundaries and ends
  - When is an HHG one-ended?
    - Can the one-endedness of an HHG be seen in the structure?
    - Can the one-endedness of an HHG be seen in the boundary?
  - Can one characterize the one-endedness of CS (in the structure that has been maximized a la Abbott–Behrstock–Durham?
  - Is there an definion of the HHS boundary which is more accessible?
  - Are there other natural boundaries for an HHS?

- Functional analytic properties of HHGs
  - Some hyperbolic groups are a-T-menable and some have property (T). Are all rank ≥ 2 HHGs a-T-menable? If not, are there nice conditions which can be imposed on an HHG to ensure a-T-menability? (Behrstock)
  - Do HHGs admit proper actions on ℓ^p or L^p spaces?
    - This is true for hyperbolic products and relatively hyperbolic products.
    - This is related to things like property T.
  - Is the Baum–Connes conjecture true for HHGs?

- Miscellaneous
  - What is the dimension of coarsely embedded flats? (Sisto)
  - Do HHGs have canonical relatively hyperbolic structures (as they do in the case of RACGs)? In particular, are all HHGs relatively hyperbolic or thick? (Behrstock–Hagen–Sisto)

    * Remark: there is no known example of a finitely presented group which is not relatively hyperbolic or thick.

  – Are all HHS/G are CHHS/CHHS. Recently Park devised a notion of combinatorial HHS with factor systems (CHHF), which seems tailor-made for coarse median spaces and CAT(0) cube complexes with factor systems. Are all HHG/S combinatorial with factor systems? Do the two combinatorial notions coincide? (Mangioni)