Skew Braces, Braids and the Yang-Baxter Equation

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1 Overview of the Field

The Yang–Baxter equation originates from Baxter's [5] approach to finding exactly solvable models in statistical mechanics and Yang's [25] construction of two-dimensional quantum field theories. Such an equation has been a source of deep and beautiful mathematics extending well beyond statistical mechanics, influencing areas such as topology (e.g., link invariants [24] and quandles [3]), Hopf algebras (e.g., Nichols algebras [1, 2, 4]), representations of the braid group and its higher-dimensional generalizations (e.g., [19, 17, 7]), and von Neumann algebras (e.g., [9]).

Specifically, the Yang–Baxter equation is given by

$$R_1R_2R_1 = R_2R_1R_2,$$

where $R \in \mathrm{Aut}(V^{\otimes 2})$ for some vector space V, and, as usual, $R_1 = R \otimes I_V$, $R_2 = I_V \otimes R$, and I_V is the identity map on V. This equation may look a bit abstract, but there is a nice way to think about it, see Figure 1.

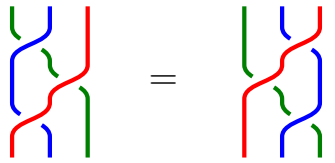

Figure 1: The Yang–Baxter equation

The arrangements of strings, representing the Yang–Baxter equation, should be read from top to bottom, with the crossing symbolizing the application of the map R and the straight line representing the identity mapping. The picture in Figure 1 itself is self-explanatory. Because of this braiding-like behavior, the equation is also known as the *braid equation*.

Recall that the braid group $\mathcal{B}_n$ can be presented by generators $\sigma_1, \dots, \sigma_n$ and relations

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && \text{for all } i \in \{1, \dots, n-1\}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{for all } i, j \text{ such that } |i-j| \geq 2. \end{aligned}$$

Given a solution $R \in \mathrm{Aut}(V^{\otimes 2})$ to the Yang–Baxter equation one immediately obtains representations of the braid group $\mathcal{B}_n$ on $V^{\otimes n}$, by sending the generators σ_i to

$$I_V^{\otimes i-1} \otimes R \otimes I_V^{n-i-1}.$$

The nature of these representations is largely unknown.

The Yang–Baxter equation unites several research groups. Broadly, the following two questions can benefit from different angles.

1) The construction and classification of families of solutions to the Yang–Baxter equation.

2) Applications of the Yang–Baxter equation in other research domains, including Hopf algebras and quantum groups, Topological quantum field theories, non-commutative ring theory, computational algebra, and low dimensional topology.

Despite the common theme of the YangBaxter equation, the research groups working on these topics are largely siloed, with minimal interaction among them. The workshop's main goal was to foster interaction among these researchers to make progress in all topics related to the Yang–Baxter equation.

At first, the problem of classifying solutions to the Yang–Baxter equation looks impossibly hard: only the solutions for $\dim(V) = 2$ have been classified (in 1992 by Hietarinta [15]). For $\dim(V) > 2$, the problem remains wide open. However, some breakthrough ideas have recently opened the way for a meaningful classification of certain restricted kinds of Yang–Baxter operators, such as charge-conserving solutions. The general philosophy here regards known low-dimensional solutions as the first member of a sequence of solutions. A family of solutions of this form is the Gaussian solutions found in the work of Jones [16].

In addition, attempts to construct solutions to the Yang–Baxter equation led to the theory of quantum groups. Since the class of linear Yang–Baxter solutions is extremely broad and currently beyond the reach of mathematicians, Drinfeld [11] suggested tackling the problem in two main steps. First, classify set-theoretic solutions, as they are more accessible yet still highly non-trivial. Second, study linear deformations of set-theoretic solutions using an appropriate cohomology theory, and describe the variety of solutions attainable this way.

Research on set-theoretic solutions has revealed deep connections with various other fields, including skew polynomial algebras and Bieberbach groups [13], Garside groups [10, 8], Artin–Shelter regular rings [12], braided homology [18], and Hopf–Galois structures [23]. Notably, certain algebraic structures introduced in the context of the Yang–Baxter equation, such as cycle sets [20], skew braces [21, 14], and trusses [6], have been found to be applicable to other mathematical problems as well.

2 Recent Developments and Open Problems

Two workshop afternoons for discussions among participants to identify and explore open problems. Given the wide diversity of topics, expertise, and geographical backgrounds of the participants — many of whom were meeting for the first time — it was essential to have time for interaction. This facilitated the establishment of connections and the initiation of new collaborations.

During the first afternoon, a plenary session was held where all participants brainstormed potential open problems within their respective fields, focusing on those that could benefit from a multidisciplinary approach.

2.1 Skew braces

Skew braces are a recently introduced algebraic structure originating from the theory of radical rings. Currently, little is known about their algebraic properties. For instance, the nature of free objects in the category of skew braces remains largely unexplored. This raises the following questions:

Problem 1. Sub-skew braces of a free skew brace.

Problem 2. Free product of skew-braces.

Problem 3. Is there a forgetful functor from skew braces to sets?

2.2 Solutions in different categories

Another important topic to discuss is the construction and study of various types of solutions to the Yang–Baxter equation. For instance, it is interesting to examine braiding operators in different categories. In this vein, the following questions have arisen:

Problem 4. Study solutions to the Yang–Baxter equation in the category $\mathrm{Mat}_N(\mathrm{Vec})$.

Problem 4 was solved for $N = 2$ by Hietarinta in [15], but apart from that, it remains wide open.

Problem 5. Study set-theoretic solutions in the category of sets and relations.

2.3 R-matrices

Smoktunowicz and Smoktunowicz [22] introduced a method to derive R-matrices from skew braces of abelian type. It is now natural to extend this construction to more general types of skew braces to explore its full potential:

Problem 6. Study R-matrices from skew braces.

A natural problem is to investigate the properties of R-matrices generated by various existing methods documented in the literature, or their variations.

Problem 7. What does applying the FRT construction to R-matrices (coming from quasi-triangular Hopf algebras) remember?

As mentioned earlier, braidings yield representations of braid groups, whose properties are not thoroughly understood. During one of the problem sessions, the following question was raised:

Problem 8. Let R be a unitary R-matrix. Is the image of the braid group under the aforementioned representation virtually abelian?

2.4 Applications

The flip map (the trivial solution on a vector space) is used to construct quantum error-correcting codes. This prompts exploring how other solutions of the Yang–Baxter equation (or R-matrices) can be applied in error-correcting codes. Natural candidates for this investigation are R-matrices derived from set-theoretic solutions.

Problem 9. (Quantum) Error Correcting Codes from R-matrices (e.g., of set-theoretic type)?

Concrete applications of braidings to low-dimensional topology were also examined during the workshop. The following questions were discussed and considered in the problem sessions:

Problem 10. Presentations of motion groups (torus links)?

Problem 11. Explain modular data reconstruction from $\mathrm{SL}(2, \mathbb{Z}/N)$-representations?

Problem 12. Study mapping class group images from (twisted) Dijkgraaf–Witten invariants for a specific finite p-group.

Problem 13. Are there topological interpretations of the Reshetikhin–Turaev invariant?

For a finite group G, we write $\mathrm{Rep}(G)$ to denote the category of (finite-dimensional) representations of G, and Vec_G to denote the category of G-graded vector spaces.

Problem 14. Classify coboundary structures on $\mathrm{Rep}(G)$ and Vec_G.

3 Presentation Highlights

Several participants commented to the organizers how much they appreciated the diversity of the workshop, both in terms of speaker demographics and topics. The informal evening talks were also mentioned as an enjoyable addition to the standard way of presenting ongoing research developments.

3.1 Panoramic talks

Various speakers delivered 50-minute overview talks on key conference topics during the morning sessions. The lineup of speakers and their respective topics, presented in chronological order, was as follows:

Ilaria Colazzo (Leeds, UK) provided an overwiew on set-theoretic solutions of the Yang-Baxter equation, with special emphasis on recent computational methods for those constructing solutions. She also explained the connection between skew braces and braidings of set-theoretic type.

Celeste Damiani (Genova, Italy) gave a panoramic talk on motion groups and current developments. She reported on recent advancements in braid representations (i.e. a strict monoidal functor from the braid category into another easier-to-handle category) and reported on recent progress made by restricting the target category and by extending the source category.

Fiona Torzewska (Bristol, UK) provided an overview of known results and explored questions, approaches, and mathematical languages for a modern theory of motion groups. The theory of motion groups began in the 1960s as a program to provide a general framework for addressing this task, drawing inspiration from Artin braid groups.

Hector Pena Pollastri (Indiana, USA) gave a general talk on the theory of Nichols algebras. He presented some known results on Nichols algebras of diagonal type and their appearance in the theory of quantum groups, and discussed some big open questions.

Dmitri Nikshych (New Hampshire, USA) provided an overview of braided fusion categories, which are semisimple tensor categories in which the tensor product is subject to a commutativity constraint called braiding; these categories admit actions of braid groups. He gave basic definitions, discussed the structural theory and examples of braided fusion categories, and outlined some of the machinery used to address current problems in this field.

Xingshan Cui (Indiana, USA) spoke on Topological Quantum Computing (TQC). In the narrowest sense, TQC refers to encoding qubits in global degrees of non-Abelian anyons/quasi-particles and executing quantum gates by manipulating anyons. In the broadest sense, TQC includes any fault-tolerant quantum computing that relies on topological properties of the underlying mechanism. He explored the relations between topological order, anyons, and quantum error correcting codes, and also addressed some questions regarding universality and leakage-free entangling gates in anyon models.

Victoria Lebed (Caen, France) discussed structure groups of set-theoretic solutions to the Yang-Baxter equation. Her talk focussed on set-theoretic solutions of the form $r(x, y) = (y, y^{-1}xy)$, where the underlying set has a group structure. She also discussed conjugation groups, which are groups admitting a presentation with only conjugation relations of type $y^{-1}xy = z$ and power relations $x^d = 1$. This vast class of groups includes free abelian groups, symmetric and braid groups and various generalizations, including Thompsons group F, cactus groups, knot groups, and groups related to set-theoretic braidings.

Paolo Bellingeri (Caen, France), building upon Lebed's presentation, discussed cacti groups (also known as quasi-braid groups) and their presentations. He explored the relationship between braid groups and cacti groups, as well as the combinatorial properties of the latter.

3.2 Research talks

There were 30-minute research talks focusing on specific recent findings. The lineup of speakers and their topics, presented in chronological order, included:

Adolfo Ballester-Bolinches (Valencia, Spain) discussed the relationship between skew braces and triply factorized groups. Based on the theory of triply factorized groups, he presented a proposal for a definition of representations of skew braces and compared it with other definitions in the literature.

Tatiana Gateva-Ivanova (Sofia, Bulgaria) discussed quadratic algebras associated with certain solutions of the Yang–Baxter equation. She explored properties of certain solutions both via Veronese subalgebras and Segre products, and in terms of noncommutative differential geometry.

Eddy Godelle (Caen, France) presented on trickle groups, which include right-angled Artin groups, cactus groups, and many other interesting groups. He explained how to solve the word problem for trickle groups.

Joshua Sussan (New York, USA) gave a talk on non-semisimple Hermitian topological quantum field theories (TQFTs). He reported on the study of certain unitary representations of the braid group and discussed when these representations have a dense image.

Geoffrey Janssens (Louvain, Belgium) talked about ways discussed ways of finding braided tensor categories. The talk was devoted to explaining basic concepts and presenting several examples, which led to the concept of a 'system of renormalized R-matrices'.

3.3 Gong-talks by young researchers

During the meeting, young researchers had the opportunity to present their findings in 10-minute talks. As organizers, we scheduled these presentations for the first day to allow young researchers to introduce themselves to more experienced colleagues, thereby increasing their visibility and fostering interaction among participants.

Melody Molander (PhD student, California, USA) reported on subfactor planar algebras and discussed presentations for subfactor planar algebras of index four associated with the affine A Dynkin diagram.

Benjamin Warren (PhD student, Texas, USA) reported on the classification of charge-conserving solutions to Yang–Baxter equation.

Kevin Piterman (Postdoctoral researcher, Marburg, Germany) discussed the topology and combinatorics of poset decompositions and their relations with group cohomology stability.

Arne Van Antwerpen (Postdoctoral researcher, Ghent, Belgium) reported on a recent joint paper on the Yang–Baxter monoid and algebra associated with a broad class of set-theoretic solutions. In particular, he discussed finiteness conditions of these algebraic objects.

Silvia Properzi (PhD student, Brussels, Belgium) reported on her recent joint paper on indecomposable set-theoretic solutions to the Yang–Baxter equation of prime-square size.

3.4 Informal evening talks

The meeting included three informal evening talks to engage and empower participants, especially PhD students and postdocs. These talks were a mix of panoramic descriptions of research areas and the presentation of possible research directions. Remarkably and not unexpectedly, these talks provided significant inspiration during the problem sessions. The speakers were **Eric Jespers** (Brussels, Belgium), **Eric Rowell** (Texas, USA), and **Hanz Wenzl** (California, USA).

4 Scientific Progress Made

Following the brainstorming session, participants voted on the proposed problems in a double-turn voting process. This method ensured that the selected problems had sufficient participants interested in exploring them. Based on the voting results, several working groups were formed, including, but not limited to, the following.

Problems 1, 2 and 3 caught the attention of several participants, who highlighted that they could be addressed under a unique framework. A group formed by Eric Jespers, Arne Van Antwerpen, Silvia Properzi, Marco Trombetti, Adolfo-Ballester Bolinches, Be'eri Greenfeld, Victoria Lebed, and Jason Bell decided to focus on the combined problem. The team obtained some partial answers on the first day and reported they were working on a joint paper.

Problem 7 was discussed among Jason Bell, Geoffrey Janssens and Eric Rowell. Jason Bell is exploring a version of this problem (see his comments below). Problem 9 was discussed in a group including Ilaria Colazzo, Agustina Czenky, Emily McGovern, Corey Jones, Melody Molander, Eric Rowell, Monique Müller.

Problem 13 was discussed among Tian Yang, Helen Wong, Jennifer Vasquez, Xingshan Cui and Qing Zhang. They reported that they had made progress already.

Problem 12 was discussed in a group including Paolo Bellingeri, Celeste Damiani, Xingshan Cui, Tian Yang, Helen Wong, Benjamin Warren, Qing Zhang and Eric Rowell.

A shorter plenary session was held on the second afternoon, during which groups shared their progress. Partial results were discussed, and several groups decided to continue their work. A group decided to change their focus, and a new team, including Eric Rowell and Jason Bell, decided to work on open questions on the loop Hecke algebra, and a follow up question on motion groups of disjoint trefoils. This included Benjamin Warren.

This flexible format and a structured voting process for problem selection encouraged effective collaboration. This allowed participants to connect, share ideas, and begin new research initiatives. Identifying and exploring open problems set the stage for continued collaboration and future workshops.

5 Outcome of the Meeting

Firstly, let us address some of the challenges we faced. Originally, our proposal heavily involved European and Spanish institutions, with Granada as our preferred location. However, due to circumstances beyond our control, we had to adapt. Key participants transitioned to virtual participation, and two of our four organizers had to move online due to family reasons. Additionally, there were last-minute cancellations from participants due to flight delays, illness, and other issues.

Despite these obstacles, the workshop proved to be successful. Many participants appreciated the structured working sessions in the afternoons, as highlighted in their feedback and testimonials.

Adolfo Ballester-Bolinches, from Universitat de Valncia, Spain, said: *I had the opportunity to participate in the workshop organised by Leandro Vendramin, Ilaria Colazzo, Julia Plavnik and Eric Rowell at the Banff International Research Station for Mathematical Innovation and Discovery in Alberta (Canada), a research centre located on a hill with spectacular views of the Rocky Mountains. Environment, accommodation, staff and the necessary facilities for the workshop were perfect. Without a doubt, this contributed enormously to the success of the meeting, which, of course, was a complete success at all levels. Thanks are due to the organisers for their work.*

Jason Bell, from University of Waterloo (Canada), said: *I learned a lot about set-theoretic solutions to the Yang-Baxter equation and have begun working on questions that came out of the open problem sessions held during the conference. Overall, it was a rewarding experience that I hope will eventually lead to new collaborations.*

Davide Ferri, PhD student in Universit di Torino (Italy), said: *The BIRS workshop "Skew Braces, Braids and the Yang–Baxter Equation" has been a great opportunity for the European and the American research communities in YBE to converge. I have personally felt it as a melting pot of ideas that were original, yet familiar: people from the other side of the world sharing their insight on my same field of research, but under a completely different perspective—an event which was, in my opinion, very impactful on the future of my field. I was personally inspired, and led to new ideas. Non-secondarily, I got to know people to whom I may refer, in the future, should I trespass into their area of expertise. Although I only attended online, I was engaged as if in person, due to the outstanding technological facilities of the BIRS.*

Be'eri Greenfeld, from University of Washington (USA), in private communication with us, said: *The workshop in Banff provided a fantastic opportunity to discuss contemporary open problems and foster connections between skew-braces, braid groups, and category theory. I gained a solid understanding of the fundamental concepts and current research directions in these areas, actively participating in fruitful discussions where we established new theorems and discovered novel examples. I believe organizing follow-up workshops would be highly beneficial to share recent progress and strengthen a collaborative research community interested in these topics. It would help consolidate our findings and further develop our collective understanding.*

Monique Mller Lopes Rocha, from Federal University of So Joo del-Rei (Brazil), said: *My participation in the BIRS workshop significantly impacted my current research. I had the opportunity to collaborate in person with a colleague and receive feedback on a completed paper from another expert in the field. While these interactions could have occurred online, the in-person experience was far more valuable. I gained*

insights into current interests within the area and made numerous informal contacts, which could lead to future collaborations.

Kevin Piterman, from Philipps-Universitt Marburg (Germany), in private communication with us, said: *I really enjoyed the conference hosted at the Banff Centre. The organizers did a very good job by bringing together researchers with different backgrounds, which allowed fruitful interactions and contributed to the conference's success. Also, the infrastructure of the center is outstanding at all levels.*

Silvia Properzi, PhD student in Vrije Universiteit Brussel (Belgium), said: *It was a rewarding experience that expanded my knowledge and provided valuable networking opportunities. The quality of the content, the engaging sessions, and the excellent venue made it a memorable event. I am looking forward to continue the collaborations started there and am excited about the future directions my research may take as a result of this inspiring workshop.*

Qing Zhang, from University of California, said: *The talks were very informative. I learned about set-theoretic solutions to the Yang-Baxter equation, motion groups, Nichols Algebras, nonsemisimple TQFTs, and the connections between braid groups, braided tensor categories, and quantum computing. The conference also gave me great opportunities to meet senior researchers and get advice about my career. The afternoon question sessions were particularly inspiring and helpful. The workshop was also a great chance to meet my current collaborators and connect with potential future collaborators.*

6 Participants

There were 73 participants from 15 countries across three continents: Argentina, Brazil, Canada, Colombia, USA, UK, Bulgaria, Germany, Belgium, Poland, Italy, France, Spain, India, and Japan.

The workshop actively encouraged participation from underrepresented groups. For instance, more than 30% of the attendees were women; this percentage increases notably when considering only those who attended in person. Additionally, approximately 35% of the onsite participants were early career researchers, primarily PhD students and junior postdocs.

List of participants

Ivan Angiono (Universidad Nacional de Crdoba, Argentina)
Adolfo Ballester-Bolinches (Universitat de Valncia, Spain)
Jason Bell (University of Waterloo, Canada)
Paolo Bellingeri (University of Caen, France)
Wade Bloomquist (Morningside University, USA)
Tomasz Brzezinski (Swansea University, UK)
Andrea Caranti (University of Trento, Italy)
Francesco Catino (Universit del Salento, Italy)
Ilaria Colazzo (University of Leeds, UK)
Teresa Crespo (Universitat de Barcelona, Spain)
Xingshan Cui (Purdue University, USA)
Agustina Czenky (University of Oregon, USA)
Celeste Damiani (Istituto Italiano di Tecnologia, Italy)
Andrew Darlington (University of Exeter, UK)
Colleen Delaney (UC Berkeley, USA)
Anastasia Doikou (Heriot-Watt University, UK)
Cain Edie-Michell (University of New Hampshire, USA)
Ramon Esteban Romero (Universitat de Valncia, Spain)
Joao Faria Martins (University of Leeds, UK)
Edouard Feingesicht (Caen University, France)
Davide Ferri (Universit di Torino, Italy)
Cecar Galindo (Universidad de los Andes, Colombia)
Tatiana Gateva-Ivanova (American University in Bulgaria, Bulgaria)
Eddy Godelle (Universite de Caen, France)

Beeri Greenfeld (University of Washington, USA)
Istvn Heckenberger (University of Marburg, Germany)
Geoffrey Janssens (UCLouvain, Belgium)
Eric Jespers (Vrije Universiteit Brussel, Belgium)
Corey Jones (North Carolina State University, USA)
ukasz Kubat (University of Warsaw, Poland)
Victoria Lebed (Universit de Caen-Normandie, France)
Thomas Letourmy (Universit Libre de Bruxelles, Belgium)
Monique Mller Lopes Rocha (Federal University of So Joo del- Rei, Brazil)
Paul Martin (University of Leeds, UK)
Isabel Martin-Lyons (Keele University, UK)
Marzia Mazzotta (Universit del Salento, Italy)
Emily McGovern (North Carolina State University, USA)
Ehud Meir (University of Hamburg, Germany)
Melody Molander (UC Santa Barbara, USA)
Dmitri Nikshych (University of New Hampshire, USA)
Manish Patnaik (University of Alberta, Canada)
Hector Martin Pena Pollastri (Indiana University, USA)
Vicent Prez Calabuig (Universitat de Valncia, Spain)
Kevin Piterman (Philipps-Universitt Marburg, Germany)
Julia Plavnik (Indiana University, USA)
Silvia Properzi (Vrije Universiteit Brussel, Belgium)
Dora Puljic (University of Edinburgh, UK)
Santiago Ramirez (Universidad de Buenos Aires, Argentina)
Nishant Rathee (IISER Mohali, India)
Anna Rio (Universitat Politcnica de Catalunya, Spain)
Charlotte Roelants (Vrije Universiteit Brussel, Belgium)
Eric Rowell (Texas A&M University, USA)
Sean Sanford (Ohio State University, USA)
Guillermo Sanmarco (University of Washington, USA)
Andrew Schopieray (Marquette University, USA)
Agata Smoktunowciz (University of Edinburgh, UK)
Noah Snyder (Indiana University, USA)
Paola Stefanelli (Universit del Salento, Italy)
Lorenzo Stefanello (Universit di Pisa, Italy)
Joshua Sussan (Medgar Evers College, USA)
Fiona Torzewska (University of Bristol, UK)
Marco Trombetti (University of Naples "Federico II", Italy)
Cindy (Sin Yi) Tsang (Ochanomizu University, Japan)
Arne Van Antwerpen (Universiteit Gent, Belgium)
Jennifer Vasquez (University of Scranton, USA)
Leandro Vendramin (Vrije Universiteit Brussel, Belgium)
Benjamin Warren (Texas A&M University, USA)
Hans Wenzl (University of California, San Diego, USA)
Magdalena Wiertel (University of Warsaw, Poland)
Helen Wong (Claremont McKenna, USA)
Manoj Kumar Yadav (Harish-Chandra Research Institute, India)
Tian Yang (Texas A&M University, USA)
Qing Zhang (University of California, Santa Barbara, USA)

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