Small Solutions to Thue Equations Over Quadratic Imaginary Fields

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The following report describes the progress made by PIMS/BIRS Team-Up collaborators Eva Goedhart, Greg Knapp, and Sumin Leem during their two week visit to BIRS from November 24, 2024 until December 7, 2024.

1 Overview of the Field

Our work lies in the subfield of number theory that studies Diophantine equations called Thue equations. A **Thue equation** is an equation of the form

$$F(X,Y) = h \tag{1}$$

where each of the following holds:

- $F(X,Y)$ can be written as
$$F(X,Y) = \sum_{j=0}^{n} a_j X^j Y^{n-j}$$
where each $a_j \in \mathbb{Z}$ and $n \geq 3$.
- $F(X,Y)$ is irreducible over $\mathbb{Z}[X,Y]$.
- h is an integer.

Thue equations are named such because Thue proved in 1909 that there only finitely many integer pairs $(x,y) \in \mathbb{Z}^2$ which satisfy equation (1) [12]. As a corollary, there are only finitely many integer pair solutions to the inequality (known as **Thue's inequality**)

$$|F(X,Y)| \leq h \tag{2}$$

where the same conditions apply to $F(X,Y)$ and h (though in this setting, we may as well assume that h is a positive integer).

The basic questions in the study of Thue equations are:

1. How many integer pair solutions are there to inequality (2)?
2. How can we find the solutions to (2)?

Indeed, it often suffices to answer these questions for $h = 1$. Let $N(F,h)$ denote the number of integer pair solutions to inequality (2). Mahler proved in [8] that if n denotes the degree of F, then $N(F,h)$ is approximately equal to $h^{2/n}N(F,1)$. Since estimating $N(F,h)$ often reduces to estimating $N(F,1)$, we only focus on $N(F,1)$ in this project.

2 Recent Developments

2.1 Special Cases

Since proving general results about Thue equations is quite difficult, many mathematicians prove results about specific families of Thue equations. For example, the following three theorems give upper bounds on $N(F,1)$ which depend on the number of summands of $F(X,Y)$.

Theorem 1 (Bennett, [1]). *Let $n \geq 3$, and let*

$$F(X,Y) = aX^n - bY^n.$$

Then $N(F,1) \leq 8$ and this is the best possible bound.

Theorem 2 (Knapp, [6]). *Let $n \geq 6$, and let*

$$F(X,Y) = aX^n + bX^kY^{n-k} + cY^n.$$

Then $N(F,1) \leq 128$. If $n \geq 219$, then $N(F,1) \leq 40$.

Theorem 3 (Grundman and Wisniewski, [4]). *Let $n \geq 9$, and let*

$$F(X,Y) = aX^n + rX^mY^{n-m} + sX^kY^{n-k} + tY^n.$$

Assume some additional restraints on the coefficients and exponents. Then $N(F,1) \leq 5688$. If $n \geq 60$, then $N(F,1) \leq 296$.

Beyond merely fixing the number of summands, one can examine Thue equations where the number of summands is fixed and the coefficients have a certain form. In particular, in [11], Thomas initiated the study of Thue equations where $F(X,Y)$ is one of the "simplest cubic forms" (as defined by Lettl et al. in [7]). The cubic, quartic, and sextic versions of these simple forms look like

$$\begin{aligned}
F_t^{(3)}(X,Y) &= X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 \\
F_t^{(4)}(X,Y) &= X^4 - tX^3Y - 6X^2Y^2 + tXY^3 + Y^4 \\
F_t^{(6)}(X,Y) &= X^6 - 2tX^5Y - (5t+15)X^4Y^2 - 20X^3Y^3 + 5tX^2Y^4 + (2t+6)XY^5 + Y^6.
\end{aligned}$$

Mignotte completed the proof of Thomas' conjecture in [10] by showing that the Thue equation $|F_t^{(3)}(X,Y)| = 1$ has only trivial solutions (the solutions with either $X = 0$ and $Y = \pm 1$ or with $X = \pm 1$ and $Y = 0$) when $t \in \mathbb{Z}$ and $|t| > 3$. This implies that $N(F_t^{(3)}, 1) \leq 4$ when $|t| > 3$. In [3], Gaál et al. give a complete classification for solutions of the Thue equations $|F_t^{(4)}(X,Y)| = 1$ and $|F_t^{(6)}(X,Y)| = 1$, and again, they prove that when $t \in \mathbb{Z}$ is sufficiently large, the only solutions are trivial (i.e. $N(F_t^{(i)}, 1) \leq 4$).

2.2 Relative Thue Equations

With the solutions to the simplest cubic/quartic/sextic equations being fully classified, one can now turn to the so-called "relative" setting. In this setting, we attempt to solve the same Thue equations, but we allow the coefficients and solutions to come from the ring of integers of an imaginary quadratic number field. The reason we require the number field to be imaginary and quadratic is that the rings of integers in these fields share an important property with the rational integers: if K is an imaginary quadratic number field with ring of integers $\mathcal{O}_K$, then any $\alpha \in \mathcal{O}_K$ with $\alpha \neq 0$ satisfies $|\alpha| \geq 1$. This fact ensures that there are still only finitely many $(x,y) \in \mathcal{O}_K^2$ which satisfy $F(x,y) = h$ whenever $h \in \mathcal{O}_K$ and $F(X,Y)$ is a homogeneous polynomial of degree at least 3, with coefficients in $\mathcal{O}_K$, and which is irreducible over $\mathcal{O}_K[X,Y]$. Hence, we can study Thue equations over imaginary quadratic number fields in much the same way that we study their solutions over $\mathbb{Z}$.

Indeed, in [5], Heuberger completely classified the solutions in $\mathcal{O}_K^2$ to $|F_t^{(3)}(X,Y)| = 1$ for $t \in \mathcal{O}_K$ where K is an imaginary quadratic number field. Further, in [3], Gaál et al. classify the solutions in $\mathcal{O}_K^2$ to $|F_t^{(4)}(X,Y)| = 1$ for any rational integer t, but they do not consider other values of $t \in \mathcal{O}_K$. In [2], Earp-Lynch et. al prove that there are no nontrivial solutions in $\mathcal{O}_K^2$ to $|F_t^{(4)}(X,Y)| = 1$ for any $t \in \mathcal{O}_K$ with $|t| \geq 100$, but they do not consider any values of t with $|t| < 100$.

The main objective of our project is to complete the classification of solutions in $\mathcal{O}_K^2$ to $|F_t^{(4)}(X,Y)| = 1$ for $t \in \mathcal{O}_K$. While it only remains to check values of t with $|t| \leq 100$ (of which there are finitely many), there are too many such values for a computer search to be practical.

Prior to attending the PIMS/BIRS Team Up visit, we had improved the efficiency of the application of hypergeometric method in [2] to obtain similar results for $|t| \geq 78$. However, this reduction is still not nearly enough to handle the remaining cases computationally. Additionally, we had reason to believe that there is a fundamental limitation of the hypergeometric method that would prevent us from improving their results any further under this approach. As a result, we need a new approach to handle the small values of t.

3 Scientific Progress Made

During our PIMS/BIRS Team Up trip, we first established the following outline for the remainder of the project:

1. Find a value of k for which we can theoretically prove that there are only trivial solutions $(x,y) \in \mathcal{O}_K^2$ to $|F_t^{(4)}(X,Y)| = 1$ whenever $|t| \geq k$.

2. For any quadratic imaginary integer t with $|t| < k$, run a computer search for solutions to $|F_t^{(4)}(X,Y)| = 1$.

In order to accomplish the steps in this outline, we worked on three primary objectives. First, we wanted to tailor Baker's method to our particular setting in order to establish upper bounds on the absolute values of the coordinates of solutions (x,y) which only depend on t. Second, we wanted to analyze a paper from Martin which describes an algorithm for producing generalized continued fractions where the components come from imaginary quadratic number fields. Third, we wanted to produce code which would allow us to quickly reprove modifications of lemmas in [2] as we adjusted our assumptions in the course of doing research.

3.1 Applying Baker's Method

The primary insight during our trip to BIRS is that Baker's method, combined with an analogue of the results in Section 7 of [2] should be enough to handle most of the values of t with $|t| < 100$.

Baker's method is an approach to solving Thue equations which gives a bound on the size of possible solutions in terms of the coefficients of the polynomial $F(X,Y)$. In our setting, a naïve application of Baker's method produces a result along the lines of the following: there is an explicit constant Y_L depending on t so that any solution $(x,y) \in \mathcal{O}_K^2$ satisfies $|y| \leq Y_L$. Since we may assume that $|t| \leq 100$, it might seem like we could simply use an application of Baker's method and run a brute force search for solutions to $|F_t^{(4)}(X,Y)| = 1$ for every $|t| \leq 100$. However, the naïve application of Baker's method does not produce a usable value of Y_L, so we must modify the method to account for our particular polynomial $F_t^{(4)}(X,Y)$.

Additionally, Baker's method is well-suited for the classical settings where the coefficients and solutions of the Thue equation are rational integers, but it must be adapted to handle the relative setting where we are allowing the coefficients and solutions to be integral in a quadratic imaginary number field. At BIRS, we modified the approach in [13] in order to account for our family of polynomials in the relative setting.

3.2 Continued Fractions for Imaginary Quadratic Number Fields

Daniel Martin recently published a paper in which he created an analogue of continued fractions where the partial numerators come from a finite set, and the partial denominators are ratios where the numerator is integral and the denominator comes from a finite set [9]. These generalizations of continued fractions share many of the properties of classical continued fractions, which are often used to solve Thue equations.

At BIRS, we were able to prove that if $(x, y) \in \mathcal{O}_K^2$ is a solution to $|F_t^{(4)}(X, Y)| = 1$ and $|y| \geq Y_S$ (we have an explicit value of Y_S that we will oppress in this discussion), then x/y is one of the convergents to some root of $F_t^{(4)}(X, 1)$ which come from Martin's generalization of continued fractions. Additionally, there is an explicit constant X_S so that for any solution $(x, y) \in \mathcal{O}_K^2$ with $|y| < Y_S$, we also have $|x| < X_S$.

Due to this work, the second step of our outline can be made more efficient. Let Y_L denote the upper bound on $|y|$ coming from Baker's method. We can now search for solutions as follows:

1. Brute force search among pairs $(x, y) \in \mathcal{O}_K^2$ with $|y| < Y_S$ and $|x| < X_S$ to see if (x, y) is a solution to $|F_t^{(4)}(X, Y)| = 1$.
2. Generate a list of convergents x/y to the roots of $F_t^{(4)}(X, 1)$ which satisfy $Y_S \leq |y| < Y_L$ using Martin's algorithm. Test each of these pairs (x, y) to see if they satisfy $|F_t^{(4)}(X, Y)| = 1$.

This new approach is much more efficient than merely searching through all pairs (x, y) which satisfy $|y| < Y_L$ (and some appropriate bound on $|x|$).

3.3 Code for Modifying Existing Lemmas

One of the subtle things about the process of conducting research that is not at all obvious when you read a complete paper is the decision about what assumptions to make for the lemmas that you need to prove. Step 1 of our outline includes finding a value of k where we can prove that the only solutions to $|F_t^{(4)}(X, Y)| = 1$ are trivial whenever $|t| > k$. In order to find such a value of k, however, we need a host of approximation results along the lines of the following:

Lemma 1. *If $|t| \geq 6$, then one of the roots α of $F_t^{(4)}(X, 1)$ satisfies $|\alpha + t^{-1}| \leq 6.9|t|^{-3}$.*

The constant 6.9 in that result comes from the assumption $|t| \geq 6$ and that constant changes if we change the assumption $|t| \geq 6$ to the assumption $|t| \geq 7$. Furthermore, we use the constant 6.9 (or its analogue) in other results that we need to prove.

However, we may not be able to accomplish step 1 from the outline with $k = 6$. Indeed, we expect that we will not be able to do so. In order to avoid manually reproving Lemma 1 every time we change the assumption $|t| \geq 6$ to something else, we spent time at BIRS developing Sage code which, on input k, outputs a value of M for which Lemma 1 holds with $|t| \geq 6$ replaced by $|t| \geq k$ and 6.9 replaced by M. Additionally, we developed similar code for several other technical lemmas that we will need.

This code is unlikely to ever be published as part of our paper, because it will not aid the reader in verifying that the proofs are true. However, this code is an invaluable part of the research process, and it will save us lots of manual effort as we continue to work on completing step 1 of the outline.

4 Outcome of the Meeting

While we do not yet have the results that we seek, our trip to BIRS was pivotal in making progress towards those results, and providing us with research momentum. We are very grateful to the staff at PIMS, BIRS, and the Banff Centre for the kindness and flexibility they afforded us for our trip.

References

[1] M. A. Bennett, *Rational approximation to algebraic numbers of small height: the Diophantine equation* $|ax^n - by^n| = 1$, J. Reine Angew. Math **535** (2001), 1–49.

[2] B. Earp-Lynch, B. Faye, E. G. Goedhart, I. Vukusic, and D. P. Wisniewski, *On a simple quartic family of Thue equations over imaginary quadratic number fields*, Acta. Arith., **208** (2023), no. 4, 355-389.

[3] I. Gaál, B. Jadrijević, and L. Remete, *Simplest quartic and simplest sextic Thue equations over imaginary quadratic fields*, Int. J. Number Theory, **15** (2019), no. 1, 11–27.

[4] H. G. Grundman and D. P. Wisniewski, *Tetranomial Thue equations*, J. Number Theory **133** (2013), no. 12, 4140–4174.

[5] C. Heuberger. *All solutions to Thomas' family of Thue equations over imaginary quadratic number fields*, J. Symbolic Comput., **41** (2006), no. 9, 980–998.

[6] G. Knapp, *The number of solutions to the trinomial Thue equation*, Funct. Approx. Comment. Math., **69** (2023), no. 2, 247–270.

[7] G. Lettl, A. Pethö, and P. Voutier, *Simple familes of Thue inequalities*, Trans Amer. Math. Soc., **351** (1999), no. 5, 1871–1894.

[8] K. Mahler, *Zur Approximation algebraischer Zahlen. I - Über den größten Primteiler binärer Formen*, Math. Ann. **107** (1933), no. 1, 691–730.

[9] D. E. Martin, *Continued fractions over non-Euclidean imaginary quadratic rings*, J. Number Theory **243** (2023), 688–714.

[10] M. Mignotte, *Verification of a conjecture of E. Thomas*, J. Number Theory, **44** (1993), no. 2, 172–177.

[11] E. Thomas, *Complete solutions to a family of cubic Diophantine equations*, J. Number Theory, **34** (199), no. 2, 235–250.

[12] A. Thue, *Über Annäherungswerte algebraischer Zahlen,* J. Reine Agnew. Math **135** (1909), no. 135, 284–305.

[13] V. Ziegler, *On a family of cubics over imaginary quadratic fields*, Period. Math. Hungar. **51** (2005), no. 2, 109–130.