Algebraic K-theory of Lawvere theories

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1 Overview of the Field

Algebraic K-theory connects to a variety of fields, techniques, and applications. Starting with the work of Grothendieck in 1957, it was quickly taken up by Bass and Milnor who defined the lower K-groups of a ring, and bloomed under Quillen—and later, Waldhausen—who introduced the rich higher invariants we know today. At its heart, it consists of a machinery that takes some flavor of algebraic structure as input and produces a space, or a spectrum, whose homotopical structure records key data about the original object. The choices of inputs that admit a K-theory construction are varied, and this results in the specific machineries, together with the foundational properties they satisfy, being finely tailored to each setting.

Historically, the interest was to study the K-theory of a ring R. In this case, $K(R)$ is defined by considering the category $\mathcal{P}(R)$ of finitely generated projective modules over R. In modern times, one way in which we can generalize rings and their modules is by considering Lawvere theories and their models, so it is natural that they would also fit into a construct of algebraic K-theory. Indeed, in recent work [2, 3], Bohmann and Szymik defined the K-theory of a Lawvere theory. Lawvere theories consist of a pair $(\mathcal{T}, T\colon \mathbb{N} \to \mathcal{T})$ where $\mathcal{T}$ is a small category with coproducts, and T is an identity-on-objects functor that preserves coproducts. In particular, this makes $\mathcal{T}$ into a monoidal category with monoidal structure induced by the coproduct, and $K(\mathcal{T})$ is defined as the K-theory of its underlying symmetric monoidal category. Just as we study rings through their modules, we study Lawvere theories through their models: functors $F\colon \mathcal{T} \to \mathsf{Set}$ taking coproducts to cartesian products.

Lawvere theories are one of the major characters in universal algebra, and they are broad enough to encompass most of the familiar structures in algebra—such as groups, R-modules, Lie algebras, and G-sets—as well as a wealth of examples from logic—such as sets, Boolean algebras, and Heyting algebras. The fact that Bohmann–Szymiks setting places examples from algebra and logic in the same footing gives a new perspective that sheds light on the foundations of algebraic K-theory. For instance, the inclusion of non-additive examples sheds light on the interaction between K-theory and Morita equivalence, as they show that the K-theory of Lawvere theories does not respect Morita equivalence and hence it depends crucially on the syntax of the theory and not solely on the semantics. We believe that there is much to learn about the K-theory of Lawvere theories, and their study was the objective of the research started in this workshop.

2 Research goals and objectives

We plan to investigate interactions between algebraic K-theory and logic through the study of the algebraic K-theory of Lawvere theories, as defined by Bohmann–Szymik. We have two main objectives:

1. Given an inclusion of Lawvere theories $\mathcal{T} \subseteq \mathcal{T}'$, study the relation between their K-theory spaces $K(\mathcal{T})$ and $K(\mathcal{T}')$.
2. Characterize which connective spectra arise as the K-theory spectra of Lawvere theories, and what this entails in relation to the study of multiplicative structures in algebraic K-theory.

2.1 Project 1: Localization for Lawvere theories

Ever since its origin, algebraic K-theory has proved to be exceedingly hard to compute, which is why results relating the K-theory groups of different categories are of vital importance in the field. A major tool in this direction consists of finding homotopy fiber sequences

$$K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{C})$$

relating the K-theory spaces (or spectra) of categories $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. The reason is that, since the n-th K-theory group $K_n(\mathcal{A})$ is defined as the n-th homotopy group of the K-theory space $K(\mathcal{A})$, such homotopy fiber sequences induce a long exact sequence

$$\cdots K_{n+1}(\mathcal{C}) \to K_n(\mathcal{A}) \to K_n(\mathcal{B}) \to K_n(\mathcal{C}) \to K_{n-1}(\mathcal{A}) \to \cdots$$

and one can use information about some of these K-theory groups to extract information about the others.

A particularly useful strategy to produce such a homotopy fiber sequence is to start with an inclusion of categories $\mathcal{A} \subseteq \mathcal{B}$ and construct some form of localization, or "quotient" category $\mathcal{C}$, whose K-theory space models the homotopy cofiber of the induced map $K(\mathcal{A}) \to K(\mathcal{B})$. Of course, this is not a simple procedure, and different K-theory machineries have different methods to produce such a localization. The goal of this project is to understand when we can obtain such a sequence from an inclusion of Lawvere theories $\mathcal{T} \subseteq \mathcal{T}'$, and how we can interpret the data encoded in the K-theory of the cofiber in the language of the syntax or the semantics of the Lawvere theories we start with.

2.2 Project 2: Connective spectra arising from Lawvere theories

When the formal study of infinite loop spaces launched in the early seventies, it quickly became known that symmetric monoidal categories produce infinite loop spectra by means of their classifying spaces. This construction is typically called the K-theory functor, as it agrees with the algebraic K-theory spectrum when applied to the symmetric monoidal category of finitely generated projective modules over a ring. This prompts the natural question: which connective spectra arise as the K-theory of symmetric monoidal categories?

Thomason answered this question by showing that symmetric monoidal categories provide an alternative model for the homotopy theory of connective spectra via their algebraic K-theory. From a practical point of view, symmetric monoidal categories can be unwieldy, as the coherence conditions require careful bookkeeping. Moreover, the category of symmetric monoidal categories is not a symmetric monoidal category, so it doesnt lend itself for the study of multiplicative structures on K-theory.

Several authors have striven to provide more user-friendly models for instance in the form of permutative (2-)categories (as in the work of Mandell and Gursky–Johnson–Osorno). These resolve the bookkeeping issues, but share the problem of not being symmetric monoidal. Addressing this requires a passage to multicategories, which allow us to express the K-theory functor as lax monoidal, but introduce an additional difficulty by straying from the 1-categorical world.

Thanks to Bohmann–Szymiks work, we can now consider a new player: Lawvere theories. Their strong ties to algebra endow the category of Lawvere theories with a symmetric monoidal structure in the form of a Kronecker product. Notably, this makes the K-theory functor from the category of Lawvere theories to the category of (connective) spectra a lax monoidal functor. This project aims to understand exactly which connective spectra can be modeled by Lawvere theories. This would provide a categorical model for a subclass of connective spectra that both makes the K-theory functor lax monoidal—thus allowing for the study of multiplicative structureswhile remaining in a user-friendly, strictly 1-categorical setting, where the tractability of the structures involved is at its greatest.

3 Scientific Progress Made

3.1 Project 1

The majority of our time at BIRS was devoted to advancing the first project described in Section 2.1. As mentioned in the first section, a Lawvere theory $\mathcal{T}$ is a symmetric monoidal category with coproduct, and in [2] Bohmann–Szymik define the K-theory of $\mathcal{T}$ using the existing machinery for symmetric monoidal categories. This perspective does not lend itself to our objectives, as there are no Localization sequences in the literature for the K-theory of symmetric monoidal categories. Then, the first step towards our goal is to find an alternate K-theory machinery that allows us to encode the K-theory of Lawvere theories and, at the same time, has a procedure to produce Localization sequences.

We found that a solution is given by a new K-theory formalism introduced very recently by Bohmann–Gerhardt–Malkiewich–Merling–Zakharevich, [1]. There, the authors define a **category with covering families**, and construct their corresponding K-theory spectrum. Then, in [4], Calle–Gould show a Localization theorem that produces a (simplicial) cofiber for any given functor $F\colon \mathcal{A} \to \mathcal{B}$ of categories with covering families.

Our first result places Lawvere theories—and more generally, any symmetric monoidal category—in the context of categories with covering families.

Proposition 3.1. *Any symmetric monoidal category $(\mathcal{C}, \otimes, I)$ defines a category with covering families.*

A priori, if we start with a symmetric monoidal category $\mathcal{C}$, we can now produce two K-theory spaces: the one we obtain from the K-theory of $\mathcal{C}$ as a symmetric monoidal category, and the one we obtain by considering $\mathcal{C}$ as a category with covering families. To justify our use of this new mahcinery, we must first show that these two K-theory spaces agree; this is our second result.

Theorem 3.2. *Given a symmetric monoidal category $(\mathcal{C}, \otimes, I)$, its K-theory as a symmetric monoidal category and as a category with covering families agree.*

Using the results from [4], we obtain the following.

Corollary 3.3. *Given any functor of Lawvere theories $(\mathcal{T}, T) \to (\mathcal{T}', T')$, there exists a homotopy fiber sequence*

$$K(\mathcal{T}) \to K(\mathcal{T}') \to K(\mathcal{T}/\mathcal{T}')$$

where $\mathcal{T}/\mathcal{T}'$ is a simplicial category with covering families.

Unfortunately, studying the K-theory of this cofiber is not straightforward, and this will require further examination. In the next section, we outline our plans for future directions.

3.2 Project 2

During our time at BIRS, we started the study of the objectives of this project at the level of K_0. This study already reveals that Lawvere theories cannot model all connective spectra, as we prove that there are limited options for the values that the K_0 group of a Lawvere theory is allowed to be.

Proposition 3.4. *For any Lawvere theory $(\mathcal{T}, T)$, we have that $K_0(\mathcal{T}) = \mathbb{Z}/n\mathbb{Z}$ for some $n \geq 0$.*

At least, we are able to show that all of these possible (non-zero) values are actually realized by Lawvere theories which are relatively well-understood: the theories of R-modules for varying rings R.

Proposition 3.5. *For each $n \geq 0$, $n \neq 1$, there exists a ring R such that the Lawvere theory of R-modules has $K_0(R{-}\mathsf{mod}) = \mathbb{Z}/n\mathbb{Z}$.*

These limitations on K_0 are not too disappointing, as results that study K-theory away from K_0 are still immensely useful in practice. With this in mind, we detail some further questions to explore in the next section.

4 Plans for future work

Our time at BIRS allowed our team to jumpstart these projects and make valuable progress, both in terms of the mathematics that we produced, and in terms of our understanding of the problems and the structures involved, as well as their features and limitations. In this section, we outline some of our plans for future work.

4.1 Project 1

Direction 4.1. *Continue to study the simplicial category with covering families produced in Corollary 3.3.*

Firstly with a direct approach described in 1. and move to 2. if that proves too hard.

1. Study the cofiber produced by Corollary 3.3, starting with some concrete examples: the inclusions of Lawvere theories $\mathsf{Set} \subseteq \mathsf{Boole}$, $\mathsf{Set} \subseteq \mathsf{Monoids}$ and $\mathsf{Set} \subseteq \mathsf{Groups}$.
2. Find another model for the cofiber of an inclusion of monoidal categories. Some avenues to explore are:
   - restricting to isomorphisms
   - formally adding duals (for instance using (co)spans)

Direction 4.2. *Try to produce a fiber, instead of a cofiber by proving a version of the Fibration theorem for categories with covering families, or directly construct fibers for a functor of monoidal categories* $F\colon \mathcal{A} \to \mathcal{B}$

4.2 Project 2

Given the results of Proposition 3.4, we know that Lawvere theories will not model all connective spectra. Then, our next direction is to widen our objects of study, and consider instead **multi-sorted Lawvere theories**. The K-theory of these structures has not yet been defined, but should generalize the work of Bohmann–Szymik.

Direction 4.3. *Define the* K*-theory of multi-sorted Lawvere theories.*

The hope is that by considering this larger class of inputs, we might be able to recover more values for K_0. Then, the next question to pursue is the following:

Direction 4.4. *Let* $\mathcal{T}$ *be a multi-sorted Lawvere theory. Is* $K_0(\mathcal{T})$ *also of the form* $\mathbb{Z}/n\mathbb{Z}$*?*

Depending on what we find, the next step is to describe the subcategory of connective spectra that we obtain from (multi-sorted) Lawvere theories.

Direction 4.5. *Characterize the image of the* K*-theory functor*

$$K\colon (\mathsf{multi-sorted})\mathsf{Lawvere} \to \mathsf{Spectra}.$$

References

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[4] M. Calle and J. Gould, A combinatorial K-theory perspective on the edge reconstruction conjecture in graph theory, *https://arxiv.org/pdf/2402.14986*.