Homological invariants of Fourier algebras (24rit022)

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1 Overview of the Field

Classical Fourier analysis on the circle or the real line was motivated by the need to analyse differential equations arising in physics. More generally, during the 20th century harmonic analysis on locally compact abelian (LCA) groups became an important tool in diverse areas of mathematics, such as operator theory and number theory. The theory of Banach algebras entered the picture in the mid 20th-century, when Gelfand gave a short algebraic proof of a version of Wieners Tauberian theorem [14].

With hindsight, Gelfands proof makes use of the structural properties of the Fourier algebras of LCA groups. In fact the Fourier algebra $\mathrm{A}(G)$ can be defined for any locally compact group G, as an algebra of certain complex-valued continuous functions on G. Structural properties of Fourier algebras have been applied to a range of phenomena in functional analysis, including: asymptotics of power-bounded operators [10]; disjointness-preservers [1]; and approximation properties for group C^*-algebras [17].

By a celebrated result of Walter [22], the Fourier algebras of two groups G and H are isometrically isomorphic as Banach algebras if and only if G and H are isomorphic as topological groups. Thus the Fourier algebra captures much more information about G than merely knowing its underlying topological space (for example, $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(2)\times\mathbb{R}^2$ are diffeomorphic as manifolds but very different as topological groups). It is therefore natural to study various invariants of Banach algebras, specialized to the setting of $\mathrm{A}(G)$, to see what information they remember about G itself.

One such invariant is the property of amenability, which has proved to be a fundamental concept for Banach algebras and operator algebras [21, 23]. The amenable Fourier algebras were classified by Forrest and Runde in an extremely influential and much-cited paper [11] from 2005. A related direction of research concerns the amenability constant of a Banach algebra, which provides a quantitative refinement of the notion of amenability. The first significant results for amenability constants of Fourier algebras were obtained in the 1990s by Johnson [18], but only for compact groups. In particular, Johnson obtained a remarkable formula for the amenability constant $\mathrm{AM}(\mathrm{A}(G))$ when G is finite, and used this to show that $\mathrm{AM}(\mathrm{A}(G)) \geq \frac{3}{2}$ for every non-abelian **finite** group G.

The next significant work on $\mathrm{AM}(\mathrm{A}(G))$ came 10 years later, and is due to Runde in [20]; he used operator-space methods to obtain improved *upper bounds* on $\mathrm{AM}(\mathrm{A}(G))$ for arbitrary G (not necessarily compact), and he also showed that $\mathrm{AM}(\mathrm{A}(G)) > 1$ for every non-abelian (locally compact) group G.

Note that Runde's lower bound applies to a much large class of groups, but for finite groups Johnson's lower bound is better. In [20], Runde himself raised the natural question of whether Johnson's lower bound remained valid for all non-abelian G, not just the finite ones. This problem remained open for 15 years, until work of the lead organizer (YC), which will be discussed in the next section.

2 Recent Developments and Open Problems

A few years ago, the lead organizer (YC) obtained new results and estimates for the amenability constants of various Fourier algebras [3], using another invariant $\mathrm{AD}(G)$ that is associated directly to a given group G. This was used to give a positive answer to the question of Runde that was mentioned in the previous section. The results of [3] were presented at the "Canadian Abstract Harmonic Analysis Symposium 2022" workshop (BIRS, June 17–19, 2022), where they attracted interest from the participants and were highlighted in the scientific report for the meeting [12]. It was therefore natural to try to capitalize on the existing momentum and make further progress, through a *Research In Teams* at BIRS.

The two members of this RIT have a proven track record of collaboration during the last 12 years [4, 5, 6, 7, 8, 9], but have also worked separately on the amenability constants of certain classes of Banach algebras [2, 15]. The goal of our stay at BIRS was to have a period of focused research on the amenability constants of Fourier algebras. In particular, we aimed to make concrete progress on the following conjecture, posed by the lead organizer (YC) in [3].

Conjecture. For every locally compact group G, the amenability constant of $\mathrm{A}(G)$, which we denote by $\mathrm{AM}(\mathrm{A}(G))$, is equal to the invariant $\mathrm{AD}(G)$.

If the conjecture is true, it would provide new tools for calculating or estimating $\mathrm{AM}(\mathrm{A}(G))$, since $\mathrm{AD}(G)$ is known to have good functorial properties.

One natural strategy is to attack the cases of G compact and G discrete in parallel. The overlap between the two cases is when G is finite, and in that case the conjecture is known to be true, by work of the lead organizer [3, Theorem 1.4]. However, the proof uses the compactness and the discreteness of finite groups in two different places, so new techniques must be developed to make progress on the general conjecture.

For countable G the lead organizer's paper [3] provides an explicit formula for $\mathrm{AD}(G)$ in terms of the Plancherel measure on the unitary dual of G. One tool that is frequently useful for describing the unitary dual is the Mackey machine for induced representations; this is machinery with which the co-organizer has particular expertise [16]. Our hope was that this would provide a fresh perspective.

Another idea, which has not been considered before in the literature, is to introduce and study a *third* quantity $\mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G))$, which is a "twisted" version of the operator-space version of the amenability constant. It enjoys better properties than $\mathrm{AM}(\mathrm{A}(G))$; for instance we always have

$$\mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G \times H)) = \mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G))\,\mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(H)) \quad \text{for all groups } G \text{ and } H. \tag{1}$$

Moreover, introducing $\mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G))$ allows us to split the original problem into two halves, in the following sense. By adapting the arguments of [11, 20], one can show that there are inequalities

$$\mathrm{AM}(\mathrm{A}(G)) \overset{(a)}{\geq} \mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G)) \overset{(b)}{\geq} \mathrm{AD}(G) \tag{2}$$

Thus if one hopes to disprove the conjecture, it suffices to find a G for which *either* (a) *or* (b) is a *strict inequality*. Conversely, if one can find new examples of G for which *both* (a) *and* (b) are *equalities*, these will provide further evidence to support the conjecture.

3 Scientific Progress Made

Both organizers were keen to use the RIT week at BIRS for sustained in-person collaboration. During the week, we worked on both the compact and the discrete cases of the main problem.

The compact case

By using an averaging argument for virtual diagonals with respect to compact group actions, we were able to show that for *every* compact group G, the inequality (a) in (2) is actually an equality. That is, we have

$$\mathrm{AM}(\mathrm{A}(G)) = \mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G)) \quad \text{for every compact group } G. \tag{3}$$

Subsequently, we realized that these calculations provide an alternative perspective on results obtained by Johnson in [18, §3], and are closely related to techniques used in [13, §4] to study Johnson's algebra $\mathrm{A}_\gamma(G)$. One advantage of our approach is that by combining the identities (1) and (3), one immediately obtains

$$\mathrm{AM}(\mathrm{A}(G\times H)) = \mathrm{AM}(\mathrm{A}(G))\,\mathrm{AM}(\mathrm{A}(H)) \quad \text{for all } \textbf{compact} \text{ groups } G \text{ and } H.$$

This appears to be a new observation, at this level of generality; neither [18] nor [13] include this result, although for finite groups it is stated as [18, Corollary 4.2].

We also discussed the problem of calculating $\mathrm{AM}(\mathrm{A}(G))$ for specific examples of compact virtually abelian groups. It turns out that many of these natural examples are covered by the lead organizer's earlier result [3, Theorem 1.10], so we did not pursue this direction further.

One afternoon was spent trying to prove that $\mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G)) = \mathrm{AD}(G_d)$ by finding a suitable lifting map from $\mathrm{B}(G_d \times G_d)$ to $\mathrm{A}(G\times G)^{**}$. Although we did not succeed, we remain optimistic that further progress could be made here by using tools from C^*-algebra theory (such as the weak expectation property) and the homological theory of Banach modules. This would be a natural line of enquiry to pursue in future work.

The discrete case

In discussions following the 2022 CAHAS meeting, we had already observed that for *every* discrete group G, the inequality (b) in (2) is actually an equality (this is a straightforward adaptation of arguments in [19]).

During the RIT week we attempted to prove $\mathrm{AM}(\mathrm{A}(G)) = \mathrm{AM}_{\mathrm{tcb}}(\mathrm{A}(G))$ for G discrete, but could not see how to do this in general. We therefore adjusted our approach and chose to focus on particular families of examples. We looked in detail at groups of the form $G = H \rtimes \mathbb{Z}_p$ for certain abelian groups H, and worked out the irreps and explicit coefficient functions of G, as well as the matrix-valued Fourier coefficients of the anti-diagonal subset of $G\times G$.

At the end of the visit, we discussed a family of groups H_q that display richer behaviour, and identified them as the next family of examples to attack. This family occurs very naturally when considering the integer Heisenberg group $H_3(\mathbb{Z})$; to be specific, H_q can be identified with the quotient of $H_3(\mathbb{Z})$ by an index-q subgroup of the centre of $H_3(\mathbb{Z})$. Thus H_q can be viewed as a discretized version of the *reduced Heisenberg group*, a nilpotent Lie group that appears in the theory of the Gabor transform for signal analysis.

We observed during the visit that, by using a result of Runde from 2006, we always have the upper bound $\mathrm{AM}(\mathrm{A}(\mathsf{H}_q)) \leq q$. To put this in context: one can use the techniques of [3] to show that $\mathrm{AD}(\mathsf{H}_q)$ is *strictly* less than q; so if the main conjecture is true, the upper bound that is stated above cannot be optimal. Therefore, an important task for future work will be to try and improve upon the existing bounds for $\mathrm{AM}(\mathrm{A}(\mathsf{H}_q))$.

4 Outcome of the Meeting

Both organizers of this RIT are mid-career researchers aiming for future promotions, and the papers [4, 5], which began our collaborative partnership, were written when we both worked at the same university and had opportunities for sustained in-person discussions. Consequently, our week in BIRS was extremely fruitful and productive, both for concrete progress on specific objectives, and also for planning further work and discussing opportunities for career development.

The results we obtained formed the basis for a follow-up project, which was a successful application to the ICMS in Edinburgh for a *Research in Groups* workshop (to be held in April 2025). Some of these results have already been presented in conference talks, for example by the lead organizer (YC) at the regional conference "Operator Algebras in the South", Southampton, September 6–7 2024.

During the week in BIRS, we also identified natural continuations of our work, with some concrete objectives. In particular, we decided that in future work we should focus on the groups H_q (described in Section 3) as the next family of test cases for the main conjecture. During the months following the RIT week in BIRS, we discussed this problem, and eventually succeeded in confirming the main conjecture for the groups H_q when q is prime. Our approach relies on calculating the explicit Plancherel measure for H_q, combined with ideas from the proofs of [20, Theorem 2.7] and [3, Proposition 3.9]. These new results are being written up as a preprint, which we aim to submit for publication later in 2025.

We spent one afternoon during the week in BIRS discussing the antidiagonal subset of $F_2 \times F_2$, where F_2 denotes the free group on two generators. The motivation for this comes from the following open problem: *if we consider the closure of* $\mathrm{A}(F_2)$ *in its cb-multiplier algebra, then this is itself a Banach algebra; is it amenable?* Our investigations were only of a preliminary nature, but we developed some ideas that could form part of a future grant application.

Finally: as remarked in Section 3, the techniques we used to obtain new results for compact groups are closely related to results from [13] concerning Johnson's algebra $\mathrm{A}_\gamma(G)$. The constructions and methods in [13] rely on G being compact, and one interesting avenue for future work would be to identify the correct analogue of $\mathrm{A}_\gamma(G)$ for (virtually abelian) non-compact groups.

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