Open questions session, Thursday

Scribe: Alex Fink

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1 Chris Eur

Question 1. Fact: if P is a lattice generalized permutahedron, then $P \cap ([0,1]^n + v)$ is also, for $v \in \mathbb{Z}^n$. Tile $\mathbb{R}^n$ by cubes; this gives a decomposition of P into translates of matroid polytopes. Q: Do this as explicitly as possible for graphical zonotopes,

$$Z_G = \sum_{(v_1,v_2) \in G} \operatorname{Conv}(e_{v_1}, e_{v_2}) \subset \mathbb{R}^{V(G)}.$$

Alex: guess: strict gammoids. A **strict gammoid** is a matroid defined from a directed graph G with ground set $V(G)$, and a distinguished basis $B \subseteq V(G)$; the bases are all sets of vertices that have a family of vertex-disjoint paths (possibly length zero) from B. [Revisiting this as I typed this document: no, they're gammoids but not strict, because the paths are only edge-disjoint, not vertex-disjoint.]

Question 2. Let $\pi : \mathbb{R}^{2n} \xrightarrow{[I_n \ -I_n]} \mathbb{R}^n$. A delta-matroid D is **envelopable** if there exists a matroid M on $[2n]$ such that $\pi(P(M)) = P(D)$, possibly with scaling depending on conventions. Not all delta-matroids are envelopable.

Q. Are all even delta-matroids envelopable? Are all delta-matroids with the strong symmetric exchange property envelopable?

Matt L: Matroids are supposed to generalize linear spaces; delta-matroids, isotropic linear spaces. Every isotropic linear space is a linear space. Envelopability is the corresponding property when not representable.

David: There are formulae which write Plücker coordinates in terms of spinor coordinates.

$$\begin{array}{ccc} SOGr(n,2n) & \hookrightarrow & \mathbb{P}^{2^n-1} \\ & \searrow & \downarrow v_2 \\ & & \mathbb{P}^{\binom{2n}{n}-1} \end{array}$$

Matt B, Chris: This doesn't work. The coordinates aren't monomial.

Matt L: Felipe gives an isotropic tropical linear spaces with multiple extensions to a tropical linear space. There's also in the literature an example of one with no extensions.

The definition of strong symmetric exchange meant here is one that does not require that D is even, as follows. Given two vertices e_{B_1}, e_{B_2} of $P(D)$, suppose that the usual exchange relation for delta-matroids requires there to exist a vertex $e_{B_1} + v$. Then strong symmetric exchange also requires $e_{B_2} - v$.

CE: The book of Borovik, Gelfand and White has an incorrect exercise on this.

Matt L: Bouchet's paper assumes that delta-matroids are even.

2 Oliver Lorscheid, interjecting

Linear spaces satisfy not just the usual Plücker relations but also **multi-exchange relations**: given bases B, B' and a set $A \subset B \setminus B'$ of size l, there exists a set $A' \subset B' \setminus B$ of size l such that $B \setminus A \cap A'$ and $B' \setminus A' \cap A$ are bases. It is also true that the single exchange relations implies the multi-exchange relations for matroids, i.e. over the Krasner hyperfield $\mathbb{K}$. Is the same true for all idylls?

Matt B:

1. To define the Grassmannian as a scheme over $\mathbb{Z}$, one needs to use all multi-exchange relations, not just the single exchanges.
2. The proof for $\mathbb{K}$-matroids can be done slickly using Edmonds' matroid intersection. I forget what paper this is in.

3 Chris Eur, resuming

Question 3. Consider

$$A^\bullet(X_E)[\delta]/\langle \delta^r + \delta^{r-1} c_1(\mathcal{S}_M) + \cdots + c_r(\mathcal{S}_M)\rangle.$$

The generator of the ideal is called a **Chern polynomial**. If M is realized by a linear space L, then this ring $\simeq A^\bullet(\mathbb{P}(\mathcal{S}_L))$.

Q: Do Hard Lefschetz and Hodge–Riemann hold for this ring with $l = c\delta + a$, for a ample on X_E?

Nick: Are there combinatorially meaningful consequences? CE: We could remove dependence on [AHK], [ADH] from the Tutte formulae in [BEST].

June: Morally this should be related to the bipermutohedral fan for $\Sigma_{X_E} \times \Sigma_M$, as a blowdown. See the book "Lefschetz Properties" by Numata, Watanabe, and others.

Matt L: By a deformation argument, taking c very small, Hard Lefschetz implies Hodge–Riemann. My conclusion from looking at the "Lefschetz Properties" book is that their techniques are ineffective: you get no control over the cone.

4 Matt Larson

Conjecture. $T_M(x+1, x+1)$ has log-concave coefficients for all matroids M. True for $|E(M)| \leq 9$. Fact:

$$T_M(x+1, x+1) = \sum_{u \in \{0,1\}^n} x^{d(P(M),u)},$$

where d is the lattice distance.

Andy: Is this known for representable matroids? ML: No.

Various people: Is this related to Merino–Welsh? ML: Not that I know.

ML: For even delta-matroids realizable in characteristic 2, this is a famous conjecture on the interlace polynomial. It's false for general delta-matroids.

Matt B: Is $T_M(x+1, y+1)$ Lorentzian? ML: I checked lots of strengthenings and found them false. I don't remember if I checked this one.

5 Andy Berget

Conjecture. The nunmber of set partitions of $E(M)$ into independent sets of M of sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$, $\lambda \vdash |E(M)|$, is at least the Kostka number $K_{\lambda,\rho^{\mathrm{t}}}$, where $\rho = \rho(M) : r_1 \geq r_2 \geq \cdots$ is the **rank partition** of M, determined by the condition that $r_1 + \cdots + r_k =$ size of the largest union of k independent sets of M, i.e. the rank of the k-fold matroid union of M. (Assume M is loopless.)

Motivation. Pick a realisation $v_1, v_2, \ldots, v_+n \in \mathbb{C}^r$ of M. Form

$$\mathfrak{S}(v) = \operatorname{span}(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} | \sigma \in \mathfrak{S}_n) \subset (\mathbb{C}^r)^{\otimes n}.$$

This is an $\mathfrak{S}_n$-representation, so it decomposes into irreducibles, indexed by partitions. It's a consequence of [Berget–Fink] that the multiplicity of each irrep is a valuative matroid invariant.

Theorem. The irrep indexed by λ appears iff $\lambda \trianglerighteq \rho^{\mathrm{t}}$, where $\trianglerighteq$ is dominance order.

Theorem. The multiplicity of $\lambda =$ a hook gives the coefficients of $\overline{\chi}_M$ up to sign.

The **Frobenius character** of a $\mathfrak{S}_n$-representation is its character written as a symmetric function.

Variant conjecture. The Frobenius character of $\mathfrak{S}(V) - e_{\rho^{\mathrm{t}}}$ is Schur-positive. Here $e_{\rho^{\mathrm{t}}}$ is an elementary symmetric function. The Gröbner degeneration $X(v) \rightsquigarrow \operatorname{in} X(v)$ from [Berget–Fink] should have a matroidal extension, and the Frobenius character should be computable from it.

6 Johannes Rau

This question is based on mork in progress by Draisma, Pendavingh, Rau, Yuen, and a student of Draisma.

Given a matroid M, we have inequalities between three numbers:

$$\begin{aligned} & d := \operatorname{rk}(M) \\ \leq & \min\{2\dim(\Sigma_M + R) - \dim R : R \text{ a rational subspace of } \mathbb{R}^n\} \\ \leq & \min\{\sum(2\operatorname{rk}_M(P_i) - 1) : P_1 \amalg \cdots \amalg P_k = E\}. \end{aligned}$$

The third number is bounded above by $\min\{n, 2d - 1\}$. The third number is the second specialized to R being a subspace in the braid arrangement. For M realizable over $\mathbb{C}$ by a subspace V, the second and third agree and both equal $\dim(\operatorname{Log}(V))$.

Q: Are the second and thord always equal?

Q: Compute these three numbers for the restriction of M to each set $S \subset E(M)$, defining set functions $f_1(S)$, $f_2(S)$, $f_3(S)$. Is f_2 a matroid rank function? f_3?

Q: Give an interpretation of f_3.

7 Federico Ardila

$T_{K_n}(1, -1) = A_{n-1}$, the number of alternating i.e. up-down permutations of $n - 1$. The only proof I know is computing generating functions of both sides. Give a better explanation.

Eric Katz: connections to [BST]?