Translating between NIP integral domains and topological fields

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The big idea

- NIP topological fields come from NIP integral domains.
- Facts about NIP integral domains imply facts about NIP topological fields.

Section 1

Ring and field topologies

Ring and field topologies

Let K be a field.

- A *ring topology* on K is a non-discrete non-trivial topology on K respecting the ring operations.
- A *field topology* on K is a ring topology respecting division.

Examples:

- The order topology on an ordered field.
- The valuation topology on a valued field.
- The standard topology on $\mathbb{C}$.

Remark

Ring topologies on K are Hausdorff.

Bounded sets

Fix a field K with a ring topology τ.

Definition

A set $B \subseteq K$ is *bounded* if for any neighborhood $U \ni 0$, there is $c \in K^\times$ with $cB \subseteq U$.

Fact

- *Finite sets are bounded.*
- *Subsets of bounded sets are bounded.*
- *If B_1, B_2 are bounded, then so are*

$$B_1 + B_2,\ B_1 \cup B_2,\ B_1 \cdot B_2,\ \overline{B_1}.$$

Locally bounded ring topologies

Definition

(K, τ) is *locally bounded* if there is a bounded neighborhood of 0.

1. Topologies from field orders, absolute values, and valuations are locally bounded.
2. The topology on $\mathbb{Q}$ induced by the diagonal embedding

$$\mathbb{Q} \hookrightarrow \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \cdots$$

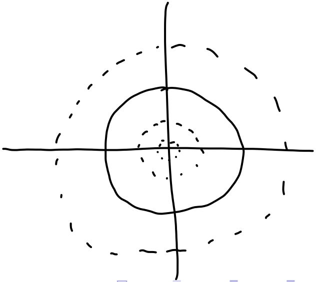
is *not* locally bounded.

Locally bounded ring topologies

Suppose (K, τ) is locally bounded and B is a bounded neighborhood of 0.

1. The family $\{cB : c \in K^\times\}$ is a neighborhood basis of 0.
2. The family $\{cB : c \in K^\times\}$ is cofinal among bounded sets—X is bounded iff $\exists c \in K^\times : X \subseteq cB$.

Example: $B = \{z \in \mathbb{C} : |z| \leq 1\}$.



Ring topologies from subrings

Suppose R is a proper subring of K and $K = \mathrm{Frac}(R)$.

Fact

There is a locally bounded ring topology τ_R on K such that

- *The family $\{cR : c \in K^\times\}$ is a nbhd basis of 0.*
- *The family of non-zero ideals in R is a nbhd basis of 0.*

R is a bounded neighborhood of 0 in τ_R.

Example

If R is a valuation ring, τ_R is the valuation topology.

Fact

τ_R is a field topology iff the Jacobson radical of R is nonzero.
If R is local or semilocal, then τ_R is a field topology.

Definable topologies

Definition

Let D be a definable set in a structure M. A topology τ on D is *definable* if there is a definable family $\{B_x\}_{x \in E}$ such that $\{B_x : x \in E\}$ is a basis of open sets for τ.

Example

In an o-minimal structure $(M, <, \ldots)$, the product topology on M^n is definable.

Example

If R is a definable subring of K, then τ_R is a definable topology on K.

Section 2

From topological fields to rings

Goal

Theorem

Let $(K, +, \cdot, \ldots)$ be an NIP expansion of a field, and τ be a definable ring topology on K.

1. *τ is locally bounded.*
2. *If K is sufficiently saturated, then $\tau = \tau_R$ for some externally definable subring $R \subseteq K$.*

Remark

The expansion $(K, +, \cdot, \ldots, R)$ is NIP by work of Shelah.
In particular, R is NIP.

The lazy path from topologies to rings

Fix a small model K and a monster model $\mathbb{M} \succeq K$ and definable ring topology τ.

- Let I_K be the "K-infinitesimals", the intersection

$$\bigcap \{U(\mathbb{M}) : U \text{ is a } K\text{-definable nbhd of } 0\}.$$

- I_K is a type-definable and externally definable subgroup of $(\mathbb{M}, +)$.
- Let $R_K = \{x \in \mathbb{M} : xI_K \subseteq I_K\}$.
- Then R_K is an externally definable proper subring of $\mathbb{M}$, and $\mathrm{Frac}(R_K) = \mathbb{M}$.

Application of the lazy path

Fact ([Joh22])

If R is an NIP integral $\mathbb{F}_p$-algebra, then R is a henselian local ring, and $\mathrm{Frac}(R)$ is "large" in the sense of Pop.

Corollary

If $(K, +, \cdot, \ldots)$ has characteristic $p > 0$, is NIP, and admits a definable ring topology τ, then K is large.

But what can we say about τ?

The better path

Theorem (to prove)

If τ is a definable ring topology on an NIP field K, then τ is locally bounded.

Take a monster $\mathbb{M} \succeq K$ and let R be the ring of "bounded elements" over K:

$$R = \bigcup \{B(\mathbb{M}) : B \text{ is } K\text{-definable and bounded}\}.$$

Theorem

R is an externally definable subring of $\mathbb{M}$, and τ_R equals the definable extension of τ to $\mathbb{M}$.

Proving local boundedness

Let τ be a definable ring topology, **not** locally bounded.

Definition (in $\mathbb{M}$)

A *special nbhd* is an intersection $G = \bigcap_{i=1}^{\infty} U_i$ where

- $U_1 \supseteq U_2 \supseteq \cdots$ and the U_i are basic nbhds of 0.
- G is a $\mathbb{Q}$-linear subspace of $(\mathbb{M}, +)$.

Key facts:

- Special nbhds are nbhds of 0.
- Special nbhds form a basis.
- Special nbhds are externally definable and type-definable.

Proving local boundedness

Lemma (to prove)

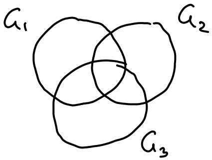
There is an independent sequence of special nbhds $G_1, G_2, G_3, \ldots$.

$$a_S \in G_i \iff i \in S \text{ for } i \in \mathbb{N},\ S \subseteq \mathbb{N}$$

Corollary

There is an independent sequence of basic nbhds $U_1, U_2, U_3, \ldots$.

This contradicts NIP.



Building the independent sequence

Given G_1, G_2, choose a_0, a_1, a_2, a_{12}.
The nbhd $G_1 \cap G_2$ isn't bounded, so there is a special nbhd G with

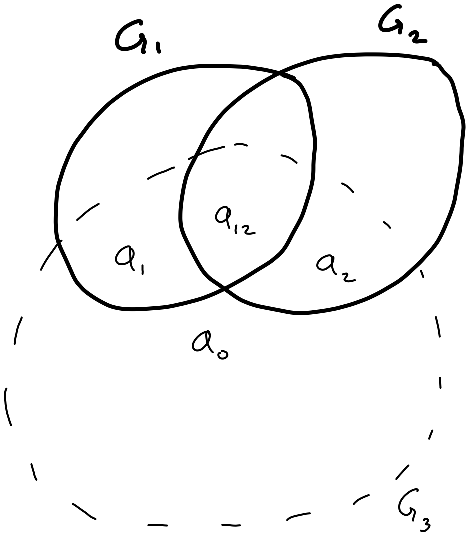
$$\forall c \in \mathbb{M}^\times : G_1 \cap G_2 \not\subseteq c^{-1} G.$$

Take c so small that

$$c \cdot \{a_0, a_1, a_2, a_{12}\} \in G,$$

and set $G_3 = c^{-1} G$.

- G_3 *intersects* each cell in the venn diagram.
- $G_3 \not\supseteq G_1 \cap G_2$, so G_3 doesn't *contain* any cell in the venn diagram.



Section 3

Applications of the main theorem

From ring topologies to field topologies

Theorem (Simon)

If R is an NIP ring, then the poset of prime ideals has finite width.

Corollary

Any NIP integral domain R is semilocal, so τ_R is a field topology.

Corollary

Any definable ring topology on an NIP field is a field topology.

Henselianity

A local ring $(R, \mathfrak{m})$ is *henselian* if it satisfies

Hensel's Lemma

If $a_0, \ldots, a_n \in R$ *and* $\alpha_i = \mathrm{res}(a_i)$ *and the polynomial*

$$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$$

has a simple root $\beta \in R/\mathfrak{m}$*, then the polynomial*

$$a_0 + a_1 x + \cdots + a_n x^n$$

has a root $b \in R$ *with* $\mathrm{res}(b) = \beta$*.*

Example

$\mathbb{Z}_p$ and $K[[t]]$ are henselian.

Generalized t-henselianity

Definition (Dittman-Walsberg-Ye)

A field topology is *gt-henselian* if it satisfies the implicit function theorem for polynomial equations.

Fact

If R is a henselian local ring, then τ_R is gt-henselian.

Fact ([Joh22, Joh23a])

Let R be a NIP integral domain. If $\mathrm{char}(R) > 0$ *or* $\text{dp-rk}(K) < \aleph_0^-$*, then R is a henselian local ring.*

Corollary

If K is an NIP field with positive characteristic or finite dp-rank, any definable field topology on K is gt-henselian.

V-topologies

A field topology τ is a *V-topology* if there is a bounded neighborhood $B \ni 0$ such that for any $x \in K$

$$x \in B \text{ or } x^{-1} \in B.$$

These are V-topologies:

1: Valuation topologies.

2: Topologies from absolute values.

3: The order topology on an ordered field.

All V-topologies come from (1) or (2).
Up to elementary equivalence, all V-topologies come from (1).

Dp-minimal fields

Theorem (d'Elbée-Halevi)

If R is a dp-minimal integral domain then R is a local ring, and if $R/\mathfrak{m}$ is infinite then R is a valuation ring.

Corollary

If τ is a definable field topology on a dp-minimal field, then τ is a V-topology.

Finite breadth

Definition

An integral domain R has $\mathrm{br}(R) \le n$ if for any $x_0, \ldots, x_n \in \mathrm{Frac}(R)$, there is i such that

$$x_i \in x_0 R + x_1 R + \cdots + \widehat{x_i R} + \cdots + x_n R.$$

$\mathrm{br}(R) = 1 \iff R$ is a valuation ring.

Definition

A field topology τ has $\mathrm{br}(\tau) \le n$ if there is a bounded neighborhood $U \ni 0$ such that for any $x_0, \ldots, x_n \in K$, there is i such that

$$x_i \in x_0 U + x_1 U + \cdots + \widehat{x_i U} + \cdots + x_n U.$$

$\mathrm{br}(\tau) = 1 \iff \tau$ is a V-topology.

Two examples

1. Let v_1, v_2 be two independent valuations on K. Consider the diagonal embedding
$$K \hookrightarrow (K, v_1) \times (K, v_2).$$
The induced topology on K has breadth 2.
2. Let $(K, \leq, \partial) \models \mathrm{CODF}$. Consider the embedding
$$K \hookrightarrow (K, \leq) \times (K, \leq)$$
$$x \mapsto (x, \partial x)$$
The induced topology on K has breadth 2.

Finite breadth

Fact ([Joh23a])

If R is a dp-finite integral domain, then R is a local ring, and if $R/\mathfrak{m}$ is infinite then $\mathrm{br}(R) \leq \text{dp-rk}(R)$.

Corollary

If τ is a definable field topology on a dp-finite field, then τ has finite breadth.

There are examples where τ has breadth 2.

Section 4

A conjecture and a question

The henselianity conjecture

Theorem (Delon, Gurevich, Schmitt)

Let $(K, \mathcal{O})$ be a valued field such that the residue field is NIP of characteristic 0. If the valuation ring $\mathcal{O}$ is henselian, then $(K, \mathcal{O})$ is NIP.

Conjecture (Henselianity conjecture)

If $(K, \mathcal{O})$ is an NIP valued field, then $\mathcal{O}$ is henselian.

This is implied by the conjectural classification of NIP fields.

The generalized henselianity conjecture

Conjecture (Henselianity conjecture)

If $\mathcal{O}$ is an NIP valuation ring, then $\mathcal{O}$ is henselian.

Conjecture ("Generalized henselianity conjecture")

If R is an NIP integral domain, then R is a henselian local ring.

Equivalent forms:

- Weaker: If R is an NIP integral domain, then R is a local ring.
- Stronger: If R is an NIP commutative ring, then R is a finite product of henselian local rings.

Evidence for the generalized henselianity conjecture

The generalized henselianity conjecture holds in the following cases:

- Frac(R) has positive characteristic [Joh22]
- R is dp-finite [Joh23a].

The henselianity conjecture implies the generalized henselianity conjecture in the following cases [Joh23a]:

- R is Noetherian.
- br(R) $< \infty$.

Topological consequences

Theorem ([Joh23b])

(Assuming GHC) If τ *is a definable field topology on an NIP field* K*, then* τ *is gt-henselian and* K *is large.*

Theorem ([Joh23b])

(Assuming HC) If τ *is a definable field topology on an NIP field* K*, and* $\mathrm{br}(\tau) < \infty$*, then* τ *is gt-henselian and* K *is large.*

A question

Question

Is there a definable field topology τ on an NIP field K with $\mathrm{br}(\tau) = \infty$*?*

Example

The ring

$$R = \mathbb{Q}^{\mathrm{alg}} \oplus t\mathbb{C}[[t]] = \{a_0 + a_1 t + a_2 t^2 + \cdots : a_0 \in \mathbb{Q}^{\mathrm{alg}},\ a_1, a_2, \ldots \in \mathbb{C}\}$$

is NIP and has $\mathrm{br}(R) = \infty$, BUT $\tau_R = \tau_{\mathbb{C}[[t]]}$ is a V-topology, so $\mathrm{br}(\tau_R) = 1$.

Reformulation in terms of rings

Question

Is there a definable field topology τ *on an NIP field* K *with* $\mathrm{br}(\tau) = \infty$*?*

is equivalent to

Question

Is there an NIP integral domain R *such that for any* n *and any* $e \in R \setminus \{0\}$*, there are* $a_0, a_1, \ldots, a_n \in R$ *with*

$$\begin{aligned} ea_0 &\notin \widehat{a_0 R} + a_1 R + a_2 R + \cdots + a_n R \\ ea_1 &\notin a_0 R + \widehat{a_1 R} + a_2 R + \cdots + a_n R \\ &\cdots \\ ea_n &\notin a_0 R + a_1 R + a_2 R + \cdots + \widehat{a_n R}. \end{aligned}$$

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Questions?