Ensemble Control for Linear and Bilinear Systems

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joint work with

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Geometry, Topology and Control System Design, 2023

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Linear Ensemble: motivation and the finite case

Some scenarios – not necessarily linear – which can be cast into the setting of ensemble control:

- "Broadcast control" in the sense that a "swarm" of (almost identical non-interacting) systems which cannot be addressed individually has to be controlled:
  - swarms of micro-satellites or micro-robots;
  - NMR-spectroscopy;
  - more general, huge number of quantum/nano particles (which are in general not accessible to measurement based feedback methods);
  - infinite platoons of vehicles (apply Fourier transform, see H. Zwart);
  - (desynchronization of) neuron populations for the treatment of epilepsy;
  - Mass transport ...
- "Robust open-loop control" in the sense that one seeks for open-loop control strategies which counteract (uniformly distributed) model uncertainties;

Linear Ensemble: motivation and the finite case

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Terminology:

ensemble control $=$ simultaneous control $=$ controlling families of systems

Motivation

Starter:

A prime example from quantum control!

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A prime example from quantum control!

It's a bilinear ensemble!

The movie "Dancing Arrows" is taken from

Steffen Glaser (TU Munich)

Motivation

Controlled Bloch Equation:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & -\omega_0 & \varepsilon_0 u_2(t) \\ \omega_0 & 0 & -\varepsilon_0 u_1(t) \\ -\varepsilon_0 u_2(t) & \varepsilon_0 u_1(t) & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \tag{B}$$

Control Inputs: $u_1(t), u_2(t)$

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Control Inputs: $u_1(t), u_2(t)$

Dispersion effects

- Lamor dispersion (results form B-field inhomogeneities)
- Transverse dispersion (results from inhomogeneities of rf-pulses)

Motivation

Controlled Bloch Equation:

$$\begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\end{pmatrix} = \begin{pmatrix} 0 & -\omega & \varepsilon u_2(t) \\ \omega & 0 & -\varepsilon u_1(t) \\ -\varepsilon u_2(t) & \varepsilon u_1(t) & 0 \end{pmatrix} \begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} \tag{B}$$

Control inputs: $u_1(t), u_2(t)$ are independent of ω and ε!

Dispersion effects = uncertain model parameters

- Lamor dispersion $\Longrightarrow$ $\omega \in [\omega_0 - \Delta\omega, \omega_0 + \Delta\omega] =: \mathcal{W}$
- Transverse dispersion $\Longrightarrow$ $\varepsilon \in [\varepsilon_0 - \Delta\varepsilon, \varepsilon_0 + \Delta\varepsilon] =: \mathcal{E}$

Motivation

Dispersion of the Bloch Equation:

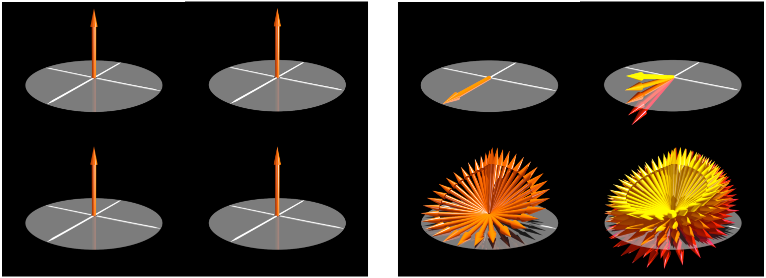

Abbildung: S. Glaser, TU München, presented 2009 at KITP

Motivation

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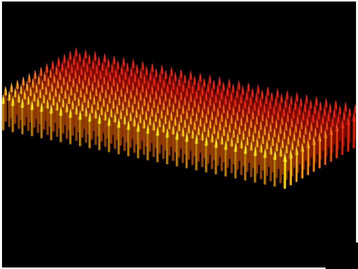

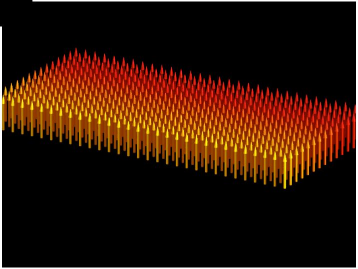

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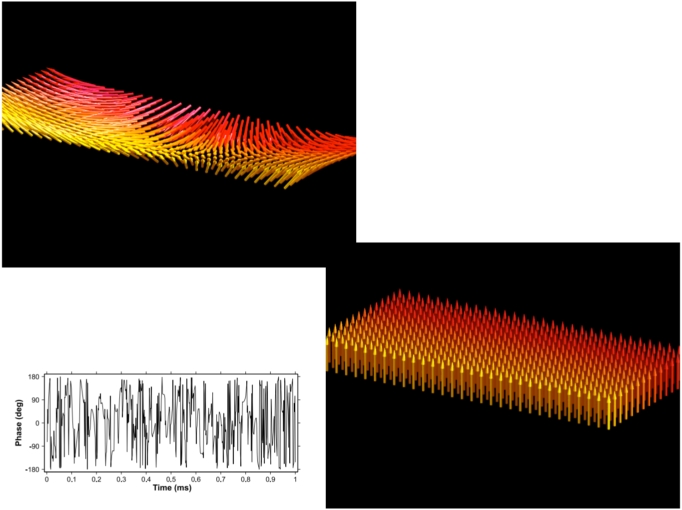

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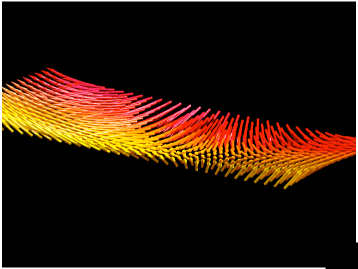

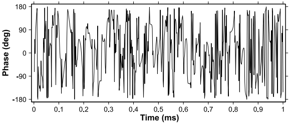

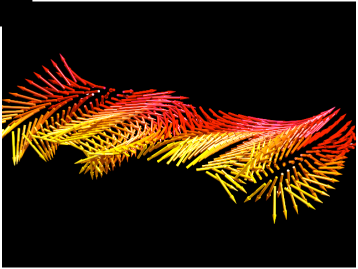

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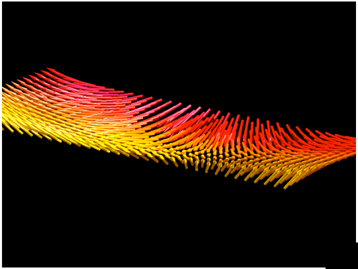

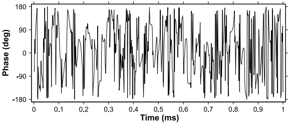

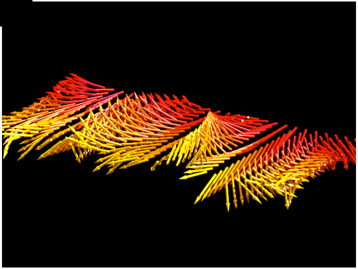

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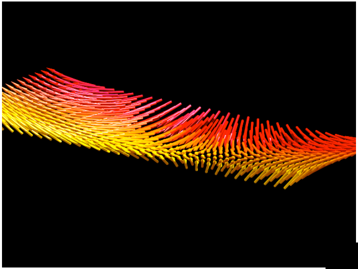

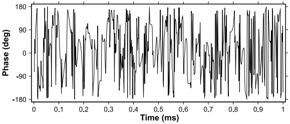

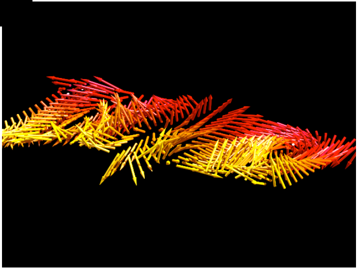

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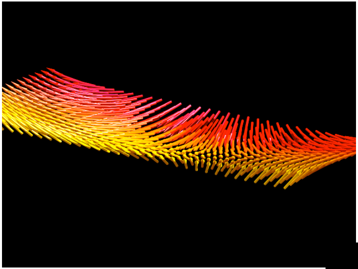
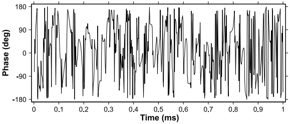
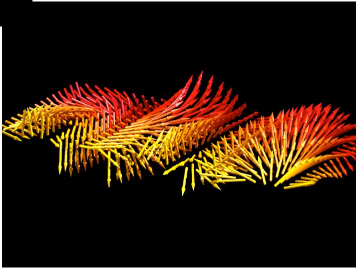
Motivation

Dispersion of the Bloch Equation:







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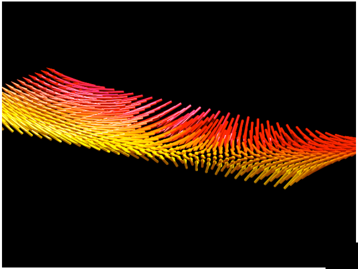

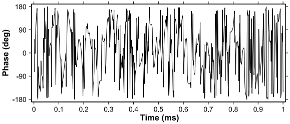

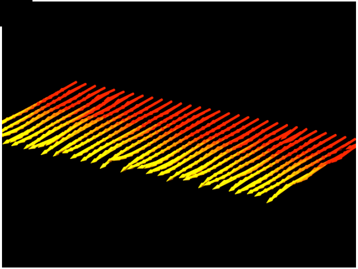

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Motivation

Bottom line (so far):

The infinite bilinear ensemble defined by

the controlled Bloch Equation (under dispersion)

seems to be (approximately) controllable

Why?

Linear Ensemble: motivation and the finite case

Back to linear ensembles – the finite case:

Consider a finite parameter set, e.g. $P := \{1, 2, \dots, N\}$ and finitely many linear systems (A_i, B_i, C_i), $i = 1, \dots, N$ with

- (possibly different) state spaces: $x_i \in \mathbb{R}^{n_i}$;
- common input space: $u := u_i \in \mathbb{R}^m$;
- common output space: $y := y_i \in \mathbb{R}^p$;

How to build the corresponding ensemble:

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- ensemble state space: $x = (x_1, \dots, x_N) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$;
- ensemble dynamics:

$$A := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix}, \quad B := \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}, \quad C := \begin{pmatrix} C_1 & \dots & C_N \end{pmatrix}. \qquad (\Sigma_{\mathrm{E}})$$

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Parallel connection!

Linear Ensemble: motivation and the finite case

Controllability[1] **condition for** (Σ_{E}):

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{pmatrix} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix} u\,. \qquad (\Sigma_{\mathrm{E}})$$

A simple test:

Lemma A (Brockett ???)

For the assertions

(a) the "ensemble" (Σ_{E}) is controllable;

(b) all subsystems (A_i, B_i) are controllable;

(c) $\sigma(A_i) \cap \sigma(A_j) = \emptyset$ for $i \neq j$;

one has the following implications:

$(a) \Longrightarrow (b)$, $(b)\,\&\,(c) \Longrightarrow (a)$ and for $m = 1$ $(b)\,\&\,(c) \Longleftrightarrow (a)$

Proof: Trivial, e.g. Hautus-Test.

[1] No observability and no discrete-time systems in this talk

Linear Ensemble: motivation and the finite case

The general case:

Recall:

- (A, B) is controllable if and only if $(zI - A)$ and B are left-coprime.
- There exists always a right-coprime factorizations

$$N_i(z)D_i(z)^{-1} = (zI - A)^{-1}B$$

of the "transfer function".

Theorem A (Fuhrmann/Helmke)

The "ensemble" (Σ_{E}) is controllable if and only if the following conditions are satisfied:

(a) all subsystems (A_i, B_i) are controllable;

(b) the matrices $D_1(z), \ldots, D_N(z)$ are mutually left coprime;

Remark: For $m = 1$ one can choose $D_i(z) = \det(zI - A_i)$ and thus Theorem A reduces to Lemma A.

Infinite Linear Ensembles – the countable/ continuum case

Let $P = \mathbb{N}$ or let $P \subset \mathbb{R}^d$ be **compact** and consider the infinite parallel connections:

Linear Ensemble

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t), \quad x_i(0) \in \mathbb{C}^n, \quad i \in \mathbb{N} \qquad (\Sigma_{\mathrm{E}}^{\infty})$$

and

$$\frac{\partial x}{\partial t}(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t), \quad x(0,\theta) = x_0(\theta) \in \mathbb{C}^n, \quad \theta \in P \qquad (\Sigma_{\mathrm{E}}^{c})$$

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Problem / Freedom of choosing the right state space?

Infinite Linear Ensembles – the countable/ continuum case

Countable Case:

Linear Ensemble

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Choose our favorite sequence space $X \subset \mathcal{S}(\mathbb{N}, \mathbb{C}^n)$, e.g.:

- Possible state spaces: $X = l_q(\mathbb{N}, \mathbb{C}^n)$ with $(1 \leq q < \infty)$;
- Ensemble matrices:

  $(A_i)_{i\in\mathbb{N}} \in l_\infty(\mathbb{N}, \mathbb{C}^{n\times n})$;

  $(B_i)_{i\in\mathbb{N}} \in l_p(\mathbb{N}, \mathbb{C}^{n\times m})$;
- Control: $u(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^+_0, \mathbb{C}^m)$;

Infinite Linear Ensembles – the countable/ continuum case

Countable Case:

Linear Ensemble

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t)\,, \quad x_i(0) \in \mathbb{C}^n\,, \quad i \in \mathbb{N} \qquad (\Sigma_{\mathrm{E}}^{\infty})$$

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  $(B_i)_{i\in\mathbb{N}} \in l_p(\mathbb{N}, \mathbb{C}^{n\times m})$;
- Control: $u(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{C}^m)$;

Remark: Real versus complex!

Infinite Linear Ensembles – the countable/ continuum case

Continuum Case:

Linear Ensemble

$$\frac{\partial x}{\partial t}(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t)\,, \quad x(0,\theta) = x_0(\theta) \in \mathbb{C}^n\,, \quad \theta \in P \qquad (\Sigma_{\mathrm{E}}^{c})$$

Again choose our favorite function space $X \subset \mathcal{F}(P, \mathbb{C}^n)$, e.g.:

- Possible state spaces: $X = C(P, \mathbb{C}^n)$ or $X = L^q(P, \mathbb{C}^n)$ with $1 \leq q < \infty$;
- Ensemble matrices:

  $A(\cdot) \in C(P, \mathbb{C}^{n\times n})$;

  $B(\cdot) = \begin{pmatrix} b_1(\cdot) & \cdots & b_m(\cdot) \end{pmatrix}$ with $b_i(\theta) \in C(P, \mathbb{C}^n)$ or $b_i(\cdot) \in L^q(P, \mathbb{C}^n)$;
- Control: $u(\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+, \mathbb{C}^m)$;

Unified notation: $x(t,i) := x_i(t)$ for $i \in \mathbb{N}$.

"The" ensemble control problem

Given a pair of initial and final states $x_0(\cdot), x_*(\cdot) \in X$.

$$\frac{\partial x}{\partial t}(t,\theta) = A(\theta)x(t,\theta) + B(\theta)u(t)\,, \quad \theta \in P \qquad (\Sigma_{\mathrm{E}})$$

Does there exist a **parameter-independent control** $u(t)$ which steers $x_0(\cdot)$ in some finite time $T \geq 0$ (approximately) to $x_*(\cdot)$?

More precisely: Given any $x_0(\cdot), x_*(\cdot) \in X$. Does there exist for all $\varepsilon > 0$ a time $T \geq 0$ and a control $u \in L^1([0,T],\mathbb{C}^m)$ such that

$$\left\| x(T,x_0,u) - x_* \right\|_X \leq \varepsilon\,?$$

Ensembles as infinite-dimensional linear systems

- State space X, e.g. $X = C(P, \mathbb{C}^n)$ or $X = L^q(P, \mathbb{C}^n)$ or $X = l_q(P, \mathbb{C}^n)$
- System operator ($=$ multiplication operator)

$$\mathcal{A} \colon X \to X, \quad (\mathcal{A}x)(\theta) = A(\theta)x(\theta)$$

- Input operator ($=$ finite rank operator)

$$\mathcal{B} \colon \mathbb{C}^m \to X, \quad (\mathcal{B}u)(\theta) = B(\theta)u$$

Resulting infinite-dimensional linear system

$$\dot{x} = \mathcal{A}x + \mathcal{B}u \qquad (\Sigma_X)$$

General assumption

Let X be a Banach space and $\mathcal{A}$ be a bounded operator.

Infinite Linear Ensembles – the countable/ continuum case

First observations I:

Lemma B (Triggiani 75)

The following assertions are equivalent:

- $\Sigma_{\mathrm{E}} = \big(A(\theta), B(\theta)\big)_{\theta\in P}$ is ensemble controllable (with respect to X);
- $\Sigma_X = \big(\mathcal{A}, \mathcal{B}\big)$ is approximately controllable;
- For every $T \geq 0$ the closure of the image of the reachability map

$$\mathcal{R}_T : u \mapsto \int_0^T e^{A(\cdot)(T-s)} B(\cdot) u(s) \mathrm{d}s$$

  is equal to X.
- The generalized Kalman condition $R(\mathcal{A}, \mathcal{B}) := \overline{\sum_{k=0}^{\infty} \mathrm{im} \mathcal{A}^k \mathcal{B}} = X$ holds;
- The approximation conditions $\overline{\{\sum_{i=1}^{m} p_i(\mathcal{A}) b_i \ : \ p_i \in \mathbb{C}[z]\}} = X$ holds;
- The the operator $\mathcal{A}$ is m-cyclic with cyclic vectors $b_1(\cdot), \dots, b_m(\cdot)$;

First observations II:

- Many standard results on approximate controllability for infinite-dimensional systems do not apply as the multiplication operator $\mathcal{A}$ has mostly continuous spectrum;
- Most infinite ensemble systems are not (exactly) controllable (Triggiani 75); therfore, only approximate notions of controllability are reasonable in general;

First observations II:

- Many standard results on approximate controllability for infinite-dimensional systems do not apply as the multiplication operator $\mathcal{A}$ has mostly continuous spectrum;
- Most infinite ensemble systems are not (exactly) controllable (Triggiani 75); therfore, only approximate notions of controllability are reasonable in general;

  Reason:

  $\mathcal{B}$ has finite-dimensional range and this results in general in a compact input-to-state operator;

A useful result for parallel connections of infinite-dimensional systems:

Theorem B (Schönlein, D. 2021)

Suppose the (possible ∞-dimensional) linear systems (A_1, B_1) and (A_2, B_2) satisfy the following conditions:

(a) (A_1, B_1) and (A_2, B_2) are approximately controllable;

(b) $\sigma(A_1)$ and $\sigma(A_2)$ have only finitely many connected components;

(c) $\sigma(A_1)$ and $\sigma(A_2)$ are non-separating (i.e. $\mathbb{C} \setminus \sigma(A_i)$ is connected);

(d) $\sigma(A_1) \cap \sigma(A_2) = \emptyset$;

Then the parallel connection $\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right)$ is approximately controllable.

Idea of Proof: ...

Some results for particular state spaces.

Case I: $X := C(P, \mathbb{C}^n)$

Lemma C

Suppose the ensemble $\big(A(\theta), B(\theta)\big)_{\theta \in P}$ is uniformly ensemble controllable. Then $\big(A(\theta), B(\theta)\big)_{\theta \in K}$ is also uniformly ensemble controllable on any compact subset of $K \subset P$.

Proof: Use Tietze's Extension Theorem

Uniform Ensemble Control, i.e. $X := C(P, \mathbb{C}^n)$

Corollary A (Helmke, Schönlein, D. 2014/2021)

Let $P \subset \mathbb{R}^d$ and suppose the single-input ensemble $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ is uniformly ensemble controllable. Then

(N1) For every $\theta \in P$ the linear system $\big(A(\theta), B(\theta)\big)$ is controllable.

(N2) For every $\theta \in P$ the eigenvalues of $A(\theta)$ have geometric multiplicity one.

(N3) The spectral map is one-to-one, i.e. $\sigma(A(\theta_1)) \cap \sigma(A(\theta_2)) = \emptyset$.

(N4) For $d \geq 2$ the set P has no interior points.

Proof:

- (N1) – (N3) follow straightforward from Lemma A and C;
- to show (N4) reduce problem to the particular case $P = \partial D$;

Uniform Ensemble Control, i.e. $X := C(P, \mathbb{C}^n)$

Lemma D (Helmke, Schönlein, D. 2014/2021)

Let $P \subset \mathbb{C}$ be a compact and contractible set with empty interior. Then the following assertions are equivalent:

(a) $\big(a(\theta), b(\theta)\big)_{\theta \in P}$ is uniformly ensemble controllable;

(b) $a : P \to \mathbb{C}$ is one-to-one and $b(\theta) \neq 0$ for all $\theta \in P$;

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Let $P \subset \mathbb{C}$ be a compact and contractible set with empty interior. Then the following assertions are equivalent:

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Proof:

- (a) $\Longrightarrow$ (b): see Corollary A;
- (b) $\Longrightarrow$ (a): For simplicity assume $a : [\theta_1, \theta_2] \to \mathbb{R}$ and w.l.lo.g. $b \equiv 1$;
- Then the approximation condition boils down to

$$\overline{\{p\big(a(\cdot)\big) \; : \; p \in \mathbb{C}[z]\}} = C([\theta_1, \theta_2], \mathbb{C}) \qquad (\star)$$

  and, since the map $a : [\theta_1, \theta_2] \to \mathbb{R}$ is one-to-one, $(\star)$ is equivalent to

$$\overline{\{p(\cdot) \; : \; p \in \mathbb{C}[z]\}} = C(a([\theta_1, \theta_2]), \mathbb{C})$$

- The above approximation problem can be solved by the Weierstraß Approximation Theorem and in the complex case by Mergelyan's Theorem.

Uniform Ensemble Control, i.e. $X := C(P, \mathbb{C}^n)$

The Magic Result (Helmke, Scherlein, Schönlein 2014/2016)

Let $P \subset \mathbb{C}$ be a compact and contractible and let $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ satisfy the necessary conditions (N1) – (N4) as well as the magic condition (MC), i.e. the characteristic polynomials $\chi(z,\theta)$ are of the form

$$\chi(z,\theta) = z^n - a_{n-1} z^{n-1} - \cdots - a_1 z - a_0(\theta) \tag{MC}$$

for some $a_1, ..., a_{n-1} \in \mathbb{C}$ and some $a_0 \in C(P, \mathbb{C})$. Then $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ is uniformly ensemble controllable.

Remark: Lemma D is obviously a special case of the “magic condition”.

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Proof:

- (a) Use the $T(\theta) = \begin{pmatrix} b(\theta) & \ldots & A^{n-1}(\theta) b(\theta) \end{pmatrix}$ to obtain the canonical from

$$A(\theta) \sim \begin{pmatrix} 0 & & & a_0(\theta) \\ 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{n-1} \end{pmatrix}, \quad b(\theta) \sim e_1 \,.$$

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- Simply start computing $A^k(\theta) b(\theta)$. – Think mathematically – act computationally!

Uniform Ensemble Control, i.e. $X := C(P, \mathbb{C}^n)$

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- Simply start computing $A^k(\theta)b(\theta)$.
- Finally, again Weierstraß / Mergelyan does the job.

Uniform Ensemble Control, i.e. $X := C(P, \mathbb{C}^n)$

Glueing Result (Schönlein, D. 2014/2021)

Let $P \subset \mathbb{C}$ be a compact and contractible and let $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ satisfy the necessary conditions (N1) – (N4). If the following conditions are additionally satisfied then $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ is uniformly ensemble controllable.

(a) $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ satisfies a technical spectral condition;

(b) The corresponding subsystems satisfy the magic condition;

Proof:

- Use the spectral condition to decompose $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ into subsystems

$$A(\theta) \sim \begin{pmatrix} A_1(\theta) & & \\ & \ddots & \\ & & A_r(\theta) \end{pmatrix}, \quad b(\theta) \sim \begin{pmatrix} b_1(\theta) \\ \vdots \\ b_r(\theta) \end{pmatrix}.$$

- Apply the magic result and "glue" things together via Theorem B.

L^q-Ensemble Control

Case II: $X := L^q(P, \mathbb{C}^n)$ with respect to some regular (Borel) measure μ

Corollary B (Schönlein, D. 2021)

Let $P \subset \mathbb{C}$ compact and suppose the single-input ensemble $\big(A(\theta), b(\theta)\big)_{\theta \in P}$ is L^q-ensemble controllable. Then

(N1) For almost all $\theta \in P$ the linear system $\big(A(\theta), B(\theta)\big)$ is controllable.

(N2) For almost all $\theta \in P$ the eigenvalues of $A(\theta)$ have geometric multiplicity one.

(N3) Every L^∞-eigenvalue selection of $A(\cdot)$ is essentially one-to-one.

Proof: similar to Corollary A

Remark: So far interior points are not excluded!

L^q-Ensemble Control

Lemma E (Schönlein, D. 2021)

Let $P \subset \mathbb{C}$ be a compact and $q \in [1, \infty)$. Then the following assertions are equivalent:

(a) $\big(a(\theta), b(\theta)\big)_{\theta \in P}$ is L^p ensemble controllable;

(b) $a : P \to \mathbb{C}$ is essentially one-to-one and $b(\theta) \neq 0$ for all almost all $\theta \in P$ and

$$\inf_{p \in \mathbb{C}[z]} \int_P \|p(a)b - \bar{a}b\|^q \mathrm{d}\mu = 0 \,.$$

A few remarks concerning the proof:

- (a) $\implies$ (b): use Corollary B, the fact that $\bar{a}b \in L^q(P, \mathbb{C})$ and the result that the multiplication operator induced by $a(\cdot)$ is cyclic if and only if $a(\cdot)$ is essentially one-to-one.
- (b) $\implies$ (a): ...

L^q-Ensemble Control

No-Go Theorem (Chen 2021)

Let $P \subset \mathbb{R}^d$, $d \geq 2$ be compact with non-empty interior and let μ be the d-dimensional Lebesgue-measure on P. If the ensemble $\big(A(\theta), B(\theta)\big)_{\theta \in P}$ is real analytic at some interior point of P then it is never L^q-controllable for $q \geq 2$.

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Corollary

For $d \geq 2$ and $q \geq 2$ cyclic vectors of the multiplication operator induced by $A(\cdot)$ are nowhere real analytic (in the interior of P).

A few remarks concerning the proof:

- Transform $A(\theta)$ locally to a block-triangular structure such that the problem can be reduced to the scalar case $P \subset \mathbb{C} = \mathbb{R}^2$ and $a : P \to \mathbb{C}$;
- A further reduction yields $a(\theta) = \theta$;
- Consider w.l.o.g. $P = \overline{\mathbb{D}}$ and assume that $B(\theta)$ is holomorphic; then the closure of $b_1(\theta)\theta^k, \ldots, b_m(\theta)\theta^k$ is contained in the Hardy $H^2(\mathbb{D})$ and thus not equal to $L^2(\mathbb{D})$;

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- The tricky part results from the assumption that the $b_i(\theta)$ are only real analytic;

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Finite Bilinear Ensembles

A few words about general bilinear systems

$$\dot{x} = (A + u(t)B)x, \qquad x(0) \in \mathbb{R}^n \tag{IS}$$

$$\dot{X} = (A + u(t)B)X, \qquad X(0) \in G \subset \mathrm{GL}_n(\mathbb{C}) \tag{L}$$

System Lie algebra: real Lie algebra $\langle A, B \rangle_{LA}$ generated by A and B

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Accessibility & Controllability (Brockett, Sussmann, Jurdjevic, ...)

Let G be a path-connected subgroup of $\mathrm{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g} \subset \mathbb{C}^{n\times n}$ and let $A, B \in \mathfrak{g}$. Then one has

(a) (L) is accessible (relative to G) $\iff$ $\langle A, B \rangle_{LA} = \mathfrak{g}$ (LARC)

(b) If G is additionally compact or e^{tA} is (almost) periodic, then

$$\text{(L) is controllable (relative to } G) \iff \langle A, B \rangle_{LA} = \mathfrak{g}$$

Toy Example

Consider the following two systems

$$\dot{x}_1 = u(t) b_1 x_1 \,, \qquad x_1 \in \mathbb{R}^+ \,, \quad u(t) \in \mathbb{R} \,, \qquad (\Sigma_1)$$

$$\dot{x}_2 = u(t) b_2 x_2 \,, \qquad x_2 \in \mathbb{R}^+ \,, \quad u(t) \in \mathbb{R} \,. \qquad (\Sigma_2)$$

Both evolve on the Lie group $\mathbb{R}^+$ and, for $b_1 \neq 0$ and $b_2 \neq 0$, both systems are controllable.

Toy Example

Consider the following two systems

$$\begin{aligned} \dot{x}_1 &= u(t) b_1 x_1 \,, \qquad x_1 \in \mathbb{R}^+ \,, \quad u(t) \in \mathbb{R} \,, & (\Sigma_1) \\ \dot{x}_2 &= u(t) b_2 x_2 \,, \qquad x_2 \in \mathbb{R}^+ \,, \quad u(t) \in \mathbb{R} \,. & (\Sigma_2) \end{aligned}$$

Both evolve on the Lie group $\mathbb{R}^+$ and, for $b_1 \neq 0$ and $b_2 \neq 0$, both systems are controllable.

However, the "parallel connection" given by

$$\begin{bmatrix} \dot{x}_1 & 0 \\ 0 & \dot{x}_2 \end{bmatrix} = u(t) \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \qquad u(t) \in \mathbb{R} \qquad (\Sigma_{||})$$

is not controllable on

$$\left\{ \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} \; : \; x_1, x_2 \in \mathbb{R}^+ \right\} \cong \mathbb{R}^+ \times \mathbb{R}^+$$

Finite Bilinear Ensembles

Finite bilinear ensembles – general setting

Given a finite parameter set $P := \{1, 2, \ldots, N\}$ and finitely many bilinear systems

$$\dot{X}_i = \Big(A_i + \sum_{k=1}^{m} u_k(t) B_{i,k}\Big) X_i\,, \quad \big(u_1(t), \ldots, u_m(t)\big) \in \mathbb{R}^m\,, \quad i \in P\,. \qquad (\Sigma_i)$$

defined on Lie groups $G_i \subset \mathrm{GL}_n(\mathbb{C})$.

Note: $u_k(t)$ is independent of $i \in P$

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defined on Lie groups $G_i \subset \mathrm{GL}_n(\mathbb{C})$.

Note: $u_k(t)$ is independent of $i \in P$

Key problem

What can be said about the controllability of the ensemble $(\Sigma_i)_{i \in P}$?

For simplicity from now on: $m \leq 2$

Finite Bilinear Ensembles

The state space of the ensemble is canonically given by the direct product

$$\mathbf{G} := G_1 \times \cdots \times G_N$$

which, for convenience, will be embedded in $\mathrm{GL}_{\bar{n}}(\mathbb{C})$ as follows:

$$\mathbf{G} \cong \left\{ \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_s \end{bmatrix} \;:\; X_i \in G_i \right\}$$

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Hence

$$\begin{bmatrix} \dot{X}_1 & & 0 \\ & \ddots & \\ 0 & & \dot{X}_s \end{bmatrix} = \left(\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{bmatrix} + u(t) \begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_s \end{bmatrix} \right) \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_s \end{bmatrix} \qquad (\Sigma_{\mathrm{E}})$$

Block structure is preserved!

Finite Bilinear Ensembles

Definition

(a) The ensemble $(\Sigma_i)_{i \in P}$ is called *simultaneously accessible* if Σ_{E} is accessible on G.

(b) The ensemble $(\Sigma_i)_{i \in P}$ is called *ensemble controllable* if Σ_{E} is controllable on G.

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Key notion:

Definition

Given $A, B \in \mathfrak{g}$ and $A', B' \in \mathfrak{g}'$, where $\mathfrak{g}$ and $\mathfrak{g}'$ are arbitrary Lie algebras. We call the pairs (A, B) and (A', B') Lie-related, if there exists a Lie algebra isomorphism $\tau : \mathfrak{g} \to \mathfrak{g}'$ such that

$$A' = \tau(A) \quad \text{and} \quad B' = \tau(B)$$

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The standard Lie algebra isomorphism/automorphism are:

$$A \mapsto TAT^{-1} \quad \text{(inner automorphism)} \quad \text{and} \quad A \mapsto -A^{\top}$$

Finite Bilinear Ensembles

A general result for semisimple Lie groups:

Theorem (D. 2012, Turinici 2014)

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ be a semisimple (matrix) Lie algebra with simple ideals $\mathfrak{g}_i$ and let G be the corresponding connected (matrix) Lie group. Then the following statements are equivalent:

(a)

$$\dot{X} = \big(A + u(t)B\big)X, \qquad u(t) \in \mathbb{R}\,. \tag{Σ}$$

is accessible on G.

1. *For all $i \in \{1, \dots, N\}$ one has $\langle A_i, B_i \rangle_L = \mathfrak{g}_i$ and for all $i \neq j$ the pairs (A_i, B_i) and (A_j, B_j) are Lie-unrelated.*

Here, A_i and B_i denote the i-th component of A and B with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$.

Finite Bilinear Ensembles

A few comments:

-
$$\text{semisimple} = \text{direct sum of simple Lie algebras}$$
$$\text{simple} = \text{no non-trivial ideals}$$

- Examples of simple Lie algebras: $\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{su}_n$, ...
- Given simple Lie algebras $\mathfrak{g}_i \subset \mathfrak{gl}_{n_i}(\mathbb{C})$, $i = 1, \ldots, N$. Then

$$\mathfrak{g} := \left\{ \begin{bmatrix} X_1 & & 0 \\ & \ddots & \\ 0 & & X_s \end{bmatrix} \quad X_i \in \mathfrak{g}_i, \right\}$$

  constitutes a semisimple Lie subalgebra of $\mathfrak{gl}_{\bar{n}}(\mathbb{C})$ with $\bar{n} := n_1 + \cdots + n_s$.
- Not every semisimple Lie algebra is of the above “block form”, for instance $\mathfrak{so}_4 \cong \mathfrak{so}_3 \oplus \mathfrak{so}_3$.
- If G is compact then accessibility can be replaced by controllability.

Finite Bilinear Ensembles

Application to bilinear ensembles:

Corollary

Let $\mathfrak{g}_i$ be simple (matrix) Lie algebras and let $G_i \subset GL_{n_i}(\mathbb{C})$ be the respective Lie subgroup. Moreover, let $A_i, B_i \in \mathfrak{g}_i$ for $i = 1, \ldots, s$. Then the following statements are equivalent:

(a) The bilinear ensemble

$$\dot{X}_i = (A_i + u(t)B_i)X_i, \quad u(t) \in \mathbb{R}, \quad i = 1, \ldots N \qquad (\Sigma_i)$$

is simultaneously accessible (ensemble controllable in the compact case).

(b) For all $i = 1, \ldots, N$ one has $\langle A_i, B_i \rangle_L = \mathfrak{g}_i$ and for all $i \neq j$ the pairs (A_i, B_i) and (A_j, B_j) are Lie-unrelated.

Proof: Apply the previous result to the Lie algebra $\mathfrak{g} := \underbrace{\mathfrak{g}_0 \times \cdots \times \mathfrak{g}_0}_{s\text{-times}}$.

Sketch of the proof of the Theorem

Proof: For simplicity assume $N = 2$ and $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \;:\; X_i \in \mathfrak{g}_i, i = 1, 2 \right\}$.

"$\Longrightarrow$": Assume that $\langle A_1, B_1 \rangle_L =: \mathfrak{s}_1 \neq \mathfrak{g}_1$. Then

$$\left\langle \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right\rangle_L \subset \mathfrak{s}_1 \oplus \mathfrak{g}_2 \neq \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Next, assume (A_1, B_1) and (A_2, B_2) are Lie-related, i.e. there exists a Lie isomorphism $\tau : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$A_2 = \tau(A_1) \quad \text{and} \quad B_2 = \tau(B_1)$$

Clearly, this implies

$$\left\langle \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right\rangle_L = \left\{ \begin{bmatrix} X & 0 \\ 0 & \tau(X) \end{bmatrix} \;\middle|\; X \in \mathfrak{g}_1 \right\} \subsetneq \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Hence the LARC fails in both cases and thus accessibility does not hold.

Sketch of the Proof of the Theorem

Proof: "$\Longleftarrow$": To prove this direction, we need the following result:

Lemma

Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be simple and assume $\langle A_2, B_2 \rangle_L = \mathfrak{g}_2$. If the Lie algebra $\mathfrak{s}$ generated by $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ is a graph over $\mathfrak{g}_1$, i.e.

$$\mathfrak{s} = \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & \Phi(X_1) \end{bmatrix} \;\middle|\; X_1 \in \mathfrak{g}_1 \right\}$$

for some map $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$, then $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra isomorphism.

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Proof of the lemma:

$\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ has to be onto due to the assumption $\langle A_2, B_2 \rangle_L = \mathfrak{g}_2$

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Proof of the lemma:

$\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ has to be onto due to the assumption $\langle A_2, B_2 \rangle_L = \mathfrak{g}_2$

The kernel of $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an ideal of $\mathfrak{g}_1$, hence $\ker \Phi = \{0\}$ or $\ker \Phi = \mathfrak{g}_1$.

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Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be simple and assume $\langle A_2, B_2 \rangle_L = \mathfrak{g}_2$. If the Lie algebra $\mathfrak{s}$ generated by $\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ and $\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$ is a graph over $\mathfrak{g}_1$, i.e.

$$\mathfrak{s} = \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & \Phi(X_1) \end{bmatrix} \;\middle|\; X_1 \in \mathfrak{g}_1 \right\}$$

for some map $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$, then $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra isomorphism.

Proof of the lemma:

$\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ has to be onto due to the assumption $\langle A_2, B_2 \rangle_L = \mathfrak{g}_2$

The kernel of $\Phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is an ideal of $\mathfrak{g}_1$, hence $\ker \Phi = \{0\}$ or $\ker \Phi = \mathfrak{g}_1$.

Since $\mathfrak{g}_2 \neq \{0\}$, we conclude $\ker \Phi = \{0\}$ and hence Φ yields an isomorphism.

Sketch of the Proof of the Theorem

Proof: Now back to the proof of "$\Longleftarrow$". Assume that the system is not accessible. Then the LARC implies

$$\mathfrak{s} := \left\langle \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \right\rangle_L \neq \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Consider the canonical projections

$$\begin{aligned} \pi_1 &: \quad \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_1, \quad \pi_1\left(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right) = X_1 \\ \pi_2 &: \quad \mathfrak{g}_1 \oplus \mathfrak{g}_2 \to \mathfrak{g}_2, \quad \pi_2\left(\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \right) = X_2. \end{aligned}$$

It is easy to see that π_1 and π_2 are Lie algebra homomorphisms. Moreover, by assumption $\pi_1|_{\mathfrak{s}}$ and $\pi_2|_{\mathfrak{s}}$ are onto.

Simplicity of $\mathfrak{g}_2$ then guarantees that the kernel of $\pi_1|_{\mathfrak{s}}$ is either $\{0\}$ or $\mathfrak{g}_2$; the later case can be excluded by the assumption $\mathfrak{s} \neq \mathfrak{g}_1 \oplus \mathfrak{g}_2$

Hence, $\mathfrak{s}$ is a graph over $\mathfrak{g}_1$ and the result follows by the previous lemma.

Infinite Bilinear Ensembles – the countable/continuum case

Given A parameter dependent family of bilinear systems (= bilinear ensemble)

$$\frac{\partial X}{\partial t}(t,\theta) = \big(A(\theta) + \sum_{k=1}^{m} u_k(t) B_k(\theta)\big) X(\theta)\,, \quad u(t) \in \mathbb{R}^m\,, \quad \theta \in P \qquad (\Sigma_{\mathrm{E}})$$

defined on a common Lie group $G \subset \mathrm{GL}_n(\mathbb{C})$ with parameter set P.

Note: $u_k(t)$ is independent of $\theta \in P$

Possible parameter sets: $P := \mathbb{N}$ or $P \subset \mathbb{R}^d$ compact

Key problems:

What's the "right" state space for the "ensemble"?

What can be said about the controllability of the "ensemble"?

“Nice” state spaces in the countable case $P := \mathbb{N}$

First approach: $\mathbf{G} = G^{\mathbb{N}}$ and $\mathfrak{g} = \mathfrak{g}^{\mathbb{N}}$

Problem: Does there exist a suitable Lie group structure for $G^{\mathbb{N}}$?

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Problem: Does there exist a suitable Lie group structure for $G^{\mathbb{N}}$?
Answer: $G^{\mathbb{N}}$ constitutes a Frechet Lie group with Lie algebra $\mathfrak{g}^{\mathbb{N}}$, but ...

BETTER: Consider suitable subgroups/subalgebras of $G^{\mathbb{N}}$ and $\mathfrak{g}^{\mathbb{N}}$, which can be equipped with a Banach Lie group/algebra structure, e.g.

$$\ell_p(\mathfrak{g}) := \Big\{ (A_k)_{k\in\mathbb{N}} \; : \; \sum_{k=1}^{\infty} \|A_k\|^p < \infty \Big\} \subset p\text{-Schatten class operators}$$

acting on $\ell_2(\mathbb{R}^n)$, if $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$.

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So far almost no results available!

“Nice” state spaces in the continuum case $P \subset \mathbb{R}^d$

First approach: $\widehat{G} = G^{[0,1]}$ and $\widehat{\mathfrak{g}} = \mathfrak{g}^{[0,1]}$

Bad idea: $\mathfrak{g}^{[0,1]}$ is “only” a locally convex space

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First approach: $\widehat{G} = G^{[0,1]}$ and $\widehat{\mathfrak{g}} = \mathfrak{g}^{[0,1]}$

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BETTER: Consider again suitable subgroups/subalgebras of $G^{[0,1]}$ and $\mathfrak{g}^{[0,1]}$, which can be equipped with a Banach Lie group/algebra structure, e.g.

$$C(P, G) \quad \text{and} \quad C(P, \mathfrak{g})$$

acting on $C(P, \mathbb{R}^n)$ or $L^p(P, \mathbb{R}^n)$ as bounded multiplication operators, if $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$.

Infinite Bilinear Ensembles – the countable/continuum case

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acting on $C(P, \mathbb{R}^n)$ or $L^p(P, \mathbb{R}^n)$ as bounded multiplication operators, if $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$.

Here some results are available!

Infinite Bilinear Ensembles – the continuous case

Theorem (Bloch Equation) [Khaneja & Li 2009]

Let $P = [a, b]$ with $a > 0$ and let $\mathbf{G} := \mathrm{C}\big([a, b], SO(3)\big)$. Then the infinite ensemble

$$\frac{\partial X}{\partial t}(t, \theta) = \big(u_1(t)\theta\Omega_1 + u_2(t)\theta\Omega_1\big)X(t, \theta), \quad \big(u_1(t), u_2(t)\big) \in \mathbb{R}^2$$

is uniformly ensemble controllable on $\mathbf{G}$. Here, Ω_1 and Ω_1 denote the standard generators of rotations around the x- and y-axis, respectively, i.e.

$$\Omega_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \Omega_2 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Remark: A similar result has been proven by Beauchard, Coron, Rouchon 2010

Uniformly ensemble controllability: For all $X_0, X_* \in \mathbf{G}$ and all $\varepsilon > 0$ there exists a $T \geq 0$ and a control $u : [0, T] \to \mathbb{R}^2$ such that

$$\max_{\theta \in [a,b]} \|X(T, X_0, u)(\theta) - X_*(\theta)\| < \varepsilon.$$

Infinite Bilinear Ensembles – the continuous case

Sketch of the proof:

- Computing commutators between the control vector fields $\theta\Omega_1$ and $\theta\Omega_2$ yields:

$$[\theta\Omega_1, \theta\Omega_2] = \theta^2\Omega_3\,, \quad [\theta^2\Omega_3, \theta\Omega_1] = \pm\theta^3\Omega_2\,, \quad [\theta^2\Omega_3, \theta\Omega_2] = \pm\theta^3\Omega_1\,,$$

$$[\theta\Omega_1, \theta^3\Omega_2] = \theta^4\Omega_3\,, \quad [\theta^4\Omega_3, \theta\Omega_1] = \pm\theta^5\Omega_2\,, \quad \ldots$$

- Again Weierstraß shows that the closure of all these vector fields yields the entire Lie algebra and thus the closure of the reachable set coincides which $C\big([a,b], SO(3)\big)$.

Infinite Bilinear Ensembles – the continuous case

Theorem [Chen 2019]

Let $P \subset \mathbb{R}^d$ be compact and $G \subset \mathrm{GL}(\mathbb{C})$ be a semisimple (matrix) Lie Group with Lie algebra $\mathfrak{g}$. Then there exist Lie algebra elements $B_i \in \mathfrak{g}$ and function $\rho_j : P \to \mathbb{R}$ such that the bilinear ensemble

$$\frac{\partial X}{\partial t}(t,\theta) = \Big(A(\theta) + \sum_{i,j} u_{ij}(t)\rho_j(\theta)B_i\Big)X(t,\theta), \quad u_{ij} \in \mathbb{R}$$

is uniformly ensemble controllable.

Idea of the proof: Use the root space decomposition of $\mathfrak{g}$ and the Stone-Weierstraß Approximation Theorem.

Infinite Bilinear Ensembles – the continuous case

Theorem (D. 2018 unpublished)

Let $P = [a, b]$ and let $\mathbf{G} := \mathrm{C}\big([a, b], SU(n)\big)$. Then the ensemble

$$\frac{\partial X}{\partial t}(t, \theta) = \mathrm{i}\big(H_0(\theta) + u_1(t)H_1(\theta) + u_2(t)H_2(\theta)\big)X(t, \theta), \quad u_1(t), u_2(t) \in \mathbb{R}$$

is uniformly ensemble controllable on **G** if none of the off-diagonal entries of $H_2(\theta)$ vanishes and

$H_1(\theta) = \begin{pmatrix} \lambda_1(p) & & \\ & \ddots & \\ & & \lambda_n(p) \end{pmatrix}$ is strongly regular in the following sense:

- $\lambda_i(\theta) - \lambda_j(\theta) \neq \lambda_k(\theta) - \lambda_l(\theta)$ for all $\theta \in P$ and $(i, j) \neq (k, l)$ with $i \neq j$, $k \neq l$.
- $\lambda_i(\theta) - \lambda_j(\theta) \neq \lambda_k(\theta') - \lambda_l(\theta')$ for all $\theta, \theta' \in P$ with $\theta \neq \theta'$ and $i \neq j$, $k \neq l$.

Note: The above results covers the previous result by Khaneja & Li.

Proof:

- Consider the linear operator

$$\mathrm{ad}_{\mathrm{i}H_1(\theta)} : \mathrm{C}\big([a,b], \mathfrak{su}(n)\big) \to \mathrm{C}\big([a,b], \mathfrak{su}(n)\big)$$

  restricted to the subspace of all $\mathrm{i}H(\cdot)$ which vanish on the diagonal. Then $\mathrm{i}H_2(\cdot)$ is a cyclic vector of $\mathrm{i}\,\mathrm{ad}_{H_1(\cdot)}$ according to part I and the strong regularity assumption.

Infinite Bilinear Ensembles – the continuous case

Proof:

- Consider the linear operator

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  restricted to the subspace of all $\mathrm{i}H(\cdot)$ which vanish on the diagonal. Then $\mathrm{i}H_2(\cdot)$ is a cyclic vector of $\mathrm{i}\,\mathrm{ad}_{H_1(\cdot)}$ according to part I and the strong regularity assumption.
- Reconstruct the diagonal elements of $\mathrm{C}\big([a,b],\mathfrak{su}(n)\big)$ as “usual” by taking further commutators.
- This shows that the closure of the system algebra coincides with $\mathrm{C}\big([a,b],\mathfrak{su}(n)\big)$ and thus we conclude uniform ensemble controllability.

Remark:

- Note that we did not use any compactness or recurrence arguments.
- If we have only one control even accessibility is not guaranteed!

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Thanks a lot for your attention!