Free p-convex Banach lattices and non-linear maps between Banach spaces

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Overview

1. Banach lattices: reminders
2. Free Banach lattices
3. Lattice isomorphisms between free lattices
4. When do we have $\mathrm{FBL}^{(p)}[E] = \mathrm{FBL}^{(p)}[F]$?
   - $\mathrm{FBL}^{(p)}[E]$ versus $\mathrm{FBL}^{(p)}[F]$ for $p = \infty$
   - $\mathrm{FBL}^{(p)}[E]$ versus $\mathrm{FBL}^{(p)}[F]$ for $p \in [1, \infty)$

Definition of a Banach lattice

Definition

A (real) Banach lattice is:

1. a Banach space $(X, \|\cdot\|)$,
2. equipped with a partial order $\leqslant$, so that if $x \leqslant y$, then, for any $z \in X$ and $a \in [0, \infty)$, $x + z \leqslant y + z$ and $ax \leqslant ay$,
3. which is also a lattice: $x, y \in X$ have their max (least upper bound) $x \vee y$ and min (greatest lower bound) $x \wedge y$,
4. such that $\|\cdot\|$ is a lattice norm: $\|x\| \leqslant \|y\|$ if $|x| \leqslant |y|$. Here, $|x| = x \vee (-x)$.

Operations on Banach lattices

Definition

Suppose X, Y are Banach lattices. $T \in B(X, Y)$ is a lattice homomorphism if it preserves lattice operations: suffices to verify that $(Tx) \vee (Ty) = T(x \vee y)$, for any $x, y \in X$.
T is a lattice isomorphism (isometry) if it is invertible (surjective isometry), and both T and T^{-1} are lattice homomorphisms-.
T is a lattice embedding (isometric lattice embedding) if $T(X)$ is a sublattice of Y, and $T : X \to T(X)$ is a lattice isomorphism (resp. lattice isometry).
T is a lattice quotient if it is a lattice homomorphic quotient map.

Example of a lattice isomorphism: a weighted composition on $C(K)$. For $x \in C(K)$, and $t \in K$, $[Tx](t) = a(t)x(\phi(t))$, where $a \in C(K)$, $a > 0$, and $\phi : K \to K$ is a topological homeomorphism.

All lattice isomorphisms of $C(K)$ are of this form.

p-convexity

Definition

Suppose $1 \leqslant p \leqslant \infty$. A Banach lattice X is *p-convex with constant C* if $\left\| \left(\sum_i |x_i|^p\right)^{1/p} \right\| \leqslant C \left(\sum_i \|x_i\|^p \right)^{1/p}$ holds for any $x_1, \ldots, x_N \in X$.

• Any Banach lattice is 1-convex with constant 1 (triangle inequality).

• Any $C(K)$-space is ∞-convex, with constant 1.
If X is ∞-convex (with constant 1), then it is lattice isomorphic (lattice isometric) to a sublattice of $C(K)$.

• $L_q(\mu)$ is p-convex with constant 1 for $p \leqslant q$.
If $p > q$, and $\dim L_q(\mu) = \infty$, then $L_q(\mu)$ is not p-convex.

• If X is q-convex, then it is also p-convex for $p \leqslant q$.

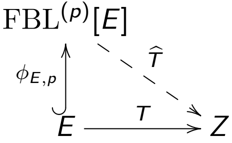
• If X is q-convex with constant C, then it can be renormed to be q-convex with constant 1.

Free p-convex Banach lattice over a Banach space

Definition

Suppose E is a Banach space, and $1 \leqslant p \leqslant \infty$. A free p-convex Banach lattice on E ($\mathrm{FBL}^{(p)}[E]$) is the unique Banach lattice X so that:

- X is p-convex with constant 1.
- There exists an isometry $\phi = \phi_{E,p} : E \to X$ so that $\phi_{E,p}(E)$ generates X as a Banach lattice.
- If Z is a Banach lattice, p-convex with constant 1, then any $T \in B(E, Z)$ extends to a lattice homomorphism $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to Z$ so that $\|\widehat{T}\| = \|T\|$, and $\widehat{T} \circ \phi_{E,p} = T$.



If Z is p-convex with constant C, we can construct an extension $\widehat{T}$ with $\|\widehat{T}\| \leqslant C\|T\|$ (renorming makes Z p-convex with constant 1).

Notation: $\mathrm{FBL}[E] := \mathrm{FBL}^{(1)}[E]$ (for $p = 1$, Z is an arbitrary lattice).

Functional representation of $\mathrm{FBL}^{(p)}[E]$

Denote by $\mathbf{H}[E^*]$ the space of positively homogeneous functions on E^* ($f(tx^*) = tf(x^*)\ \forall t \geqslant 0$). $\mathbf{H}_p[E^*]$ consists of those $f \in \mathbf{H}[E^*]$ for which $\exists C > 0$ s.t. $\sum_{i=1}^N |f(x_i^*)|^p \leqslant C^p$ when $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} \leqslant 1$. $\|f\|_p := \inf C$.

Here $\|(x_i^*)_{i=1}^N\|_{p,\mathrm{weak}} = \sup_{x\in\mathbf{B}(E)} \big(\sum_i |\langle x_i^*, x\rangle|^p\big)^{1/p}$.

Let $\phi_{E,p} : E \to \mathbf{H}[E^*] : x \mapsto \delta_x$; $\delta_x(x^*) = \langle x^*, x\rangle$.

Theorem

$\phi_{E,p} : E \to \mathbf{H}_p[E^*]$ is an isometry. $\mathrm{FBL}^{(p)}[E]$ is the Banach lattice generated by $\phi_{E,p}(E)$ in $\mathbf{H}_p[E^*]$.

Note: All functions from $\mathrm{FBL}^{(p)}[E]$ are weak* continuous on $\mathbf{B}(E^*) := \{e^* \in E^* : \|e^*\| \leqslant 1\}$.

Why does $\mathrm{FBL}^{(p)}[E]$ look the way it does?

Recall: $\mathbf{B}(E^*) = \{e^* \in E^* : \|e^*\| \leqslant 1\}$. $C(\mathbf{B}(E^*))$ contains E, is "large enough" for many purposes. $\mathrm{FBL}^{(p)}[E]$ is "something like" $C(\mathbf{B}(E^*))$.

Any p-convex lattice Z embeds into $(\oplus_i L_p(\mu_i))_\infty$, so we need to show that any $T : E \to L_p(\mu)$ extends to $\widehat{T} : \mathrm{FBL}^{(p)}[E] \to L_p(\mu)$, with $\|\widehat{T}\| = \|T\|$. Discretize, and replace $L_p(\mu)$ by ℓ_p^n.

$T = (e_i^*)_{i=1}^n$, $e_i^* \in E^*$, $1 = \|T\| = \|(e_i^*)\|_{p,\mathrm{weak}}$. Each $e_i^* : E \to \mathbb{R}$ extends to a lattice homomorphism $\widehat{e_i^*} : \mathrm{FBL}^{(p)}[E] \to \mathbb{R}$ (point evaluation at e_i^*). Then $\widehat{T} = (\widehat{e_i^*})_{i=1}^n$. For $f \in \mathrm{FBL}^{(p)}[E]$,

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} \geqslant \|\widehat{T}f\| = \|(f(e_i^*))\|_{\ell_p^n} = \Big(\sum_i |f(e_i^*)|^p\Big)^{1/p}.$$

Take sup over $\|(e_i^*)\|_{p,\mathrm{weak}} \leqslant 1$:

$$\|f\|_{\mathrm{FBL}^{(p)}[E]} := \sup\Big\{\Big(\sum_i |f(e_i^*)|^p\Big)^{1/p} : \|(e_i^*)\|_{p,\mathrm{weak}} \leqslant 1\Big\}.$$

Examples of free lattices

If $E = \mathbb{R}$ $(\dim E = 1)$, then $\mathrm{FBL}^{(p)}[E] = C(\{1, -1\}) = \ell_\infty^2$. $\phi(1) = \delta_1 = (1, -1)$. For a Banach lattice Z and $T : E \to Z$, write $T1 = z_+ - z_-$, where $z_+ = (T1)_+$ and $z_- = (T1)_-$ (disjoint). Then $\widehat{T}(a, b) = az_+ + bz_-$ $(\widehat{T}(1, 0) = z_+,\ \widehat{T}(0, 1) = z_-)$.

Theorem

$\mathrm{FBL}^{(\infty)}[E]$ coincides with the space $\mathbf{CH}[E^*]$ of weak* continuous positively homogeneous functions on $\mathbf{B}(E^*)$, with sup *norm $\| \cdot \|_\infty$.*

For $p \in [1, \infty)$, $\mathrm{FBL}^{(p)}[E]$ need not be an ideal in "the weak* continuous part of" $\mathbf{H}_p[E^*]$.

Theorem (Avilés, Rodríguez, Tradacete)

For $1 \leqslant p < \infty$ $\exists$ weak* continuous $0 \leqslant g \leqslant f$ s.t. $g, f \in \mathbf{H}_p[\ell_1^*] \backslash \{0\}$ s.t. $f \in |\mathrm{ran}\, \phi_{\ell_1, p}| \subset \mathrm{FBL}^{(p)}[\ell_1]$, $g \notin \mathrm{FBL}^{(p)}[\ell_1]$.

Structure of lattice homomorphisms between free lattices

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$*;* $\|\Phi_T y^*\| \leqslant \|T\|\|y^*\|$*. Moreover:*

- *Any positively homogeneous* $\Phi : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, determines* $T : \mathrm{FBL}^{(\infty)}[E] \to \mathrm{FBL}^{(\infty)}[F]$ *s.t.* $\Phi_T = \Phi$, $\|T\| = \sup_{\|y^*\|\leqslant 1} \|\Phi_T y^*\|$.
- *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\|\|(y_i^*)\|_{q,\mathrm{weak}}$.
- *If* T *is a lattice isomorphism, then* $\Phi_{T^{-1}} = \Phi_T^{-1}$.

Recall: $\|(y_i^*)_{i=1}^N\|_{q,\mathrm{weak}} = \sup_{y\in\mathbf{B}(F)} \left(\sum_i |\langle y_i^*, y\rangle|^q\right)^{1/q}$.

Remark. (1) $\forall y^* \|\Phi_T y^*\| \leqslant \|T\|\|y^*\|$. (2) If T is a lattice quotient, then T^* is bounded below, hence $\exists c > 0$ s.t. $\forall y^* \|\Phi_T y^*\| \geqslant c\|y^*\|$.

Lattice homomorphisms: a construction

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$; $\|\Phi_T y^*\| \leqslant \|T\| \|y^*\|$. *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\| \|(y_i^*)\|_{q,\mathrm{weak}}$.

Construction of Φ_T. $\mathrm{FBL}^{(p)}[F]^* \supset \{\widehat{y^*} : y^* \in F^*\}$ = set of atoms ($\widehat{y^*}$ = evaluation at y^*). T^* is interval preserving, so $T^* \widehat{y^*}$ is an atom, say $\widehat{\Phi_T y^*}$.

$\|(y_i^*)_{i=1}^N\|_{q,\mathbf{weak}} = \|(\widehat{y_i^*})_{i=1}^N\|_{q,\mathbf{weak}}$.

Indeed, consider $T : F \to \ell_q^N : f \mapsto (\langle y_i^*, f \rangle)_{i=1}^n$. Extension:

$\widehat{T} : \mathrm{FBL}^{(p)}[F] \to \ell_q^N : \varphi \mapsto (\varphi(y_i^*))_{i=1}^n = (\widehat{y_i^*}(\varphi))_{i=1}^n$.

$\|(y_i^*)_{i=1}^N\|_{q,\mathrm{weak}} = \|T\| = \|\widehat{T}\| = \|(\widehat{y_i^*})_{i=1}^N\|_{q,\mathrm{weak}}$.

$\|(\widehat{T^* y_i^*})_{i=1}^N\|_{q,\mathrm{weak}} \leqslant \|T\| \|(\widehat{y_i^*})_{i=1}^N\|_{q,\mathrm{weak}}$. ∎

When do we have $\mathrm{FBL}^{(p)}[E] = \mathrm{FBL}^{(p)}[F]$?

$$\begin{array}{ccc} \mathrm{FBL}^{(p)}[F] & \overset{\overline{T}}{\dashrightarrow} & \mathrm{FBL}^{(p)}[E] \\ {\scriptstyle \phi_{F,p}} \big\uparrow & & \big\uparrow {\scriptstyle \phi_{E,p}} \\ F & \xrightarrow{\quad T \quad} & E \end{array}$$

$\overline{T}$ is the canonical lattice homomorphic extension of T.
If T is isomorphic (isometric), then so is $\overline{T}$.

Can $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ be "similar," while E and F are "different?"

The answer for $p = \infty$ differs from what we get for $p \in [1, \infty)$.

The isometric and isomorphic settings are different.

Case of $p = \infty$: structure of lattice homomorphisms

Theorem

$\mathrm{FBL}^{(\infty)}[E]$ *coincides with the space* $\mathbf{CH}[E^*]$ *of weak** *continuous positively homogeneous functions on* $\mathbf{B}(E^*)$*, with* sup *norm* $\|\cdot\|_\infty$.

Corollary (Lattice homomorphism vs compositions)

(1) *If* $\Phi : F^* \to E^*$ *is a positively homogeneous and weak** *continuous on bounded sets, then* $\exists$ *a lattice homomorphism* $T : \mathrm{FBL}^{(\infty)}[E] \to \mathrm{FBL}^{(\infty)}[F]$ *s.t.* $\Phi = \Phi_T$ *– that is,* $Tf = f \circ \Phi$.

(2) *If, moreover,* Φ *is surjective, and* $\forall y^* \in F^*$ $C_1\|y^*\| \geqslant \|\Phi y^*\| \geqslant \|y^*\|/C_2$,
then T *is a lattice isomorphism, with* $\|T\| \leqslant C_1$ *and* $\|T^{-1}\| \leqslant C_2$.

Comparing $\mathrm{FBL}^{(\infty)}[E]$ with $\mathrm{FBL}^{(\infty)}[F]$, and FDDs

A Banach space E has Finite Dimensional Decomposition (FDD) if $\exists$ a sequence of finite rank projections $P_n \in B(E)$ s.t. (i) $P_i P_j = P_{\min\{i,j\}}$, and (ii) $\forall x \in E$, $\lim_N P_N x = x$ (then $\sup_N \|P_N\| < \infty$).

Example: a Schauder basis (e_i) gives rise to an FDD (P_i = canonical basis projection).

An FDD (P_i) is monotone if $\sup_N \|P_N\| = 1$.
Example: canonical basis in ℓ_p.

Theorem

If E is a Banach space with a monotone FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isometric to $\mathrm{FBL}^{(\infty)}[c_0]$.

$\mathrm{FBL}^{(\infty)}[E] = \mathrm{FBL}^{(\infty)}[c_0]$, construction in particular case

Suppose E has monotone normalized basis (e_i), and (e_i^*) is strictly monotone: $\| \sum_{i=1}^{N} \alpha_i e_i^* \| \leqslant \| \sum_{i=1}^{N+1} \alpha_i e_i^* \|$, with equality iff $\alpha_{N+1} = 0$.

Goal: construct positively homogeneous norm-preserving surjection $\Phi : E^* \to \ell_1 = c_0^*$ s.t. Φ and Φ^{-1} are weak* continuous on bounded sets.

Let f_i^* be the canonical basis of ℓ_1. Let P_n be the n-th basis projection in E, then $P_n^* : E^* \to X_n = \mathrm{span}[e_i^* : 1 \leqslant i \leqslant n]$ canonically.

Construct maps $\Phi_n : X_n \to Y_n = \mathrm{span}[f_i^* : 1 \leqslant i \leqslant n] \subset \ell_1$.
Let $\Phi_1 \alpha_1 e_1^* := \alpha_1 f_1^*$.
Define Φ_n recursively: for $e^* = x^* + \alpha e_n^*$ ($x^* \in X_{n-1}$), let
$\Phi_n e^* := \Phi_{n-1} x^* + t f_n^*$, where $t = \mathrm{sign}\alpha \cdot \big(\|e^*\| - \|x^*\|\big)$.

The $\Phi = \mathrm{weak}^* - \lim \Phi_n P_n^*$ has the desired properties. ■

Question. Can we avoid the FDD assumption?

Properties of $\mathrm{FBL}^{(\infty)}[c_0]$

Theorem

If E is a Banach space with a monotone FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isometric to $\mathrm{FBL}^{(\infty)}[c_0] =: \mathfrak{U}$ (for "universal").

Corollary

(1) *If E has FDD, then $\mathrm{FBL}^{(\infty)}[E]$ is lattice isomorphic to $\mathfrak{U}$.*
(2) *If E is a separable Banach space, then $\exists$ linear isometry $J : \mathrm{FBL}^{(\infty)}[E] \to \mathfrak{U}$ and a contractive lattice homomorphism $P : \mathfrak{U} \to \mathrm{FBL}^{(\infty)}[E]$ so that $PJ = id_{\mathrm{FBL}^{(\infty)}[E]}$.*
(3) *If E has the BAP, J above can be a lattice isomorphism.*

Proof of (2). $X = \mathrm{FBL}^{(\infty)}[E]$ is an $\mathcal{L}_{\infty,1+}$ space, hence it has a monotone basis, hence $\mathrm{FBL}^{(\infty)}[X]$ is lattice isometric to $\mathfrak{U}$.
$id : X \to X$ extends to a lattice homomorphism $\widehat{id} : \mathrm{FBL}^{(\infty)}[X] \to X$.
$\widehat{id} \circ \phi_X$ is the desired factorization via $\mathrm{FBL}^{(\infty)}[X] = \mathfrak{U}$. ∎

Properties of $\mathfrak{U}$

Question. What can we say about $\mathfrak{U}$?

Proposition ($\mathfrak{U}$ is not homogeneous (Gurarii))

If $1 \leqslant p \leqslant \infty$ *and* $\dim E \geqslant 2$, *there exist norm one* $\phi, \psi \in \mathrm{FBL}^{(p)}[E]_+$ *s.t.* $\|T\phi - \psi\| \geqslant 1/3$ *for any lattice isometry* T.

Proposition

$\mathfrak{U}$ *is Banach space isomorphic to* $C[0,1]$.

$\mathrm{FBL}^{(\infty)}[E]$ vs. $\mathrm{FBL}^{(\infty)}[F]$, non-separable case

Theorem (Keller)

If E and F are separable Banach spaces, then $(\mathbf{B}(E^*), w^*)$ is homeomorphic to $(\mathbf{B}(F^*), w^*)$. In fact, both are homeomorphic to $[0,1]^\omega$.

No analogue exists in the non-separable setting.

Proposition

For Γ with $|\Gamma| \geqslant \mathfrak{c}$, and $p \in (1,\infty)$, let $E = \ell_1(\Gamma)$ and $F = \ell_p(\Gamma)$. Then:

1. $(\mathbf{B}(E^*), w^*)$ *is not homeomorphic to* $(\mathbf{B}(F^*), w^*)$.
2. $\mathrm{FBL}^{(\infty)}[E]$ *is not lattice isomorphic to* $\mathrm{FBL}^{(\infty)}[F]$.

Lattice homomorphisms of $\mathrm{FBL}^{(p)}$ for $1 \leqslant p < \infty$

Theorem (N. Laustsen, P. Tradacete)

Suppose $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ *is a lattice homomorphism. Then* $\exists$ *a positively homogeneous* $\Phi_T : F^* \to E^*$, *weak** *to weak** *continuous on bounded sets, s.t.* $[Tf](y^*) = f(\Phi_T y^*)$ *for* $y^* \in F^*$, $f \in \mathrm{FBL}^{(p)}[F]$; $\|\Phi_T y^*\| \leqslant \|T\|\|y^*\|$. *If* $q \geqslant p$, *then, for any* $y_1^*, \ldots, y_N^* \in F^*$, *we have* $\|(\Phi_T y_i^*)\|_{q,\mathrm{weak}} \leqslant \|T\|\|(y_i^*)\|_{q,\mathrm{weak}}$.

Although any lattice homomorphism is a composition operator, a composition need not generate a lattice homomorphism.

Proposition

$\forall p \in [1, \infty) \, \exists$ *positively homogeneous* $\Phi : \ell_1^* \to \ell_1^*$, *weak** *continuous on bounded sets, s.t.* $\forall q \in [p, \infty]$, $\forall y_1^*, \ldots, y_N^* \in \ell_1^*$ *we have* $\|(\Phi y_i^*)\|_{q,\mathrm{weak}} \leqslant \|(y_i^*)\|_{q,\mathrm{weak}}$, *but s.t.* $f \circ \Phi \notin \mathrm{FBL}^{(p)}[\ell_1]$ *for some* $f \in \mathrm{FBL}^{(p)}[\ell_1]$.

Lattice quotients $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

Theorem

Fix $u, v \in [2, \infty]$, $p \in [1, \infty]$, and $u < \min\{v, p'\}$, $1/p + 1/p' = 1$. Suppose E^* has cotype u, and F^* does not have cotype less than v. Then $\mathrm{FBL}^{(p)}[F]$ is not lattice isomorphic to a lattice quotient of $\mathrm{FBL}^{(p)}[E]$.

Cotype u: $\exists K$ s.t. $\forall x_1^*, \ldots, x_N^* \in E^*$, $\mathbb{E}\| \sum_i \pm x_i^* \| \geqslant K(\sum_i \|x_i^*\|^u)^{1/u}$.

L_r has cotype $\max\{r, 2\}$.

Corollary

Suppose $\beta \in [1, 2]$, and $\alpha \in (\beta, \infty]$. For σ-finite Radon measures μ and ν, $\mathrm{FBL}[L_\beta(\nu)]$ is not a lattice quotient of $\mathrm{FBL}[L_\alpha(\mu)]$.

Known: $L_\beta(\nu)$ is not a quotient of $L_\alpha(\mu)$.

Proof. Apply theorem with $E = L_\alpha(\mu)$, $F = L_\beta(\nu)$, $p = 1$. E^* has cotype $\max\{2, \alpha'\}$, F^* has cotype $\beta' > \max\{2, \alpha'\}$. ∎

Lattice quotients $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

Theorem

Suppose $\infty \geqslant u > \max\{v, p\} \geqslant v \geqslant 1$, $E = (\sum_j E_j)_u$ ($E_1, E_2, \ldots$ *are finite dimensional*), *and* F^* *contains a copy of* $\ell_{v'}$, *with* $1/v + 1/v' = 1$. *Then* $\mathrm{FBL}^{(p)}[F]$ *is not a lattice quotient of* $\mathrm{FBL}^{(p)}[E]$.

Questions. (1) Do there exist non-isomorphic E and F s.t. $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ are lattice isomorphic ($1 \leqslant p < \infty$)?

(2) If $\mathrm{FBL}^{(p)}[E]$ and $\mathrm{FBL}^{(p)}[F]$ are lattice isomorphic, which properties do E and F share?

Proposition

Suppose $\mathrm{FBL}[E]$ *is lattice isometric to* $\mathrm{FBL}[F]$.

- *If* E *contains a complemented copy of* ℓ_1, *then so does* F.
- *If* E *contains* ℓ_1^n*'s complementably uniformly, then so does* F.

Indirect proof: relate properties of E to those of $\mathrm{FBL}[E]$.

Lattice isometries $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$, $1 \leqslant p < \infty$

A Banach space Z is called smooth if $\forall z \in Z \backslash \{0\}$ the norming functional $z^* \in Z^*$ is unique. Meaning: the unit sphere of Z has no "corners."

Theorem

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces s.t. either (i) E^*, F^* are smooth, or (ii) E, F are reflexive, and either E^* or F^* is smooth. Then $\mathrm{FBL}^{(p)}[E]$ is lattice isometric to $\mathrm{FBL}^{(p)}[F]$ iff E is isometric to F.

Idea. Quantities $\|(e_i^*)\|_{p,\mathrm{weak}} = \|(\widehat{e_i^*})\|_{p,\mathrm{weak}}$ are preserved by lattice isometries.

For $x \in Z$, denote by $\mathcal{F}(x)$ the set of support functionals for x – that is, of $x^* \in Z^*$ for which $\|x^*\| = \|x\|$ and $\|x\|^2 = \langle x^*, x \rangle$.

Lemma

Suppose $x, y \in \mathbf{S}(Z)$, and $1 \leqslant p < \infty$. Let $\kappa = \sup_{x^* \in \mathcal{F}(x)} |\langle x^*, y \rangle|$. Then, for $t \to 0$, $\|(x, ty)\|_{p,\mathrm{weak}} = 1 + \frac{\kappa^p}{p}|t|^p + o(|t|^p)$.

Lattice isometries when E^*, F^* are smooth

Theorem

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces so that E^*, F^* are smooth. Then $T : \mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ is a lattice isometry iff $T = \overline{U}$, for some isometry $U : E \to F$.

Proof. If Z is smooth, then $\forall x \in Z\ \exists f_x \in Z^*$ s.t. $\mathcal{F}(x) = \{f_x\}$. Define semi-inner product $[y, x] := \langle f_x, y \rangle$. This has the same properties as the inner product, except symmetry and additivity in second variable.

Let $[\cdot, \cdot]_E$ and $[\cdot, \cdot]_F$ be the semi-inner products on E^* and F^* respectively.

For $x^*, y^* \in \mathbf{S}(F^*)$, $\|(x^*, ty^*)\|_{p,\mathrm{weak}} = 1 + \frac{|t|^p}{p} |[y^*, x^*]_F| + o(|t|^p)$, hence $|[y^*, x^*]_F| = |[\Phi_T y^*, \Phi_T x^*]_E|$.

[Ilisevic & Turnsec]: $\exists \sigma : F^* \to \{-1, 1\}$ and $V : F^* \to E^*$ s.t. $\forall x^* \in F^*$ $\Phi_T x^* = \sigma(x^*) V x^*$. Show that $\sigma = \mathrm{const}$ and V is weak* continuous, so $V = U^*$. ■

Lattice isometries and the lack of smoothness

Definition

An element x of a Banach space X is called an ℓ_1-point if the equality $\max_{\pm} \|x \pm y\| = \|x\| + \|y\|$ holds for any $y \in X$.

Proposition

Suppose $1 \leqslant p < \infty$. x *is an* ℓ_1*-point iff* $\|(x, y)\|_{p,\mathrm{weak}} = (\|x\|^p + \|y\|^p)^{1/p}$ *for any* $y \in X$.

Proposition

Suppose $T : \mathrm{FBL}^{(p)}[F] \to \mathrm{FBL}^{(p)}[E]$ *is a lattice isometry. Then* Φ_T *is a bijection between* ℓ_1*-points of* E^* *and* F^*.

Application: free lattices on AM-spaces

Proposition

Suppose E and F are AM-spaces. For $1 \leqslant p < \infty$, TFAE:

1. $\mathrm{FBL}^{(p)}[E]$ *is lattice isometric to* $\mathrm{FBL}^{(p)}[F]$.
2. E *and* F *are isometric.*
3. E *and* F *are lattice isometric.*

Idea of proof. Realize E as a set of functions of $\overline{\Omega_E}$ (the weak* closure of the state space). Identify ℓ_1 points with point masses. Do the same for F. Any lattice isometry $\mathrm{FBL}^{(p)}[E] \to \mathrm{FBL}^{(p)}[F]$ will generate a homeomorphism between Ω_E and Ω_F. ■

Lattice isometries and strict convexity

A Banach space Z is called strictly convex if $\forall$ norm one $z_1, z_2 \in Z$ we have $\|z_1 + z_2\| < 2$ unless $z_1 \neq z_2$.

Proposition

Suppose $1 \leqslant p < \infty$, and E, F are Banach spaces s.t. $\mathrm{FBL}^{(p)}[E]$ is lattice isometric to $\mathrm{FBL}^{(p)}[F]$. Then E^* is strictly convex iff F^* is.

Thank you for your attention! Questions welcome!