Questions about phase retrieval for subspaces of Banach lattices

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1. Real vs complex phase retrieval.
2. Continuity and convergence structures.
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Real vs complex phase retrieval

Let X be a Banach lattice and $E \subseteq X$ be a subspace.

We say that E does phase retrieval in X if for all $f, g \in E$,

$$|f| = |g| \iff f = \lambda g \text{ for some scalar } |\lambda| = 1.$$

If X is a real Banach lattice then $E \subseteq X$ does phase retrieval in X if and only if E does not contain a pair of non-zero disjoint vectors.

Real vs complex phase retrieval

Suppose X is a complex Banach function space.
Q1. How can we characterize when a subspace $E \subseteq X$ does phase retrieval?

Q2. What are necessary or sufficient conditions for $E \subseteq X$ to do phase retrieval?

Known necessary conditions for $E \subseteq X$ to do phase retrieval:

1. E cannot contain a disjoint pair of non-zero vectors $f, g \in E$.
2. E cannot contain an independent pair of entirely real vectors $f, g \in E$.

Theorem (Alharbi-Alshabhi-F-Ghoreishi '22, F-Oikhberg-Pineau-Taylor '22)

Let X be a Banach lattice, $E \subseteq X$ be a subspace, and $\langle \cdot, \cdot \rangle$ be an inner product on E. Then E fails to do phase retrieval in X if and only if there are orthogonal non-zero vectors $f, g \in X$ with $|f| = |g|$.

Phase retrieval, continuity, and convergence structures

Let X be a Banach lattice and $E \subseteq X$ be a subspace.

$E \subseteq X$ does phase retrieval means that for all $f, g \in E$,
$|f| = |g|$ if and only if $f = \lambda g$ for some scalar $|\lambda| = 1$.

Define an equivalence relation $\sim$ on E by $f \sim \lambda f$ for every scalar $|\lambda| = 1$.

$E \subseteq X$ does phase retrieval if and only if the map $f \mapsto |f|$ is one-to-one on $E/\sim$.

We say that $E \subseteq X$ does C-stable phase retrieval if the recovery map $|f| \mapsto f$ is C-Lipschitz. That is,

$$\min_{|\lambda|=1} \|f - \lambda g\|_X \leq C \big\| |f| - |g| \big\|_X \qquad \text{for all } f, g \in E.$$

If X is a real Banach lattice then $E \subseteq X$ does phase retrieval if and only if E does not contain a pair of non-zero disjoint vectors.

Theorem (F-Oikhberg-Pineau-Taylor '22)

Let X be a real Banach lattice and let $E \subseteq X$ be a subspace. Then E does stable phase retrieval in X if and only if there exists $K > 0$ such that

$$\Big\| |f| \wedge |g| \Big\|_X \geq K \min\left(\|f\|_X, \|g\|_X\right) \qquad \textit{for all } f, g \in E.$$

That is, E does stable phase retrieval in X if and only if X does not contain a sequence of almost disjoint pairs.

For every $1 \leq p < \infty$, it is possible to build infinite dimensional subspaces $E \subseteq L_p[0,1]$ which do stable phase retrieval. (Calderbank-Daubechies-F-Freeman '22, Christ-Pineau-Taylor '22, F-Oikhberg-Pineau-Taylor '22)

Phase retrieval, continuity, and convergence structures

Let X be a vector lattice and suppose that $E \subseteq X$ does phase retrieval. That is, the recovery map $|f| \mapsto f$ is well defined from $|E| \subseteq X$ to $E/\sim$.

We say that phase retrieval for $E \subseteq X$ preserves a convergence structure η if whenever $(x_\alpha)_{\alpha \in I}$ is a net in E and $x \in E$ is such that $|x_\alpha| \to_\eta |x|$ then there exist scalars $(\lambda_\alpha)_{\alpha \in I}$ with $|\lambda| = 1$ so that $\lambda_\alpha x_\alpha \to_\eta x$

Q3. Given a convergence structure η, what properties of $E \subseteq X$ imply that phase retrieval preserves η convergence?

Q4. Given a convergence structure η, what properties of $E \subseteq X$ are necessary for phase retrieval to preserve η convergence?

Q5. Which subspaces of $E \subseteq c$ do phase retrieval which preserve order convergence?

Q6. Given two convergence structures η_1 and η_2, what are examples of $E \subseteq X$ where phase retrieval preserves one convergence structure but not the other.

Every infinite dimensional Banach lattice X has an infinite dimensional subspace $E \subseteq X$ which does phase retrieval. (F-Oikhberg-Pineau-Taylor '22)

Q7. Given a convergence structure η, what properties of X guarantee the existence of an infinite dimensional subspace $E \subseteq X$ where phase retrieval preserves η convergence?

Q8. Let X be a Banach lattice and let $E \subseteq X$ be an infinite dimensional subspace. Does there exist a further infinite dimensional subspace $F \subseteq E$ so that F does phase retrieval in X.

Q9. Let X be a Banach lattice and let $E \subseteq X$ be an infinite dimensional subspace. Given a convergence structure η, what properties of $E \subseteq X$ guarantee the existence of a further infinite dimensional subspace $F \subseteq E$ where phase retrieval for $F \subseteq X$ preserves η convergence?

We have been considering phase retrieval for $E \subseteq X$ where X is a Banach lattice and E is a subspace.

We say that a subset $\mathcal{A} \subseteq X \times X$ does C-stable phase retrieval in X if

$$\min_{|\lambda|=1} \|f - \lambda g\|_X \leq C \big\| |f| - |g| \big\|_X \qquad \text{for all } (f, g) \in \mathcal{A}. \tag{1}$$

Q10. What are interesting examples of Banach lattices X and subsets $\mathcal{A} \subseteq X \times X$ such that A does stable phase retrieval in X?

If H is a Hilbert space and $\mathcal{F} : H \to L_2(\mu)$ is a continuous transform then $\mathcal{F}(H) \subseteq L_2(\mu)$ cannot do stable phase retrieval in $L_2(\mu)$. (Alaifari-Grohs '17)

There are nice examples of continuous transforms $\mathcal{F} : H \to L_2(\mu)$ and subsets $\mathcal{A} \subseteq \mathcal{F}(H) \times \mathcal{F}(H)$ such that $\mathcal{A}$ does stable phase retrieval in $L_2(\mu)$! (Chen-Cheng-Sun-Wang '20, Cheng-Daubechies-Dym-Lu '21, Grohs-Rathmair '22)

We say that a subset $\mathcal{A} \subseteq X \times X$ does C-Hölder stable phase retrieval in X with parameter $\gamma \geq 1$ if

$$\min_{|\lambda|=1} \|f - \lambda g\|_X \leq C\big(\|f\|_X + \|g\|_X\big)^{1-1/\gamma} \big\||f| - |g|\big\|_X^{1/\gamma} \qquad \text{for all } (f, g) \in \mathcal{A}.$$

The case $\gamma = 1$ corresponds to Lipschitz stable phase retrieval.

If $E \subseteq X$ is a subspace which does Hölder stable phase retrieval in X then E does Lipschitz stable phase retrieval in X because the stability is worst at orthogonal vectors. (F-Oikhberg-Pineau-Taylor '22)

There are interesting subsets $\mathcal{A} \subseteq L_2 \times L_2$ which do Hölder stable phase retrieval. (Cahill-Casazza-Daubechies '16, Christ-Pineau-Taylor '22)

Q11. Do these subsets do Lipschitz stable phase retrieval?

Q12. How can we construct $\mathcal{A} \subseteq X \times X$ such that $\mathcal{A}$ does Hölder stable phase retrieval in X for some $\gamma > 1$ but $\mathcal{A}$ does not do Lipschitz stable phase retrieval in X.

Phase retrieval and larger relations

In applications, we have measured $|f|$ and we want to recover either f or $-f$.
There is often a much larger class of functions $\mathcal{G}$ where we are happy to recover any $g \in \mathcal{G}_f$ instead of just f or $-f$.

Example: Suppose $f = \psi + \phi$ is a sound wave consisting of a 2 second sound wave ψ followed by 1 second of silence and then a 2 second sound wave ϕ.
Then $\psi + \phi$ sounds exactly the same as $\psi - \phi$
When doing phase retrieval, we are happy to recover any of
$\psi + \phi$, $\psi - \phi$, $-\psi + \phi$, or $-\psi - \phi$ from $|f|$.

Q13. Suppose that $E \subseteq X$ and $\sim_G$ is a larger equivalence relation on X. How can we characterize when $f \mapsto |f|$ is one-to-one on $E/\sim_G$?

If $E \subseteq \mathbb{R}^N$ is an n-dimensional subspace which does phase retrieval then $N \geq 2n - 1$.
Furthermore, almost every n-dimensional subspace of $\mathbb{R}^{2n-1}$ does phase retrieval.

Q14. Suppose that $\sim_G$ is a larger equivalence relation on $\mathbb{R}^N$. How big must N be for it to be possible that $f \mapsto |f|$ is one-to-one on $E/\sim_G$?

Q15. How big must N be so that $f \mapsto |f|$ is one-to-one on $E/\sim_G$ for almost every n-dimensional $E \subseteq \mathbb{R}^N$?

Continuous transforms often do stable phase retrieval on certain local subsets. These local subsets can then be pieced together so that given $|f| \subseteq L_2(\Omega)$ it is possible to stably recover $\sum_{j=1}^{n} \lambda_j f 1_{\Omega_j}$ for some $|\lambda_j| = 1$ and certain subsets $(\Omega_j)_{j=1}^{n}$ of Ω. (Alaifari-Daubechies-Grohs-Yin '19, Chen-Cheng-Sun-Wang '20, Cheng-Daubechies-Dym-Lu '21, Grohs-Rathmair '22)

Q16. What are interesting examples of Banach lattices X, subspaces $E \subseteq X$, and equivalence relations $\sim_G$ such that the recovery map $|f| \mapsto f$ is Lipschitz continuous from $|E|$ to $E/\sim_G$?

We have a characterization of when a subspace of a real Banach lattice does stable phase retrieval.

Q17. What are necessary and sufficient conditions for $|f| \mapsto f$ to be Lipschitz continuous from $|E|$ to $E/\sim_G$?

Phase retrieval and positive bases

Let H be a Hilbert space and $(x_j)_{j=1}^{\infty} \subseteq H$ so that the map $\Theta(x) = (\langle x, x_j \rangle)_{j=1}^{\infty}$ is an embedding of H into ℓ_2.

$\Theta(H) \subseteq \ell_2$ does phase retrieval in ℓ_2 means that for all $x, y \in H$

$$(|\langle x, x_j \rangle|^2)_{j=1}^{\infty} = (|\langle y, x_j \rangle|^2)_{j=1}^{\infty} \Leftrightarrow x = \lambda y \text{ for some } |\lambda| = 1.$$

$$(\langle x \otimes x, x_j \otimes x_j \rangle_{HS})_{j=1}^{\infty} = (\langle y \otimes y, x_j \otimes x_j \rangle_{HS}|)_{j=1}^{\infty} \Leftrightarrow x \otimes x = y \otimes y.$$

$$(\langle x \otimes x - y \otimes y, x_j \otimes x_j \rangle_{HS})_{j=1}^{\infty} = 0 \Leftrightarrow x \otimes x - y \otimes y = 0.$$

$$(\langle T, x_j \otimes x_j \rangle_{HS})_{j=1}^{\infty} = 0 \Leftrightarrow T = 0 \text{ for every s.a. } T \text{ with rank at most 2.}$$

We have that phase retrieval for $\Theta(H) \subseteq \ell_2$ is equivalent to whenever T is a non-zero self-adjoint operator with rank at most 2 then the orthogonal projection of T onto the closed span of $(x_j \otimes x_j)_{j=1}^{\infty}$ is non-zero.

Doing phase retrieval in ℓ_2 is equivalent to constructing a sequence $(x_j)_{j=1}^{\infty} \subseteq H$ so that whenever T is a non-zero self-adjoint operator with rank at most 2 then the orthogonal projection of T onto the closed span of $(x_j \otimes x_j)_{j=1}^{\infty}$ is non-zero.

Q18. Let H be an infinite dimensional separable Hilbert space. Does there exist a conditional Schauder basic sequence $(x_j \otimes x_j)_{j=1}^{\infty}$ and $C > 0$ so that for every self-adjoint operator T with rank at most 2, $\|T\|_{HS} \leq C\|P_{\overline{span} x_j \otimes x_j}\|_{HS}$.

Such a sequence cannot be unconditional as stable phase retrieval is not possible for infinite dimensional subspaces of ℓ_2. (Casazza)

Q19. Does there exist a conditional Schauder basis for the self-adjoint Hilbert-Schmidt operators on H consisting of positive rank one operators?

The Faber-Schauder system is a basis of positive functions in $C[0,1]$. There exists a conditional Schauder basis for $L_1(\mathbb{R})$ consisting of positive functions (Johnson-Schechtman '15). There exists a conditional Schauder basis for $L_2(\mathbb{R})$ consisting of positive functions (F-Powell-Taylor '21).

Q20. What other Banach lattices have a conditional Schauder basis of positive vectors, but not an unconditional basis of positive vectors?

Q21.(Vladimir Kadets) What are examples of cones in Banach spaces which contain a Schauder basis?

In applied harmonic analysis, researchers work with discrete samplings of a continuous transform. This corresponds to given some $E \subseteq L_2(\Omega) \cap L_\infty(\Omega)$ finding $(t_j)_{j \in J} \subseteq \Omega$ and uniform constants $0 < A \leq B$ such that

$$A\|f\|^2_{L_2(\Omega)} \leq \sum_{j \in J} |f(j)|^2 \leq B\|f\|^2_{L_2(\Omega)} \quad \text{for all } f \in E.$$

The L_2-norm on $E \subseteq L_2(\Omega) \cap L_\infty(\Omega)$ can always be discretized. (F-Speegle '19)

In approximation theory, it is important to discretize a norm on a finite dimensional subspace $E \subseteq L_p(\Omega)$ where Ω is a probability space and we use a number of sampling points which is close to the order of the dimension.
(Limonova-Temlyakov '22, Kosov '21, Dai-Prymak-Temlyakov-Tikhonov '19)

This corresponds to finding sampling points $(t_j)_{j=1}^n \subseteq \Omega$ and uniform constants $0 < A \leq B$ such that

$$A\|f\|^p_{L_p(\Omega)} \leq \frac{1}{n} \sum_{j=1}^n |f(j)|^p \leq B\|f\|^p_{L_p(\Omega)} \quad \text{for all } f \in E.$$

We are interested in discretizing the norm on a finite dimensional subspace of a Banach lattice in a way that preserves stable phase retrieval.

Q22. Let $A, B, C, \kappa > 0$ be some uniform constants. Suppose that $E \subseteq L_p(\Omega)$ is N-dimensional and does C-stable phase retrieval where Ω is a probability space. When can we find sampling points $(t_j)_{j=1}^n \subseteq \Omega$ so that the subspace $\big\{(n^{-1/p} f(t_j))_{j=1}^n : f \in E\big\} \subseteq \ell_p^n$ does κ-stable phase retrieval and

$$A\|f\|_{L_p(\Omega)}^p \leq \frac{1}{n}\sum_{j=1}^n |f(j)|^p \leq B\|f\|_{L_p(\Omega)}^p \quad \text{for all } f \in E,$$

where n is on the order of N, $N\log(N)^p$, or something similar?

For $p = 2$, if E is the span of independent Gaussian random variables or uniformly sub-Gaussian random variables which do stable phase retrieval then sampling at n random points in Ω works with high probability when n is on the order of N. (Candès-Li '14, Krahmer-Liu '21)

One difficulty is that a discretization of the L_2-norm on $E \subseteq L_2(\Omega)$ which preserves stable phase retrieval will also be a discretizing of the L_1-norm on $E \subseteq L_1(\Omega)$. (F-Ghoreishi '23)

For more open questions about phase retrieval in Banach lattices see

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