Worst Case Complexity of Solving (Structured) Linear Systems

Richard Peng

University of Waterloo

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Solving Systems of Linear Equations

- Input:
  - n-by-n matrix **A** with m non-zeros
  - vector b
- Output x such that **Ax** = **b**

- Goal: design an algorithm whose worst case complexity as function of size of **A** & output error tolerance is small

Worst Case analysis

- Why? Linear solvers are often used as inner loops:
  - Preconditioning
  - Use in inner loop
  - Complexity-theoretic reductions

Parameterizing the complexity of **Ax** = **b**

- Model: exact arithmetic, bit complexity, low memory, distributed
- Output: norm bound, entry-wise, eigenvector, determinant
- Representation of numbers: finite field, fractions, rounding

Assumptions on A

- Symmetry
- Small condition number
- Low rank
- Low factor width: graph Laplacians, finite element methods
- Displacement structure: Toeplitz/Hankel, hierarchical
- Geometric correspondence

Some rather strange worst-case complexities

- Krylov space methods: over finite fields/exact arithmetic, sparse linear systems can be solved in O(nm) time.
- Over finite fields & poly-conditioned floats, solving 1 linear system in **A** is as expensive as solving n systems (**AX** = **B**).
- [Storjohann `05]: bit-complexity of finding exact fractional solution to integer **Ax** = **b** is same as matrix multiplication over finite fields ($O(n^{\omega})$).
- Laplacian solvers: over fixed-point numbers, poly-conditioned factor width 2 systems can be solved in nearly-linear time.
- Dimensionality reduction/matrix concentration: in poly conditioned floats, can reduce problem size close to rank.
- [Kyng-Zhang `17] & subsequent: many well-studied classes of structured systems are complete for solving general linear systems.

Combinations of Strange Complexities

- Does structure help for computing exact rational solutions?
- Exact rational solution of low-rank problems?
- Solving sequences of systems over low displacement matrices?
- Empirical comparisons of shifted 2-adics / Hansel lifting vs. floats?
- Complexity of solving Laplacians over floating point representation?
- Are there other (than Laplacians / low displacement) classes of systems that can be solved in sub-CG time in some model?