Banff International Research Station
for Mathematical Innovation and Discovery

Comments on Krylov Complexity in Field Theory

2023. 06. 6

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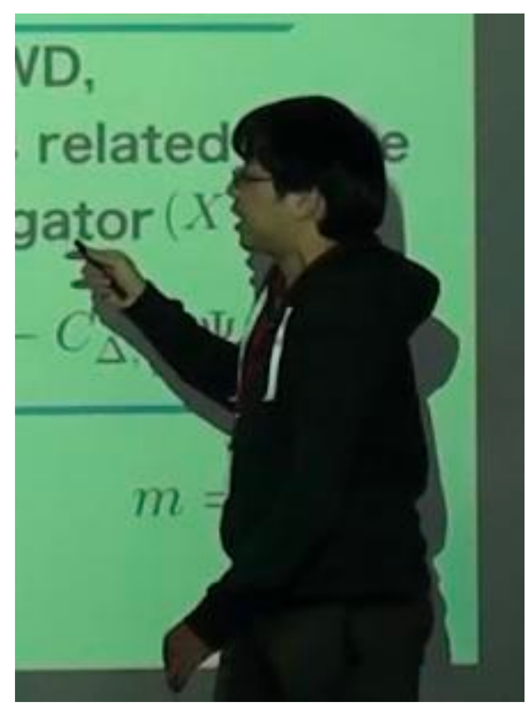
arXiv > hep-th > arXiv:2212.14702

High Energy Physics – Theory

[Submitted on 30 Dec 2022 (v1), last revised 25 Jan 2023 (this version, v2)]

Krylov Complexity in Free and Interacting Scalar Field Theories with Bounded Power Spectrum

Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim, Mitsuhiro Nishida



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arXiv > hep-th > arXiv:2212.14429

High Energy Physics – Theory

[Submitted on 29 Dec 2022]

Krylov complexity in quantum field theory, and beyond

Alexander Avdoshkin, Anatoly Dymarsky, Michael Smolkin

A Universal Operator Growth Hypothesis

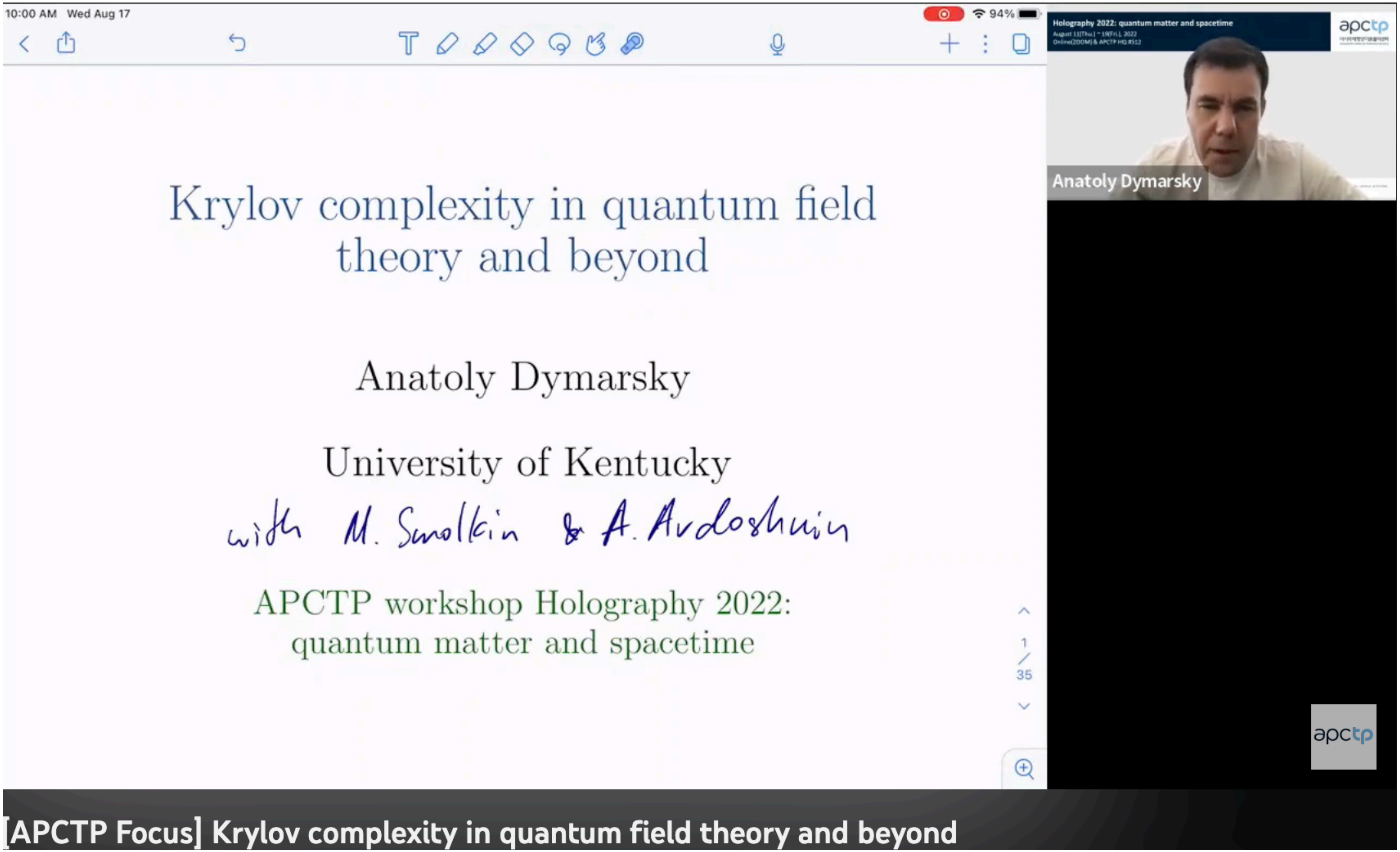
#1

Daniel E. Parker (UC, Berkeley), Xiangyu Cao (UC, Berkeley), Alexander Avdoshkin (UC, Berkeley), Thomas Scaffidi (UC, Berkeley), Ehud Altman (UC, Berkeley) (Dec 20, 2018)

Published in: *Phys.Rev.X* 9 (2019) 4, 041017 • e-Print: 1812.08657 [cond-mat.stat-mech]

pdf DOI cite claim reference search 180 citations

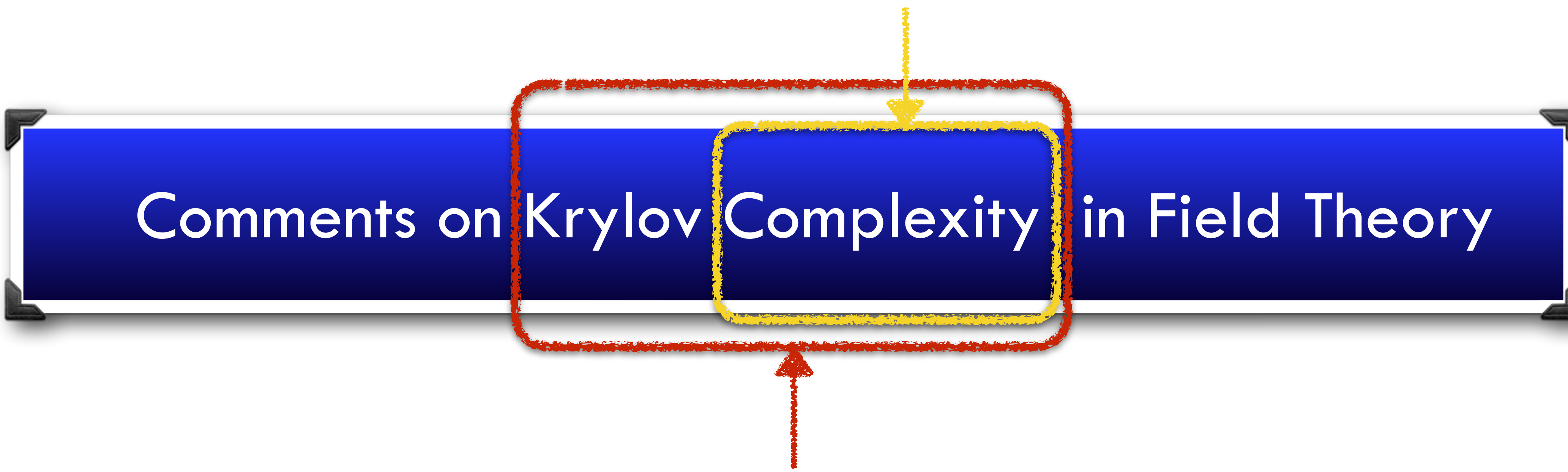
Comments on Krylov Complexity in Field Theory



13talks (6 + 7) /28 ~ 46%

- Nick Hunter-Jones: Progress on quantum complexity growth conjectures
- Anthony Munson: Quantum (Un)complexity: A Resource for Quantum Computation
- Poulami Nandi: Complexity and BMS
- Mohsen Alishahiha: One Quantum Complexity
- Bret Underwood: Cosmological Complexity
- Shubho R. Roy: Gravitational Singularities and Holographic Complexity

Quantum (circuit/cumputational/holographic) complexity



A diagnose of quantum chaos

- Khushboo Dixit: Quantum Spread Complexity in Neutrino Oscillations
- Javier Magan: Long times, chaos, and spread complexity
- Keun-Young Kim: Comments on Krylov Complexity in Field Theory
- Anatoly Dymarsky: REVIEW TALK -4 Chaos and complexity through thelens of dynamics in Krylov space
- Hendrik J R Van Zyl: Spread Complexity and Toplogical Transitions in the Kitaev Chain
- Ryota Watanabe: Krylov complexity and chaos in quantum mechanics
- Nikolaos Angelinos: Temperature dependence of Lanczos coefficients and integrability

Quantum complexity

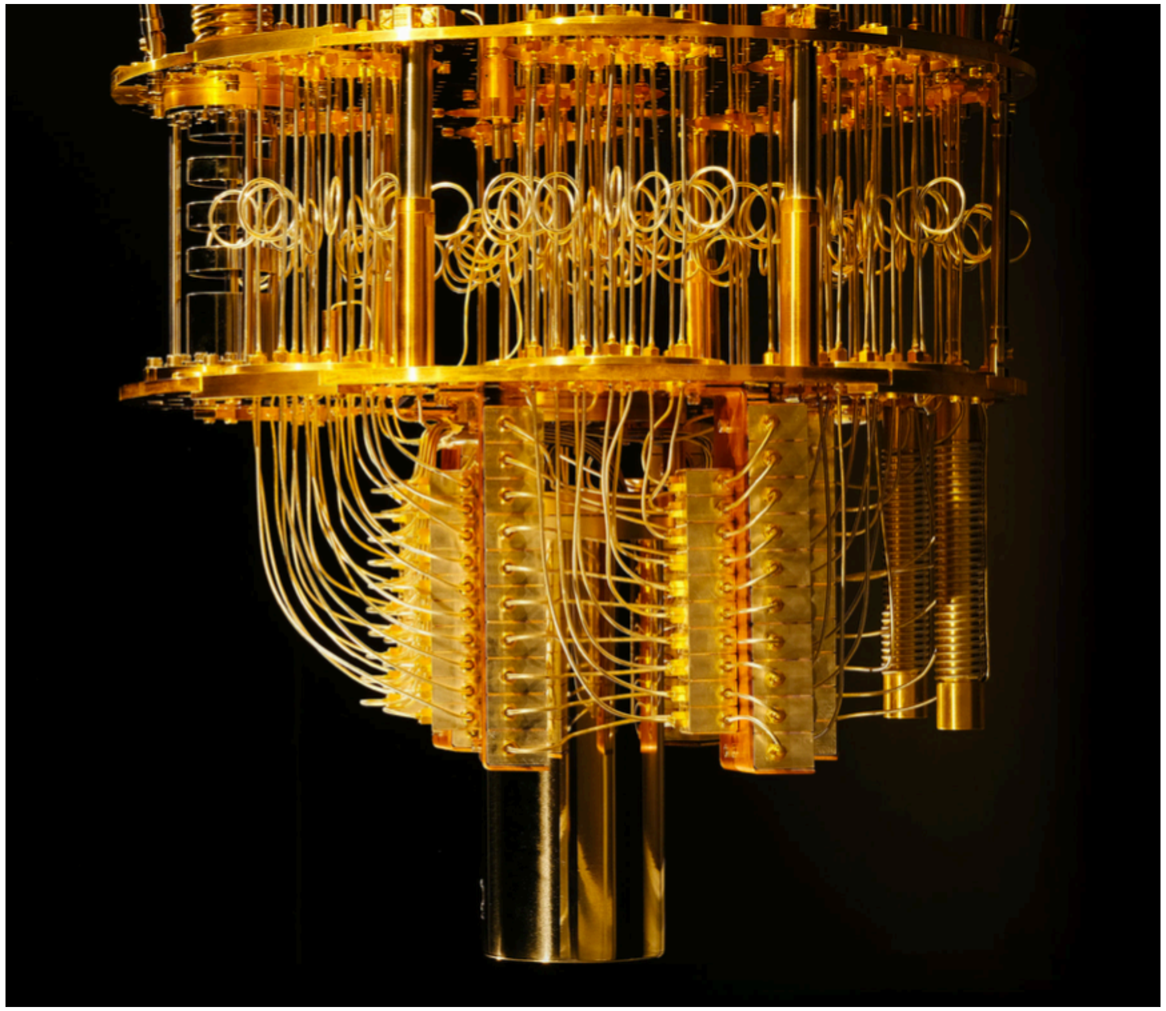
Nick

Complexity

(Computational) complexity — [Computer science] quantifying the difficulty of carrying out a task.

Quantum Computer

Input state →



→ output state

Complexity

(Computational) complexity — [Computer science] quantifying the difficulty of carrying out a task.

(Circuit) complexity — Quantum Computer ~ Quantum Circuit

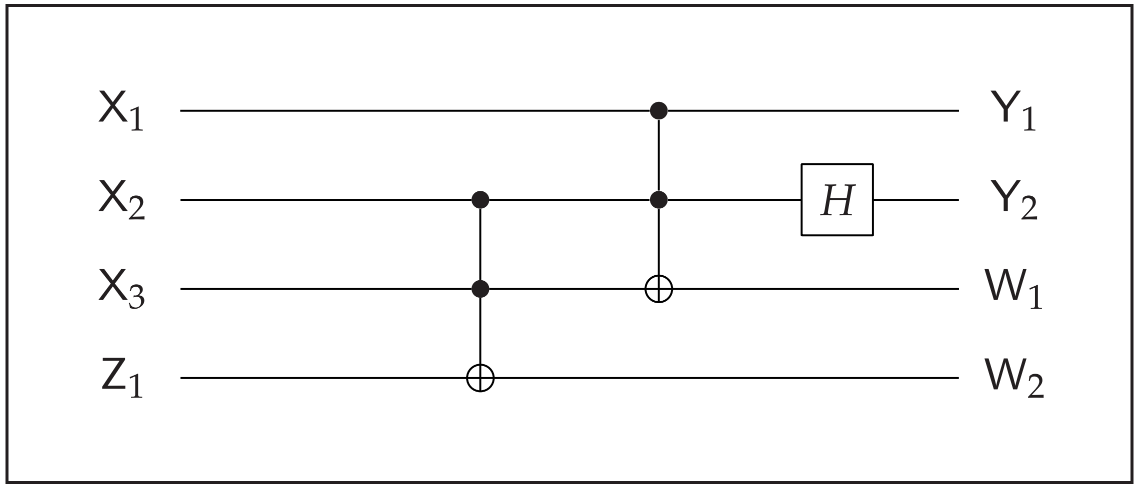
Minimal number of gates for the transformation from the reference to target state

$$|\psi_T\rangle = U|\psi_R\rangle = g_n g_{n-1} \cdots g_2 g_1 |\psi_R\rangle$$

Ambiguity

Universal gate sets = {a,b,c,d,e,f}

ex)



$$G = dbe$$
$$G = ceab$$
$$G = abefa$$

complexity = 3

"Distance" between two sates?

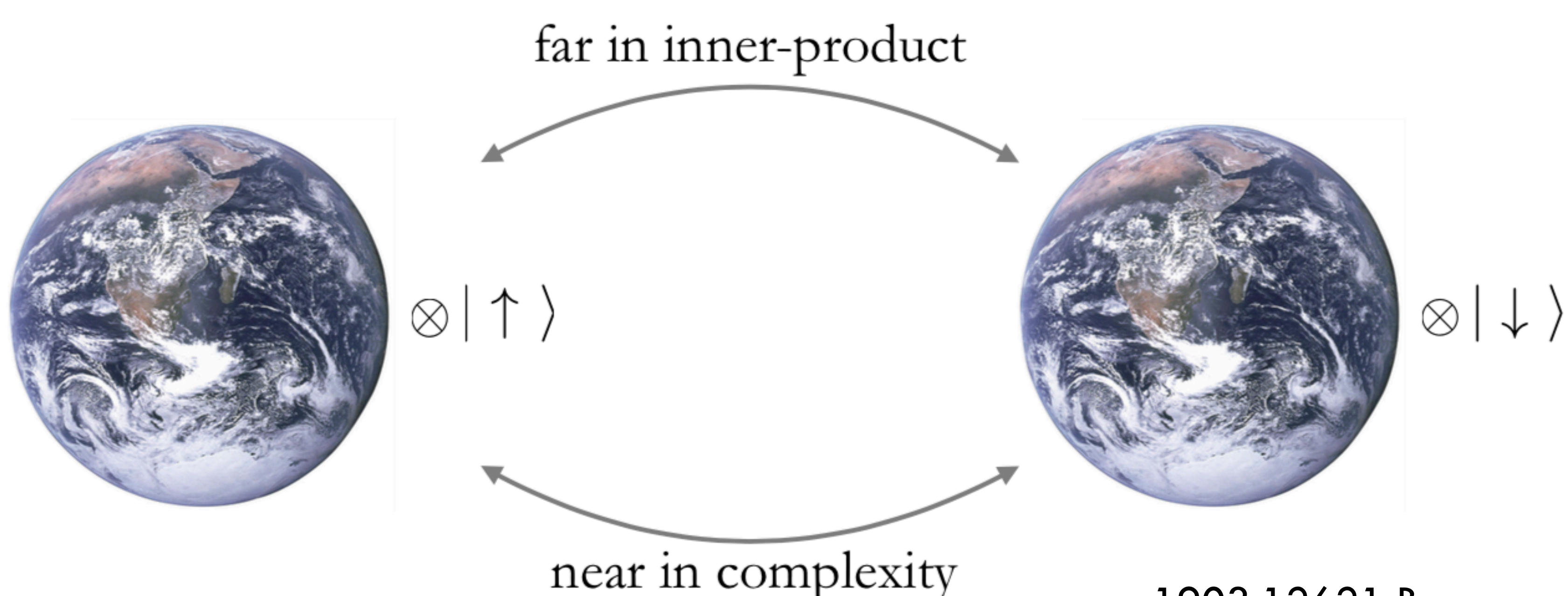
(inner-product) distance: $d_{AB} = \arccos |\langle B|A\rangle|$ (closest) $0 \sim \pi/2$ (farthest)

Are these close or far? $|0000000000\rangle \longrightarrow |0000000001\rangle$

Far in the inner-product sense

However, in some sense they are close → "easy" or "difficult" transform

Need a new distance reflecting this sense: "Complexity distance?"



1903.12621 Brown and Susskind

Complexity

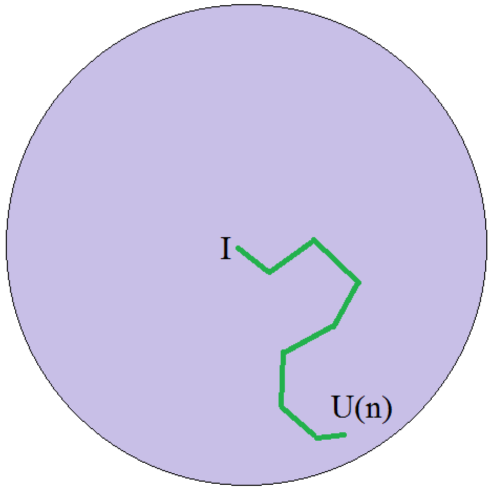
Complexity of quantum states New distance in Hilbert space Spread complexity

For given states $|\psi_T\rangle = U|\psi_R\rangle$ ~How hard (minimal number of gates) from the reference to target state

Complexity of operator (unitary transformation) New distance in Unitary group

For a given operator $U = g_n g_{n-1} \cdots g_2 g_1$ $\sim$ minimum number of gates

$$\mathbb{I} \longrightarrow U$$



Krylov complexity

Relation between two

$$\mathcal{C}(|\psi_1\rangle, |\psi_2\rangle) = \min \left\{ \mathcal{C}(U) \mid \forall\, \hat{U} \in \mathcal{O}, \quad |\psi_2\rangle = \hat{U}|\psi_1\rangle \right\}$$

Quantum Chaos

Quantum Chaos

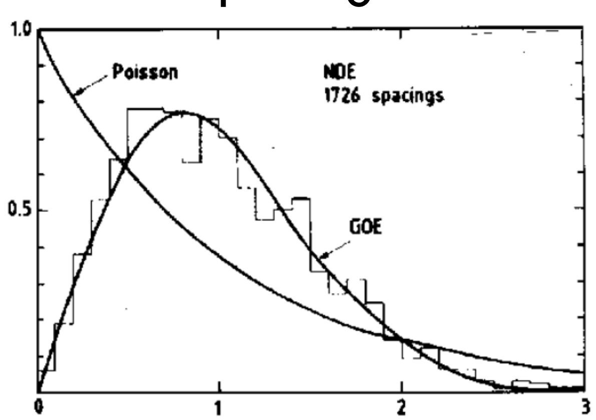
$$|\{q^i(t),p^j(0)\}_{PB}=\left|\frac{\partial q^i(t)}{\partial q^j(0)}\right|\sim e^{\lambda t}$$

$$-\langle[q^i(t),p^j(0)]^2\rangle_\beta,$$

$$-\langle[V(t),W(0)]^2\rangle_\beta \quad \sim e^{\lambda t}$$

Out-of-time-order correlator (OTOC)

Level spacing statistics



Random Matrix Theory

Thermalization
(ETH, Quantum device)
Quantum black holes
Quantum gravity

FO What is quantum chaos?

ChatGPT

Quantum chaos is a field of physics that studies the behavior of quantum systems that are classically chaotic. Classically chaotic systems are those that exhibit sensitive dependence on initial conditions, meaning small changes in the initial conditions lead to vastly different outcomes over time.

In quantum mechanics, chaotic behavior can manifest in the statistical properties of the system's energy spectrum or in the time evolution of its wave function. The study of quantum chaos is concerned with understanding how the behavior of the system changes as classical chaos is introduced or increased, and how this behavior can be observed and measured experimentally.

Quantum chaos has applications in various fields, including solid-state physics, quantum computing, and quantum information theory. It has also led to new insights into the fundamental nature of quantum mechanics and the connections between classical and quantum physics.

Krylov complexity?

Quantum chaos and complexity

Midjourney

Comments on Krylov Complexity in Field Theory

Complexity: how much things are complex

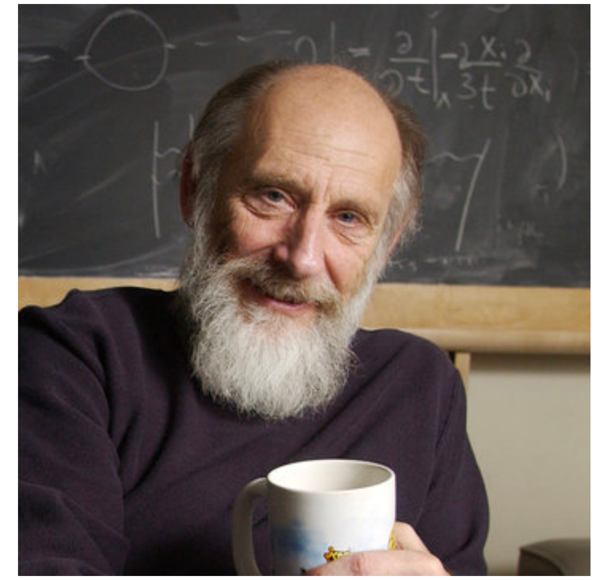
Chaos: how fast things get complex

~ fast increase of complexity

*Circuit complexity is not well-defined

"Krylov complexity" is a well-defined concept
proposed as a diagnose of quantum chaos (which is not-well defined)

Entanglement is not enough!
Black hole interior?



Krylov complexity in conformal field theory

Anatoly Dymarsky (Kentucky U. and Skoltech), Michael Smolkin (Hebrew U.) (Apr 19, 2021)

Published in: *Phys.Rev.D* 104 (2021) 8, L081702 • e-Print: 2104.09514 [hep-th]

Comments on Krylov Complexity in Field Theory

Complexity: how much things are complex

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~ fast increase of complexity

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"Krylov complexity" is a well-defined concept
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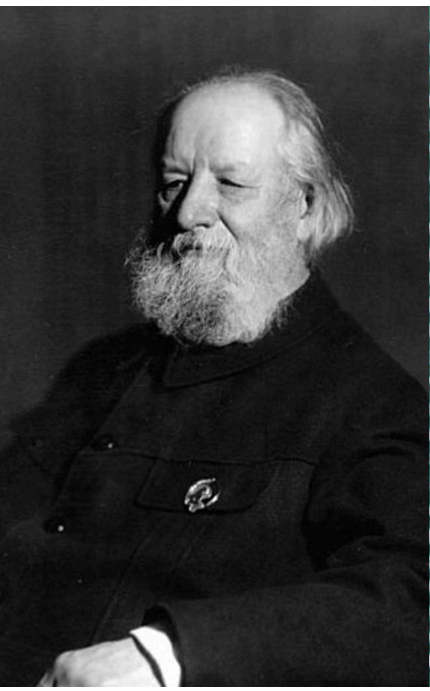
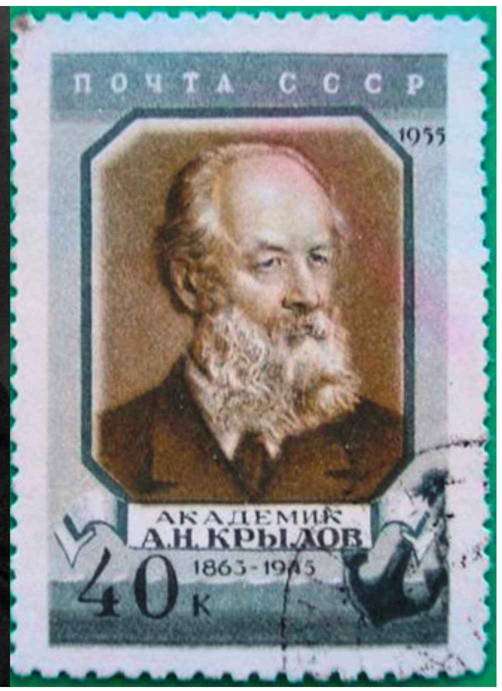
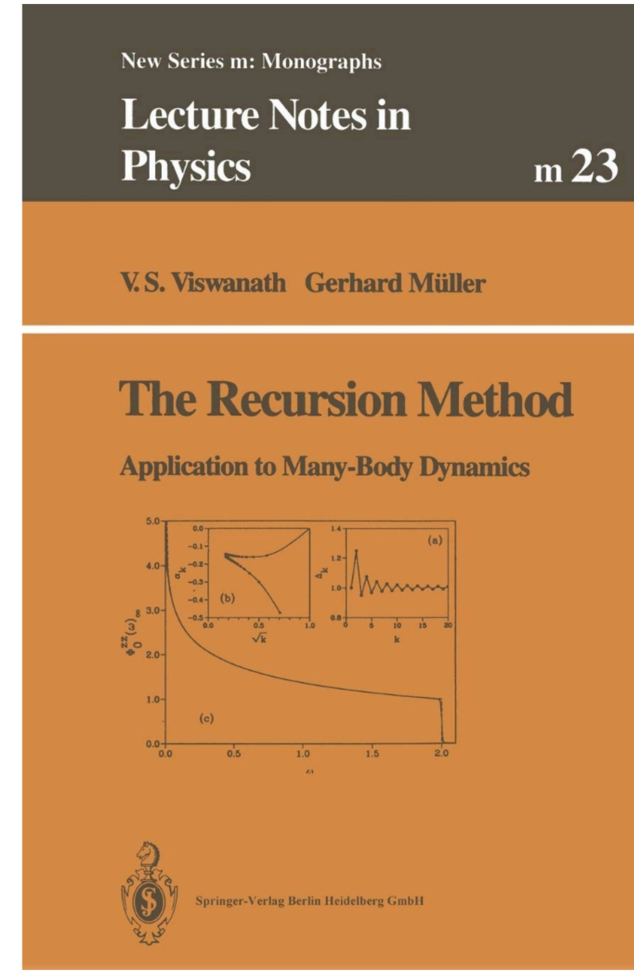
Contents

Aleksey Nikolaevich Krylov (1863 –1945)
Russian naval engineer, applied mathematician and memoirist.





- Short Review on Krylov Complexity
  - Operator growth
  - Krylov space
  - Lanczos coefficient
  - Krylov complexity
- Success in lattice systems
- Towards field theory
  - Too good to be true
  - How to extract info from the power spectrum (IR/UV cutoff effect)



1994

Cornelius (Cornel) Lanczos (1893-1974):
a Hungarian-American and later Hungarian-Irish mathematician and physicist.

Short Review on Krylov Complexity

Khushboo

Javier

Operator growth

The time evolution of an operator O by a time independent Hamiltonian *H*

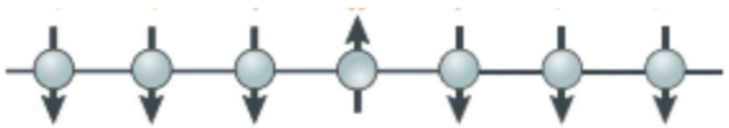
$$\partial_t \mathcal{O}(t) = i\ [H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\ \mathcal{O}(0)\ e^{-itH}$$

Baker-Campbell-Hausdorff (BCH) formula $e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_X^n Y}{n!}$

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H, \mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

ex) 1D spin chain



$$H = -\sum (Z_i Z_{i+1} + gX_i + hZ_i)$$

$$Z_1(t) = Z_1 + it[H, Z_1] - \frac{t^2}{2!}[H,[H,Z_1]] - \frac{it^3}{3!}[H,[H,[H,Z_1]]] + \ldots$$

$$\begin{aligned}
[H, Z_1] &\sim Y_1 \\
[H,[H,Z_1]] &\sim Y_1 + X_1 Z_2 \\
[H,[H,[H,Z_1]]] &\sim Y_1 + Y_2 X_1 + Y_1 Z_2 \\
[H,[H,[H,[H,Z_1]]]] &\sim X_1 + Y_1 + Z_1 + X_1 X_2 + Y_1 Y_2 + Z_1 Z_2 + X_1 Z_2 + \\
&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

Operator growth

The time evolution of an operator O by a time independent Hamiltonian H

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$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

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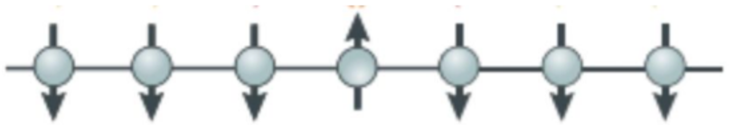
$$\mathcal{O}(t) = \mathcal{O}_0 + it[H, \mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \; .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!} \tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

ex) 1D spin chain



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&\quad + Z_3 Y_1 + Y_1 Z_2 Y_2 + Z_1 X_2 X_1 + X_2 Z_3 X_1
\end{aligned}$$

Operator growth

1812.08657: Parker et al.

The time evolution of an operator O by a time independent Hamiltonian H

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- The set of operators $\{\tilde{\mathcal{O}}_n\}$ defines a basis of the so-called *Krylov space* associated to the operator $\mathcal{O}$
- Regard the operator as a state $\mathcal{O} \to |\mathcal{O})$ in the Hilbert space of operators

Inner product: Wightman inner product

$$(A|B) := \langle e^{\beta H/2} A^\dagger e^{-\beta H/2} B\rangle_\beta = \frac{1}{\mathcal{Z}_\beta}\mathrm{Tr}(e^{-\beta H/2} A^\dagger e^{-\beta H/2} B) \qquad \mathcal{Z}_\beta := \mathrm{Tr}\left(e^{-\beta H}\right)$$

Krylov basis

$(\mathcal{O}_m|\mathcal{O}_n) = \delta_{mn}$ (Lanczos algorithm: Gram–Schmidt procedure)

$$|\mathcal{O}_0) := |\tilde{\mathcal{O}}_0) := |\mathcal{O}(0))$$

$\{b_n\}$: Lanczos coefficients

$$|\mathcal{O}_1) := b_1^{-1}\mathcal{L}|\tilde{\mathcal{O}}_0) \qquad b_1 := (\tilde{\mathcal{O}}_0\mathcal{L}|\mathcal{L}\tilde{\mathcal{O}}_0)^{1/2}$$

$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

The time evolution of an operator O by a time independent Hamiltonian H

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\mathcal{O}(t) = e^{itH}\,\mathcal{O}(0)\,e^{-itH}$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\,\mathcal{O}(0)$$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t)|\mathcal{O}_n)$$

$$\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$$

"probability amplitudes"

$$\mathcal{O}(t) = \mathcal{O}_0 + it[H,\mathcal{O}] + \frac{(it)^2}{2!}[H,[H,\mathcal{O}]] + \frac{(it)^3}{3!}[H,[H,[H,\mathcal{O}]]] + \cdots \quad .$$

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(i\,t)^n}{n!}\tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \qquad \mathcal{L} := [H, \cdot\,]$$

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$$|\mathcal{O}_n) := b_n^{-1}|A_n) \qquad b_n := (A_n|A_n)^{1/2}$$

$$|A_n) := \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$$

Discrete "Schrodinger equation"

$$\partial_t \mathcal{O}(t) = i\,[H, \mathcal{O}(t)]$$

$$\partial_t |\mathcal{O}(t)) = i\mathcal{L}|\mathcal{O}(t))$$

"probability amplitudes" $\sum_{n=0}^{\infty} |\varphi_n(t)|^2 = 1$

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} i^n \varphi_n(t) |\mathcal{O}_n) \qquad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t))$$

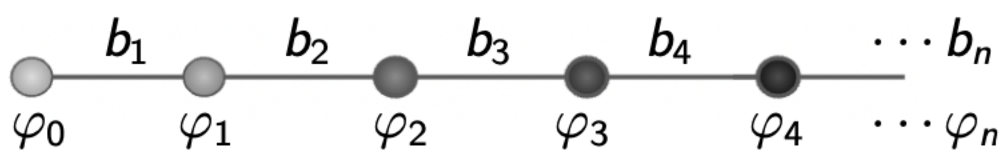
$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = b_n \delta_{m,n-1} + b_{n+1}\delta_{m,n+1}$$

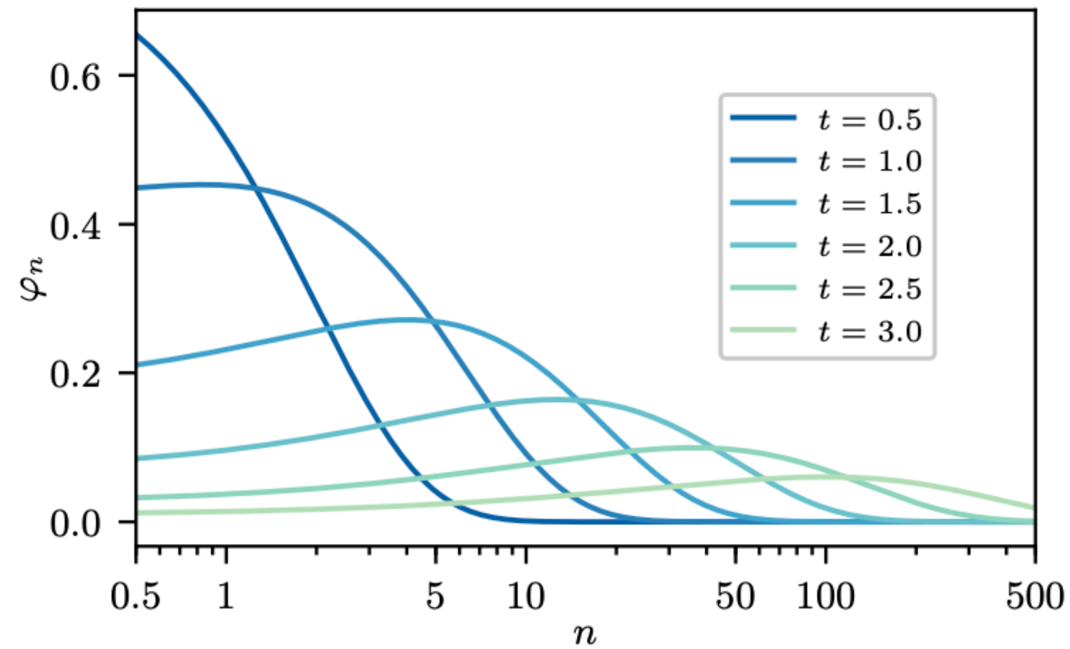
$$\frac{\mathrm{d}\,\varphi_n(t)}{\mathrm{d}t} = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0} \qquad \varphi_{-1}(t) \equiv 0 \equiv b_0$$

$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1 \varphi_1(t)$$
$$\dot{\varphi}_1(t) = b_1 \varphi_0(t) - b_2 \varphi_2(t)$$
$$\vdots$$
$$\dot{\varphi}_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

a quantum-mechanical particle on a 1- dimensional chain.

b_n = hopping amplitudes





Krylov complexity

average position over the chain

$$K_{\mathcal{O}}(t) := (\mathcal{O}(t)|n|\mathcal{O}(t)) = \sum_{n=0}^{\infty} n |\varphi_n(t)|^2$$

K-complexity from the auto-correlation function

1812.08657: Parker et al.

Auto-correlation function

$C(t) = \Pi^W(t) = \varphi_0(t)$

$$C(t) := (\mathcal{O}(t)|\mathcal{O}(0)) = \varphi_0(t)$$
$$= \langle e^{i(t-i\beta/2)H}\mathcal{O}^\dagger(0)e^{-i(t-i\beta/2)H}\mathcal{O}(0)\rangle_\beta$$
$$= \langle \mathcal{O}^\dagger(t-i\beta/2)\mathcal{O}(0)\rangle_\beta =: \Pi^W(t)\ .$$

$$\langle\cdots\rangle_\beta = \mathrm{Tr}(e^{-\beta H}\cdots)/\mathrm{Tr}(e^{-\beta H})$$

Power spectrum

$f^W(\omega)$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

Moments

μ_{2n}

$$\Pi^W(t) := \sum_{n=0}^{\infty}\mu_{2n}\frac{(it)^{2n}}{(2n)!} \qquad \mu_{2n} := \frac{1}{i^{2n}}\frac{d^{2n}\Pi^W(t)}{dt^{2n}}\Big|_{t=0}$$

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

Lanczos coefficients from moments

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_2 = b_1^2\,, \quad \mu_4 = b_1^4 + b_1^2 b_2^2\,, \quad \cdots\,.$$

Hankel matrix constructed from the moments.

$$H_n = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix}$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j,\ldots,n\,,$$
$$M_{2l}^{(0)} = \mu_{2l}\quad,\quad b_{-1}\equiv b_0 := 1\quad,\quad M_{2l}^{(-1)} = 0\,.$$

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$C(t) = \Pi^W(t) = \varphi_0(t) \qquad b_n$

K-complexity

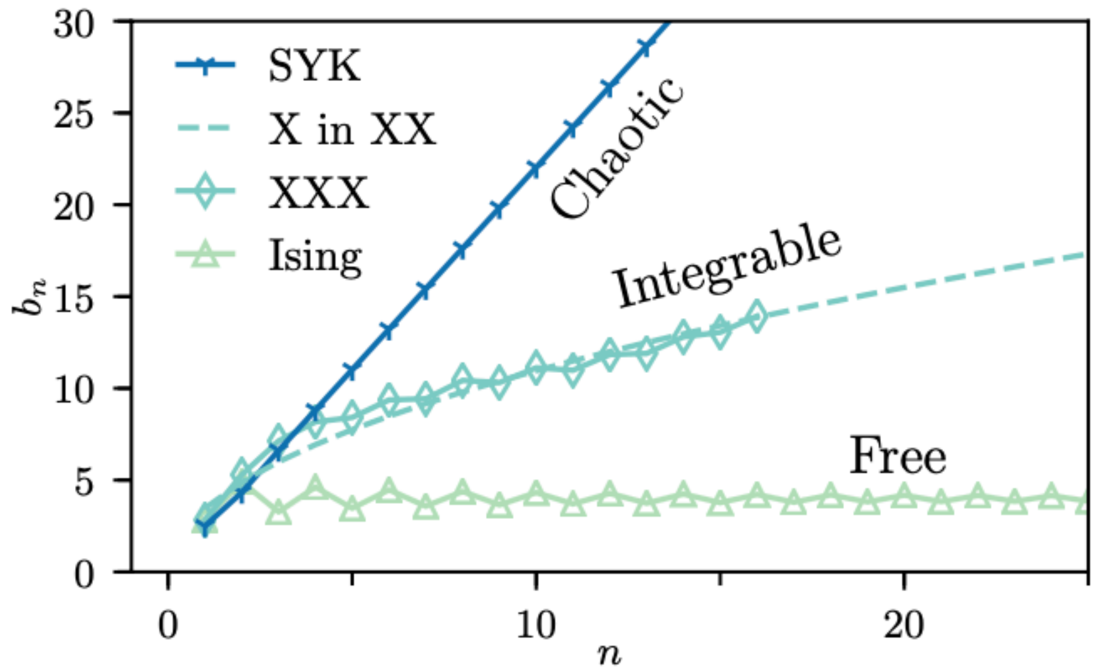
$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2\,, \quad n_{\max} = 200.$$

Success in lattice systems

$$b_n \sim n^\delta \iff f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$$\delta \le 1$$



D. S. Lubinsky, "A survey of general orthogonal polynomials for weights on finite and infinite intervals," Acta Applicandae Mathematica **10**, 237–296 (1987).
A. Magnus, "The recursion method and its applications: Proceedings of the conference, imperial college, london, england september 13–14, 1984," (Springer Science & Business Media, 2012) Chap. 2, pp. 22–45.

Signatures of chaos in time series generated by many-spin systems at high temperatures

Tarek A. Elsayed, Benjamin Hess, and Boris V. Fine
Phys. Rev. E **90**, 022910 – Published 20 August 2014

$f^W(\omega) \sim e^{-\frac{\omega}{\omega_0}}$ Is a signature of classical chaos

Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

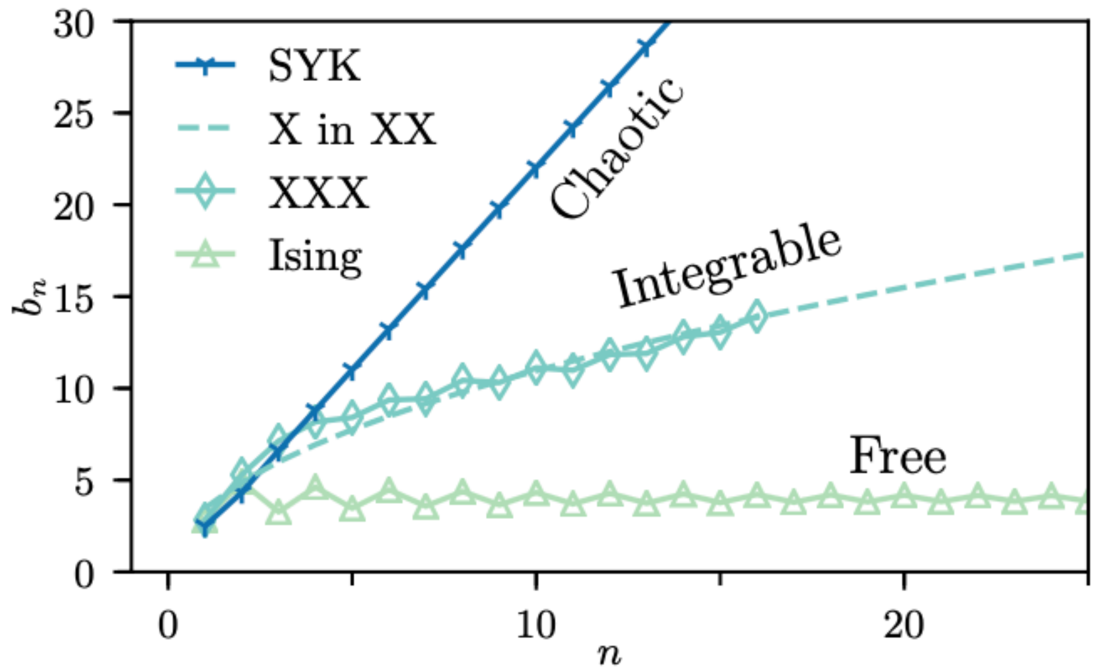
Krylov complexity grows exponentially

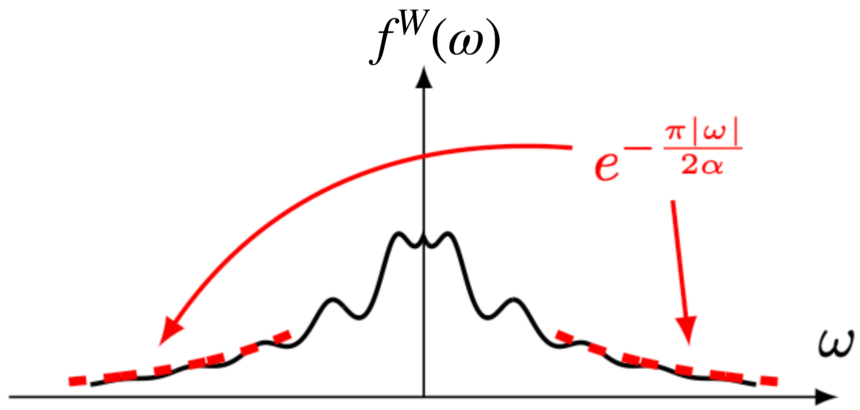
$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

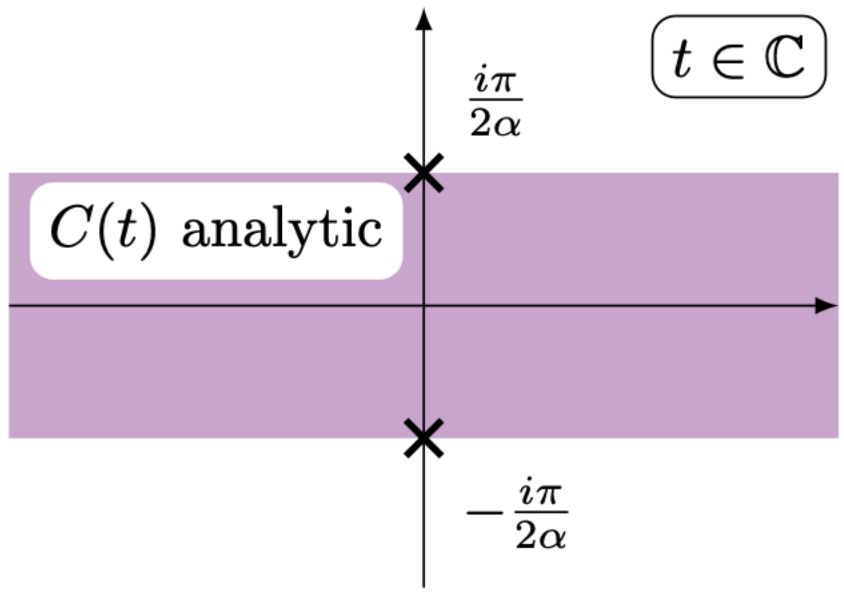
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$b_n \sim n^\delta \iff f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$$\delta \le 1$$







Universal operator growth hypothesis

In a chaotic quantum system

Lanczos coefficients $\{b_n\}$ grow as fast as possible

$$b_n \sim \alpha n$$

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Towards Field theory

Free massive scalar in d-dimensions

$$\mathcal{L}_E^{\rm free} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t-i\beta/2,\mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \ ,$$

$$\Pi^W(\omega,\mathbf{k}) := \int \mathrm{d}t \int \mathrm{d}^{d-1}\mathbf{x}\, e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}\, \Pi^W(t,\mathbf{x})$$

$$\Pi^W(\omega,\mathbf{k}) = \frac{1}{\sinh[\beta\omega/2]}\rho(\omega,\mathbf{k}).$$

$$\rho(\omega,\mathbf{k}) = \frac{N}{\epsilon_k}[\delta(\omega-\epsilon_k) - \delta(\omega+\epsilon_k)],$$

$$\epsilon_k := \sqrt{|\mathbf{k}|^2 + m^2}.$$

Power spectrum

$$C(t) = \Pi^W(t,\mathbf{0})$$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t} = \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\Pi^W(\omega,\mathbf{k}).$$

$$f^W(\omega) = N(m,\beta,d)\,\frac{(\omega^2-m^2)^{(d-3)/2}}{|\sinh\left(\frac{\beta\omega}{2}\right)|}\Theta(|\omega|-m)$$

$$\int \frac{\mathrm{d}\omega}{2\pi} f^W(\omega) = 1$$

m=0, d=4

$$f^W(\omega) = \frac{\beta^2\omega}{\pi\sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

$$f^W(\omega) \longrightarrow \mu_{2n} \longrightarrow b_n$$

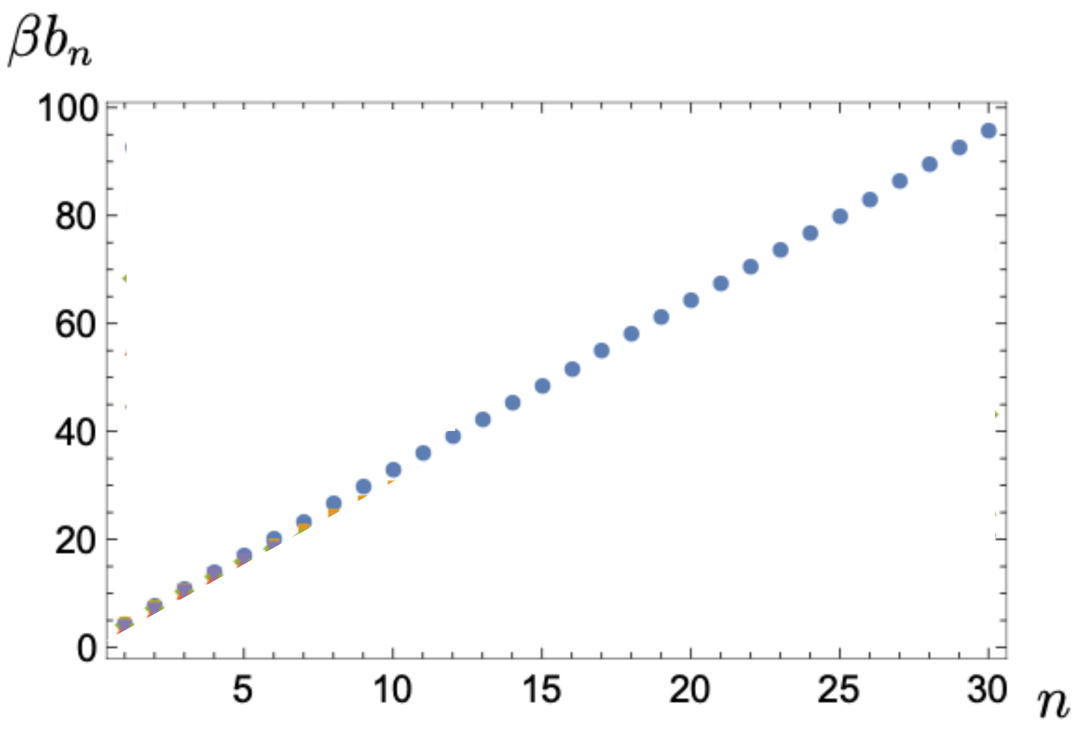
$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) \qquad b_1^{2n}\cdots b_n^2 = \det(\mu_{i+j})_{0\le i,j\le n}$$

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Power spectrum (m=0, d=4)

$$f^W(\omega) = \frac{\beta^2\omega}{\pi \sinh(\frac{\beta\omega}{2})}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$



Free theory is chaotic?

~~In a chaotic quantum system~~ In free QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible??

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

?

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

2212.14429: Avdoshikin, Dymarsky, Smolkin
2212.14702: Camargo, Jahnke, KYK, Nishida

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

Power spectrum (m=0, d=4)

$$f^W(\omega) = \frac{\beta^2\omega}{\pi \sinh(\frac{\beta\omega}{2})}$$

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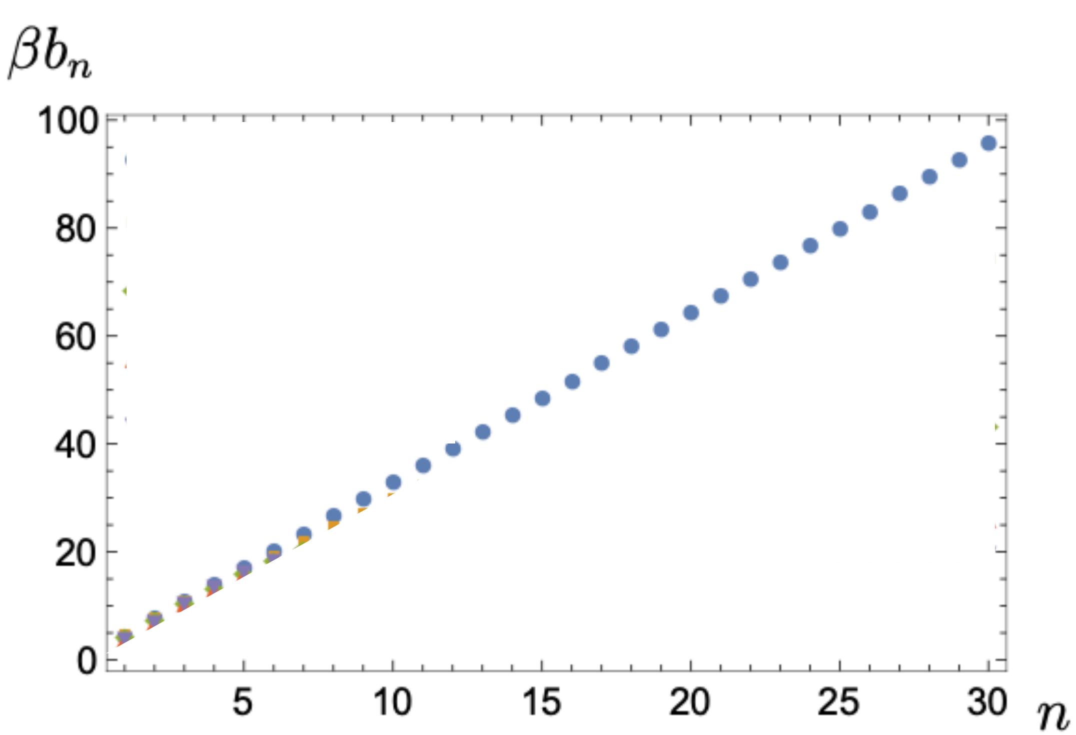


Wightman 2-point function

$$\Pi^W(t,\mathbf{x}) := \langle \phi(t - i\beta/2, \mathbf{x})\phi(0,\mathbf{0})\rangle_\beta \qquad \left(t = \frac{i\beta}{2}\right)$$

Power spectrum $C(t) = \Pi^W(t,\mathbf{0})$

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t} = \int \mathrm{d}t\, \Pi^W(t,\mathbf{0}) e^{i\omega t}$$

$$f^W(\omega) \sim e^{-\frac{\beta|\omega|}{2}} \sim e^{-\frac{\pi|\omega|}{2\alpha}} \qquad \left(\alpha = \frac{\pi}{\beta}\right)$$

2104.09514: Dymarsky, Smolkin

General QFT is chaotic? No

~~In a chaotic quantum system~~ In general QFT

Lanczos coefficients $\{b_n\}$ grow as fast as possible!

$$b_n \sim \alpha n \sim \frac{\pi}{\beta} n$$

?

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_\mathcal{O}(t) \sim e^{2\alpha t}$$

Too good to be true

Towards Field theory

Counter example:

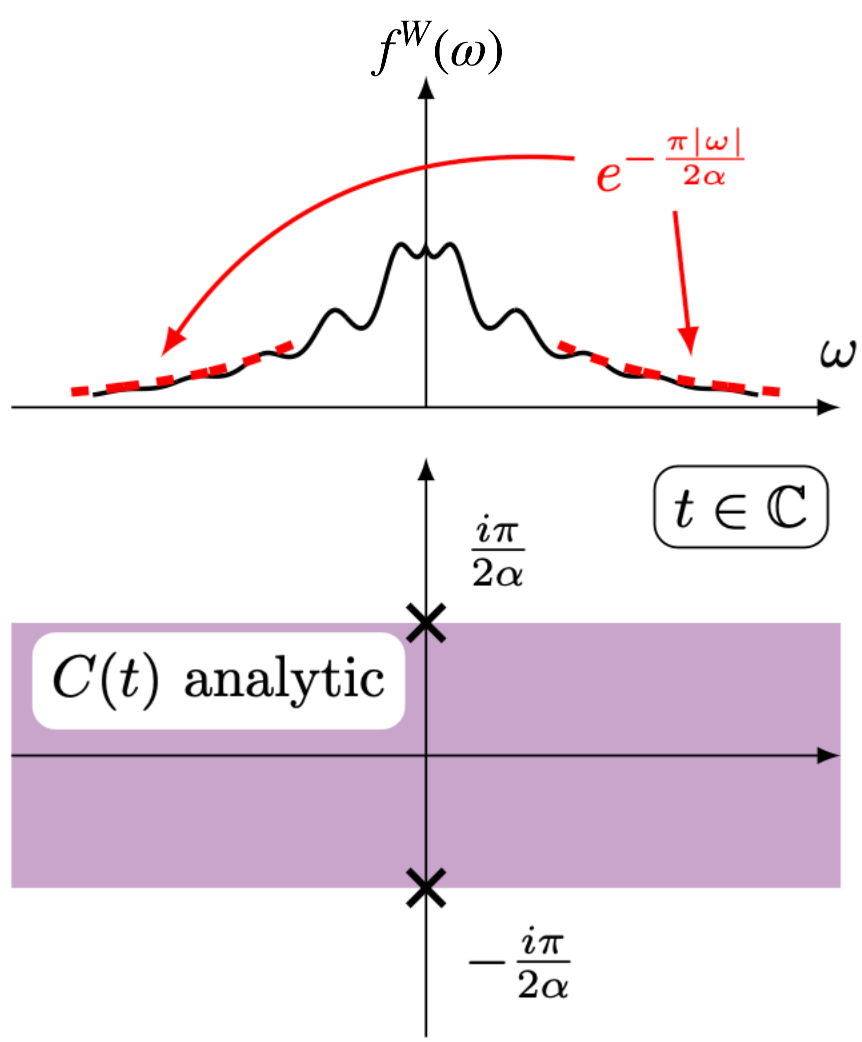
- Field theory
- Krylov complexity in saddle-dominated scrambling (2203.03534: Bhattacharjee, Cao, Nandy, Pathak)

Too good to be true

$$\text{Chaos} \Longleftrightarrow \boxed{f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}}$$

Only if b_n is a smooth function of n, Otherwise

$$\text{Chaos} \overset{?}{\Longleftrightarrow} \boxed{f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \overset{?}{\Longleftrightarrow} b_n \not\sim \alpha n \overset{?}{\Longleftrightarrow} K_{\mathcal{O}}(t) \sim e^{2\alpha t}}$$



Counter example:

- Field theory
- Krylov complexity in saddle-dominated scrambling (2203.03534: Bhattacharjee, Cao, Nandy, Pathak)

Too good to be true

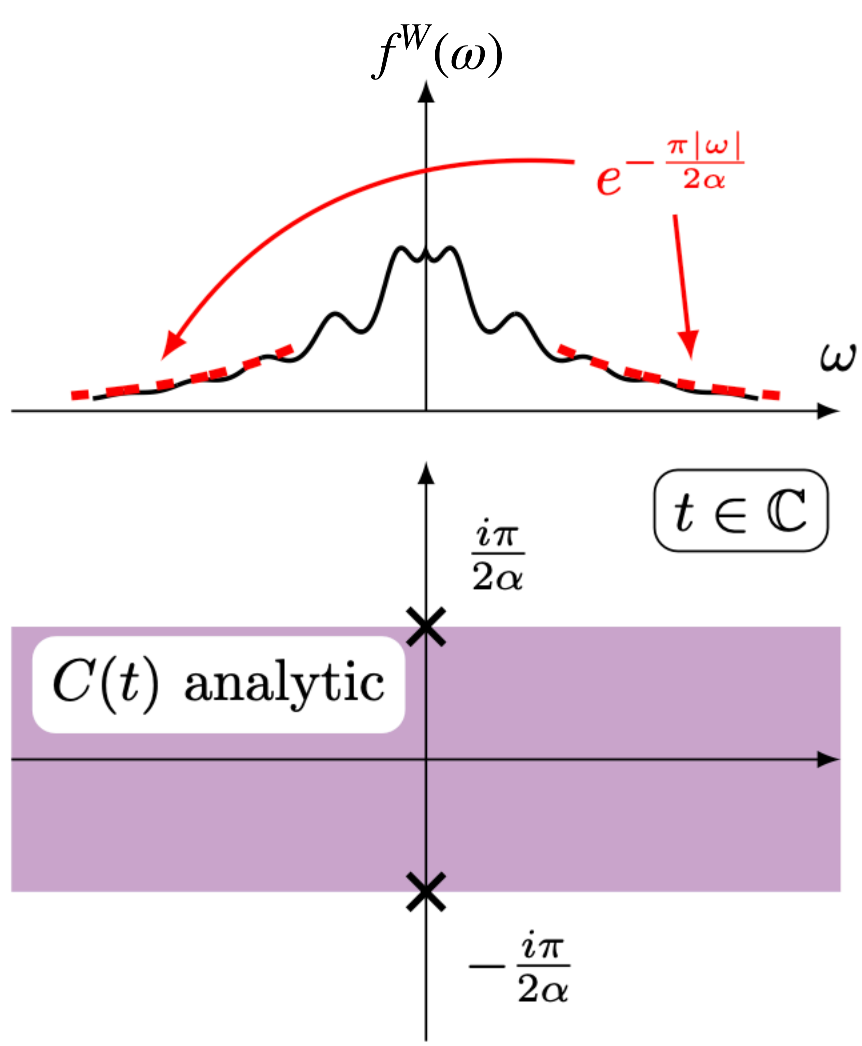
$$\text{Chaos} \Longleftrightarrow \quad f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \Longleftrightarrow b_n \sim \alpha n \Longleftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

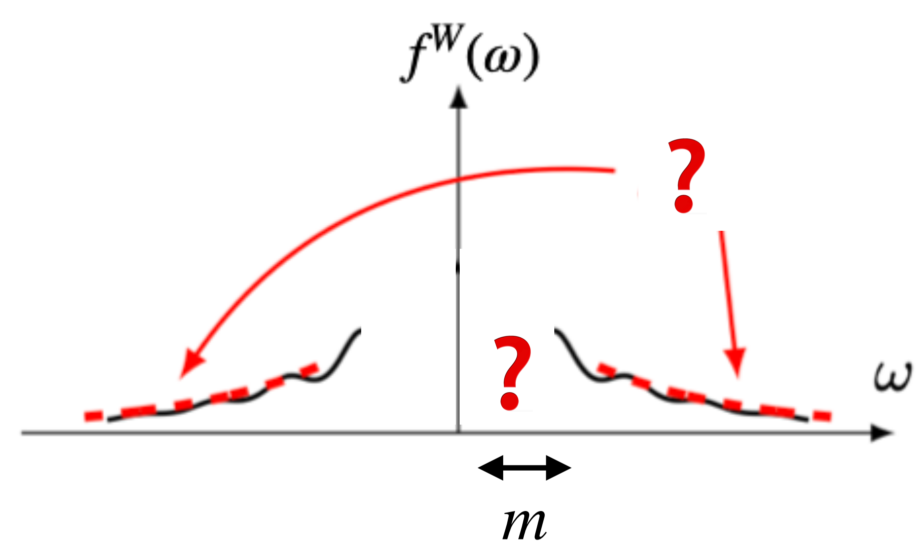
Only if b_n is a smooth function of n, Otherwise

$$\text{Chaos} \overset{?}{\Longleftrightarrow} \quad f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \overset{?}{\Longleftrightarrow} b_n \not\sim \alpha n \overset{?}{\Longleftrightarrow} K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Need to investigate these relations further.

How to extract (chaotic) information from the power spectrum?





Computation method

1812.08657: Parker et al.

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega)$$

$$f^W(\omega) \longrightarrow \mu_{2n}$$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$$\mu_{2n} \longrightarrow b_n$$

$$C(t) = \Pi^W(t) = \varphi_0(t)$$

K-complexity

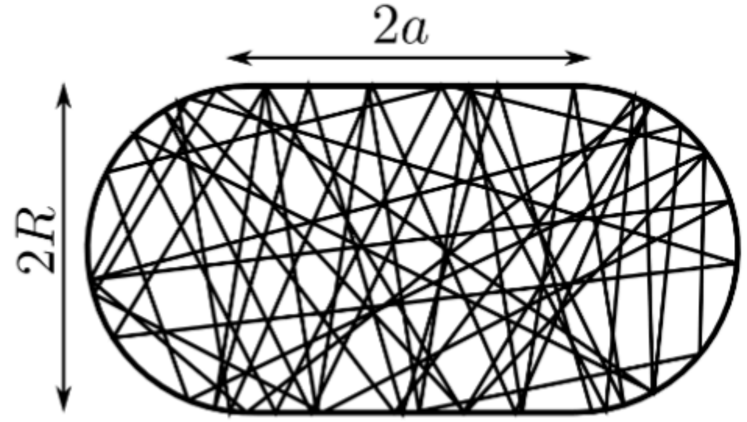
$$\dot{\varphi}_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$

$$\dot{\varphi}_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$

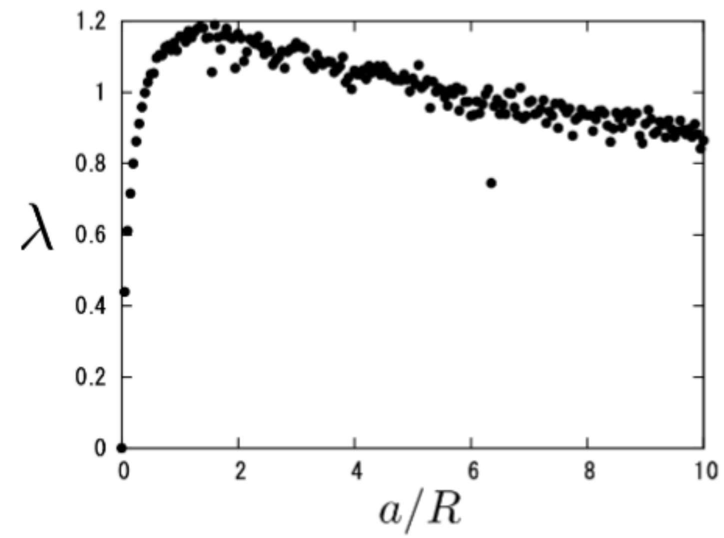
$$\vdots$$

$$\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

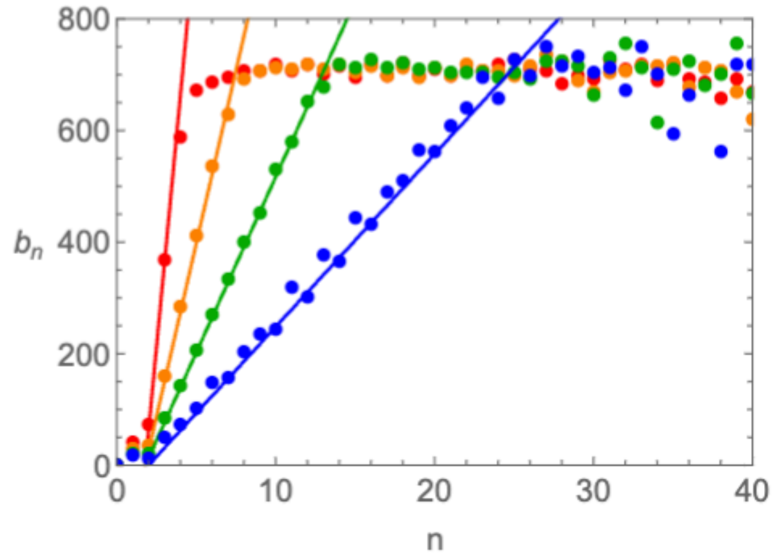
$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$



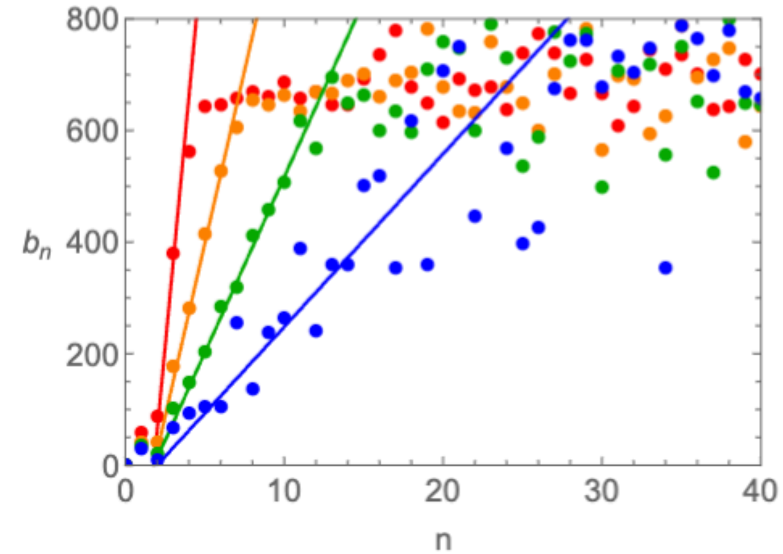
(a) A typical trajectory



(b) Lyapunov exponent



(a) Stadium billiard ($a/R = 1$)



(b) Circle billiard ($a/R = 0$)

Unpublished: Camargo, Jahnke, Jeong, KYK, Nishida

2305.16669: Hashimoto, Murata, Tanahashi, Ryota Watanabe

2112.12128: Rabinovici, Sanchez-Garrido, Shir, Sonner

Power spectrum

$\beta m \gg 1$

$$f^W(\omega) \approx N(m,\beta,d)\, e^{-\beta|\omega|/2} \left(\omega^2 - m^2\right)^{(d-3)/2} \Theta(|\omega| - m) \qquad N(m,\beta,d) = \frac{\pi^{3/2}\beta^{(d-2)/2}}{2^{d-2}\, m^{(d-2)/2} K_{\frac{d-2}{2}}\left(\frac{m\beta}{2}\right)\Gamma\left(\frac{d-1}{2}\right)}$$

$K_n(z)$ is the modified Bessel function of the second kind

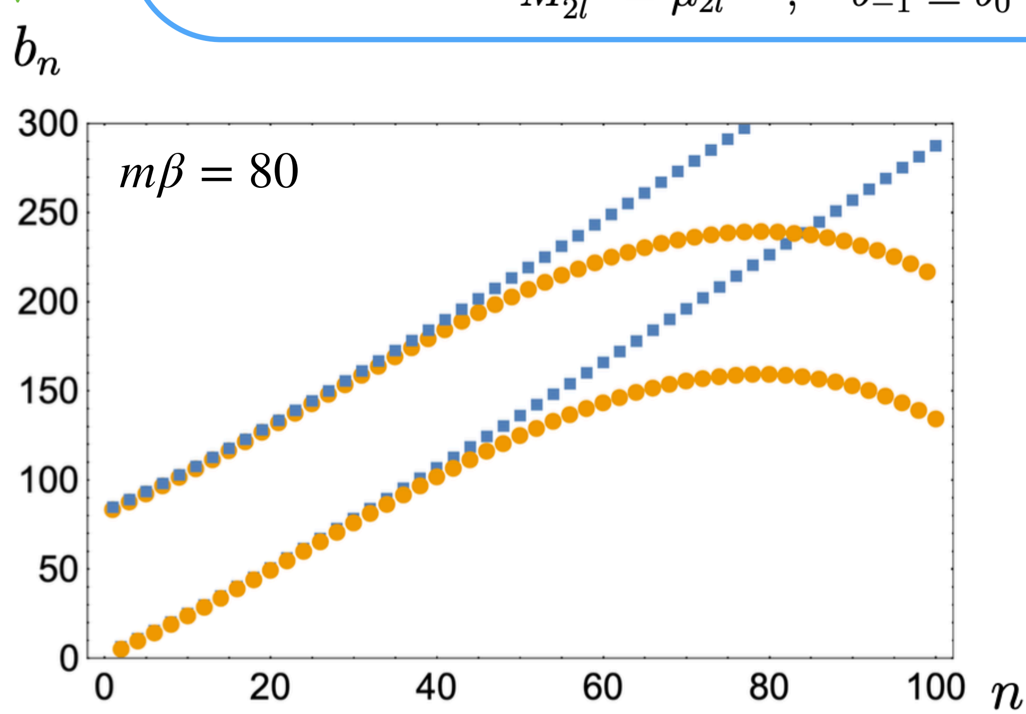
Moments to Lanczos coefficients (d=5)

$\tilde{\Gamma}(n,z)$ is the incomplete Gamma function.

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega^{2n} f^W(\omega) = \frac{2^{-2} e^{\frac{m\beta}{2}}}{2 + m\beta}\left(\frac{2}{\beta}\right)^{2n}\left[-m^2\beta^2\, \tilde{\Gamma}\left(2n+1, \frac{m\beta}{2}\right) + 4\tilde{\Gamma}\left(2n+3, \frac{m\beta}{2}\right)\right]$$

$$b_n = \sqrt{M_{2n}^{(n)}}\,, \qquad M_{2l}^{(j)} = \frac{M_{2l}^{(j-1)}}{b_{j-1}^2} - \frac{M_{2l-2}^{(j-2)}}{b_{j-2}^2} \quad \text{with} \quad l = j, \ldots, n\,,$$

$$M_{2l}^{(0)} = \mu_{2l}\,, \quad b_{-1} \equiv b_0 := 1\,, \quad M_{2l}^{(-1)} = 0\,.$$



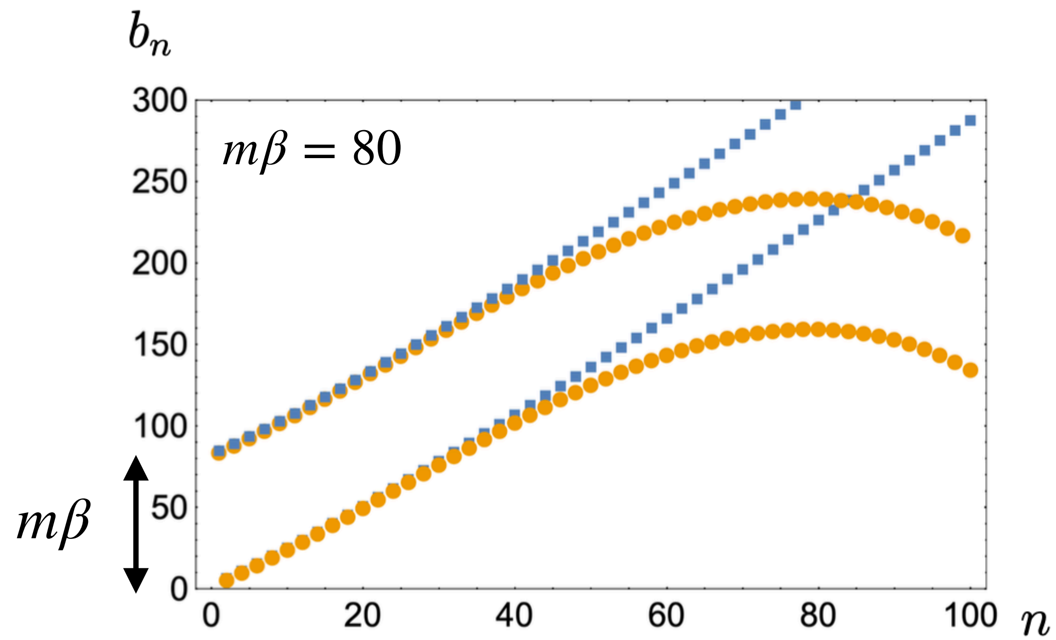
$$\beta^2 b_n^2 = m^2\beta^2 \begin{cases} 1 + 4\frac{1+n}{m\beta} + 8\frac{(n+1)^2}{m^2\beta^2} + 12\frac{(n+1)^3}{m^3\beta^3} + \cdots\,, & \text{for } n \text{ odd}\,, \\ 4\frac{n(n+2)}{m^2\beta^2} + 8\frac{n(n+1)(n+2)}{m^3\beta^3} + \cdots\,, & \text{for } n \text{ even}\,, \end{cases}$$

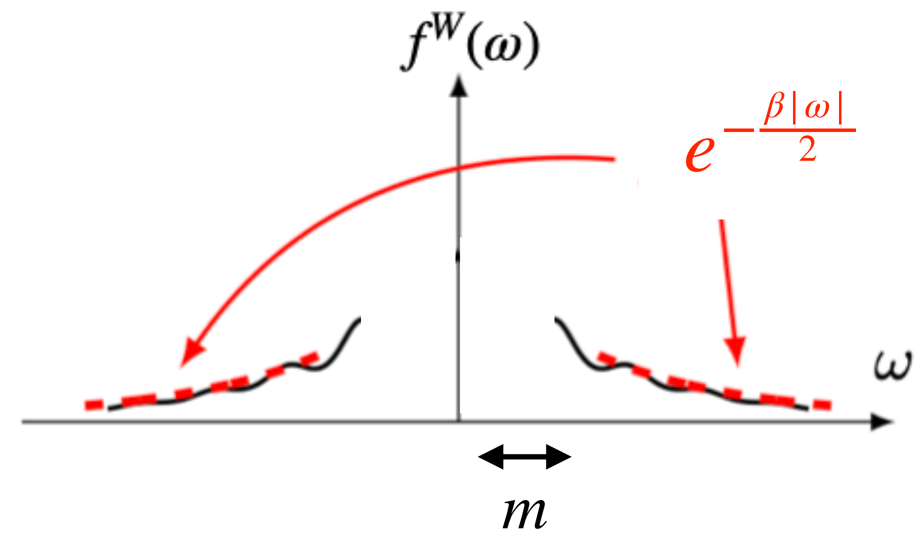
Staggering: two families for even *n* and odd *n*

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

Non-trivial mass (IR-cutoff) effect: staggering

2212.14702: Camargo, Jahnke, KYK, Nishida





Staggering

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
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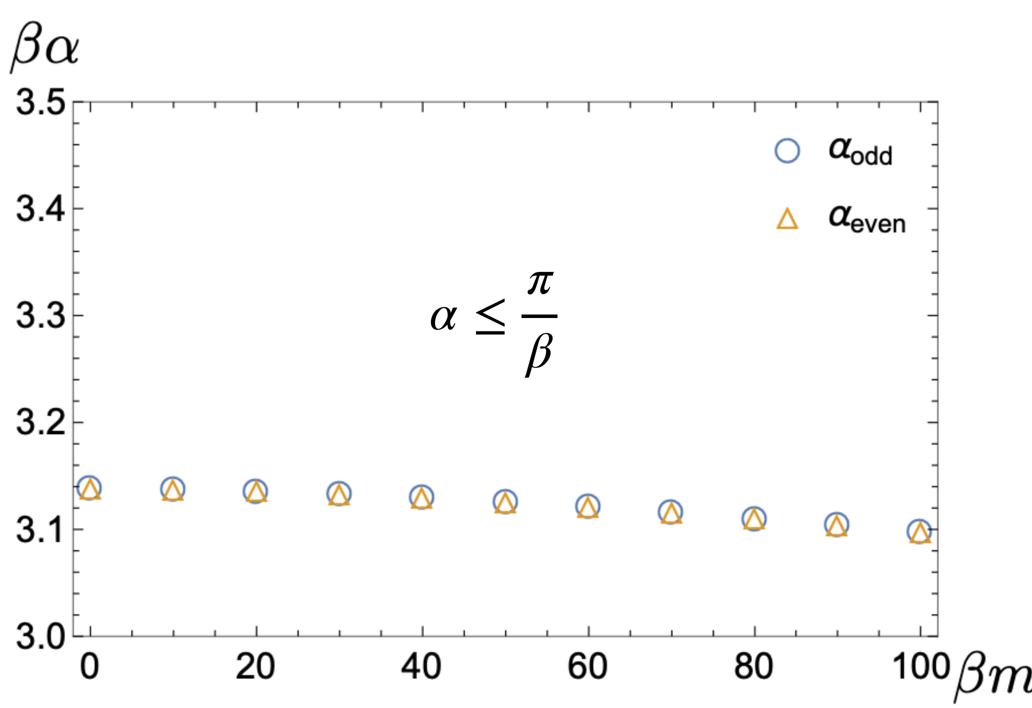
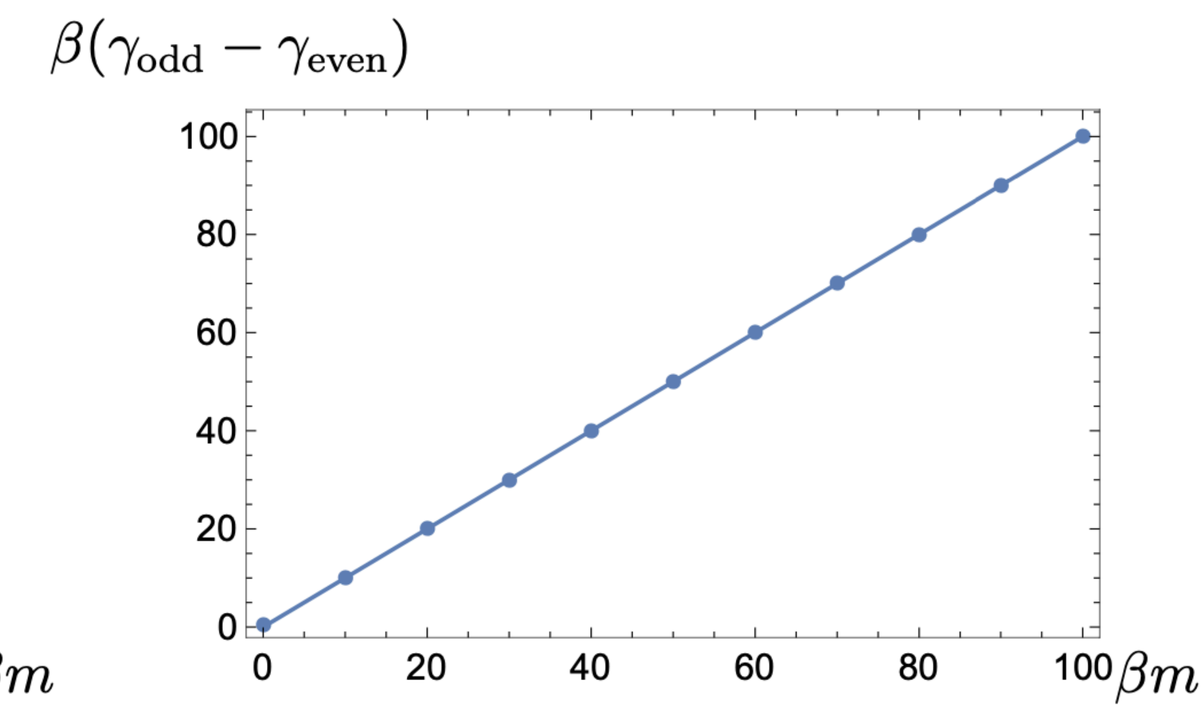


(a) Mass-dependence of α_{odd} and α_{even}



(b) Mass-dependence of $\gamma_{\text{odd}} - \gamma_{\text{even}}$

Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

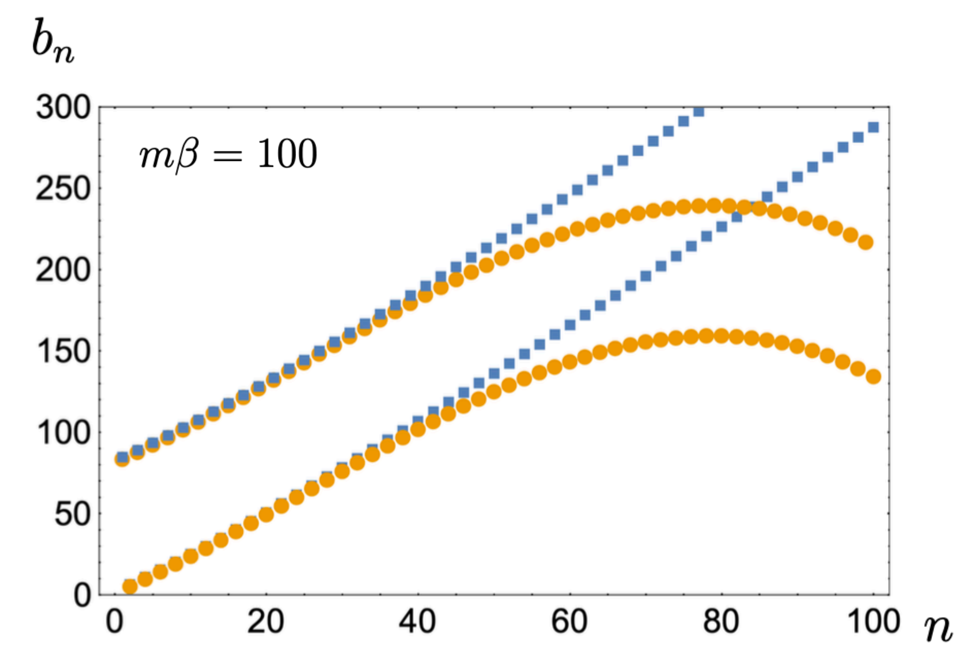
$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$\longrightarrow b_n$

$$C(t) = \Pi^W(t) = \varphi_0(t)$$

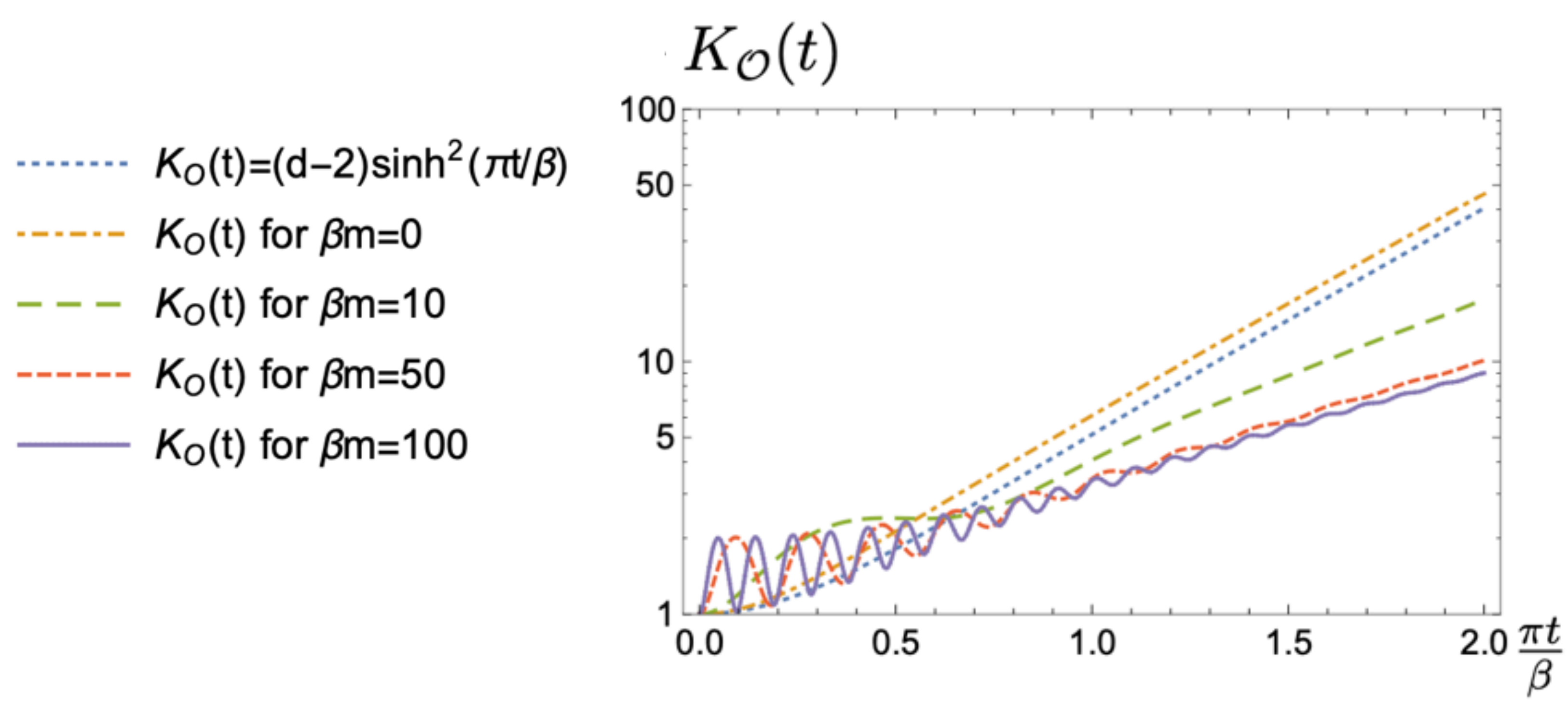
$$C^{(d)}(t) \equiv \varphi_0^{(d)}(t) = c_1^{(d)}(t)\left(c_2^{(d)}(t)\sin(m\,t) + c_3^{(d)}(t)\cos(m\,t)\right)$$



K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$



Non-trivial mass (IR-cutoff) effect: K-complexity

2212.14702: Camargo, Jahnke, KYK, Nishida

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

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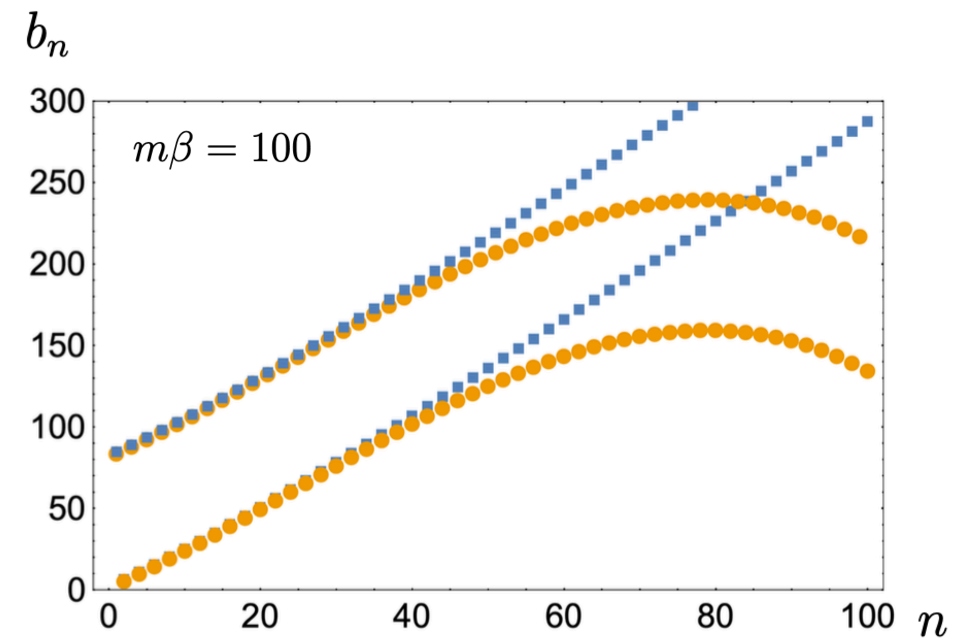
$\mu_{2n} \longrightarrow b_n$

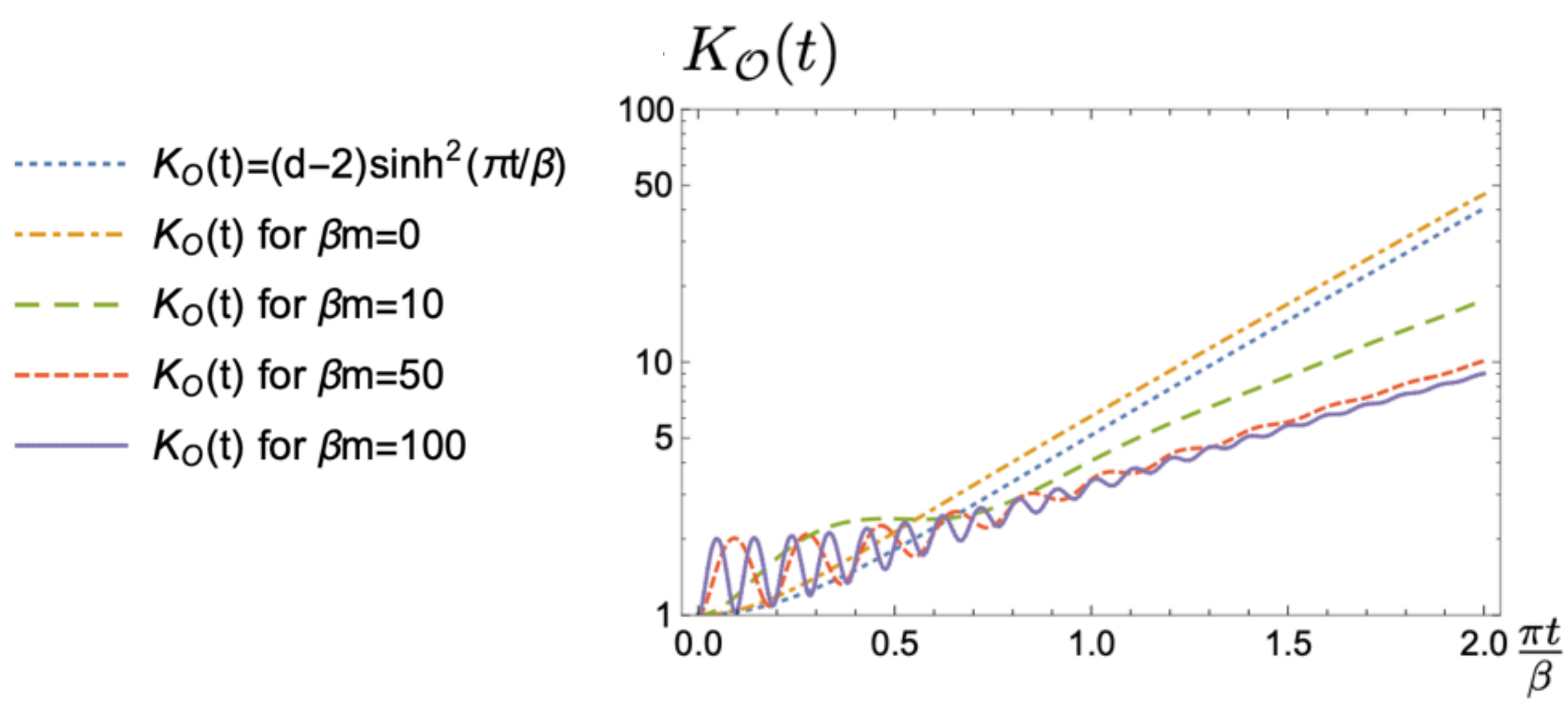
$$C(t) = \Pi^W(t) = \varphi_0(t)$$

$$C^{(d)}(t) \equiv \varphi_0^{(d)}(t) = c_1^{(d)}(t)\left(c_2^{(d)}(t)\sin(m\,t) + c_3^{(d)}(t)\cos(m\,t)\right)$$



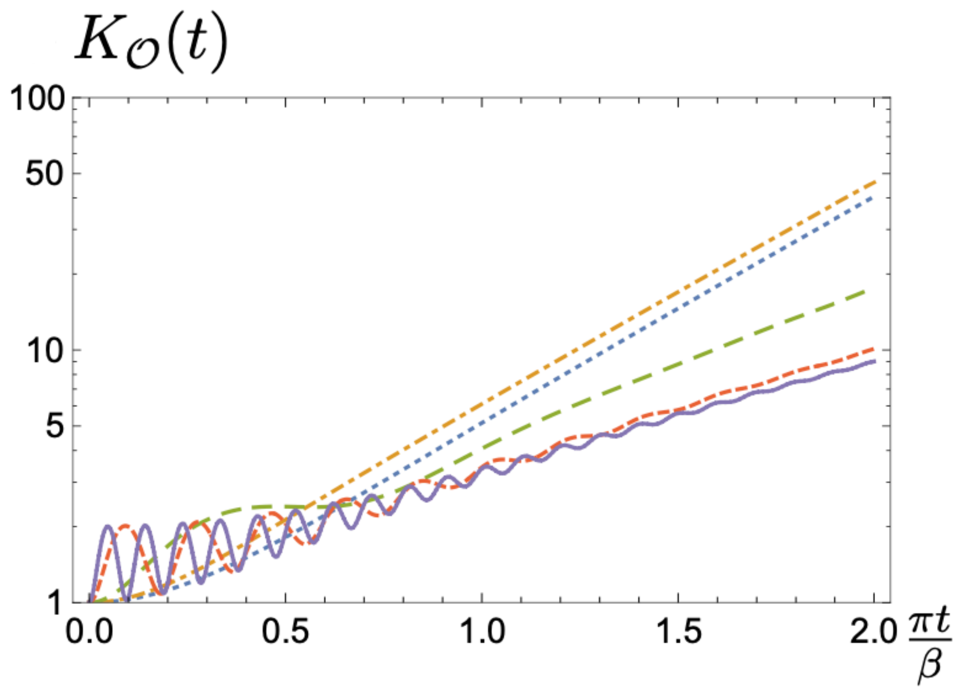
K-complexity

- Early time: oscillation:
  - larger m, shorter period
- Late time: oscillation disappears
  - cancelation due to large n
- Exponential increase
  - larger m, slower increase
  - mass effect

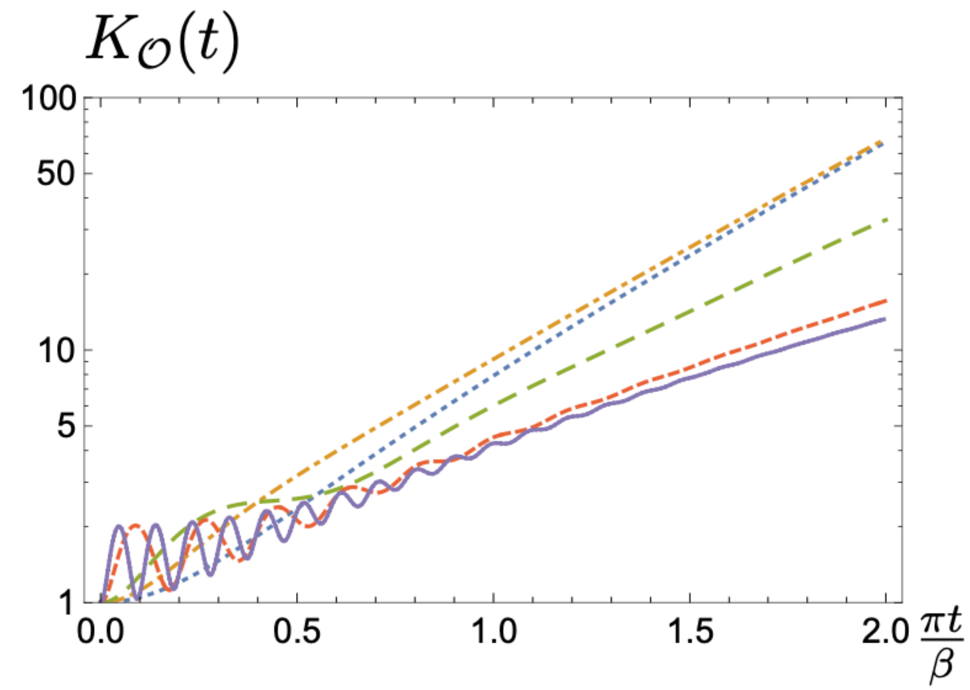


Non-trivial mass (IR-cutoff) effect: K-complexity

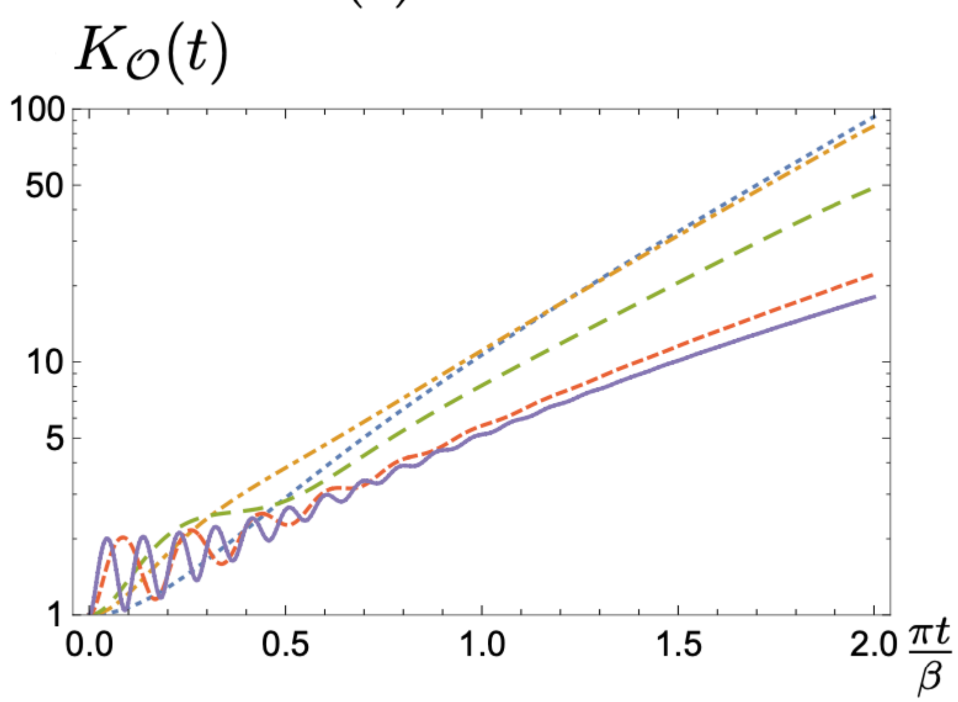
2212.14702: Camargo, Jahnke, KYK, Nishida



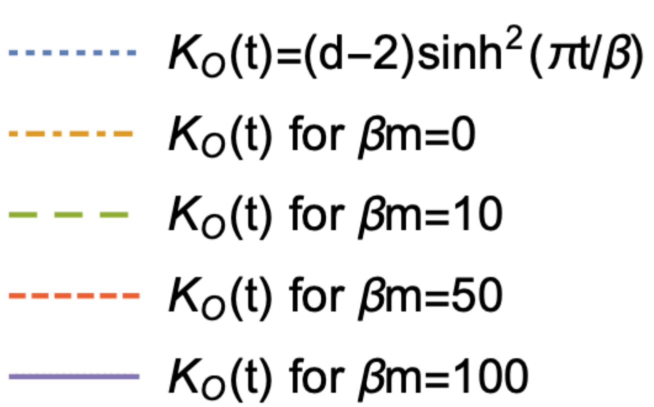
(a) $d = 5$



(b) $d = 7$

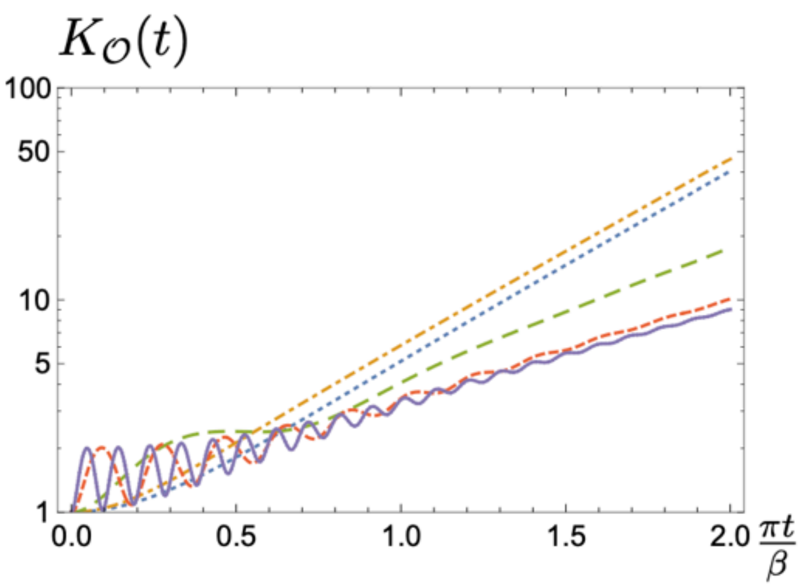


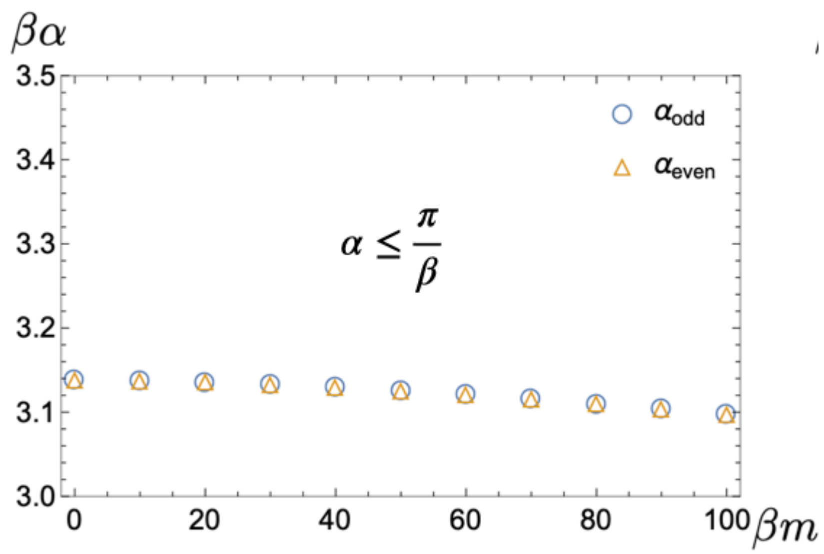
(c) $d = 9$

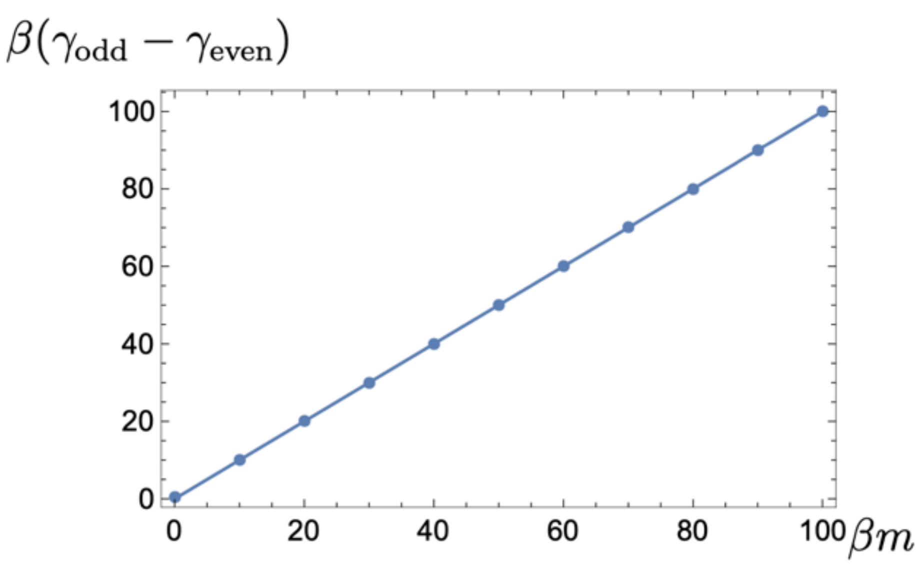


Non-trivial mass (IR-cutoff) effect: K-complexity

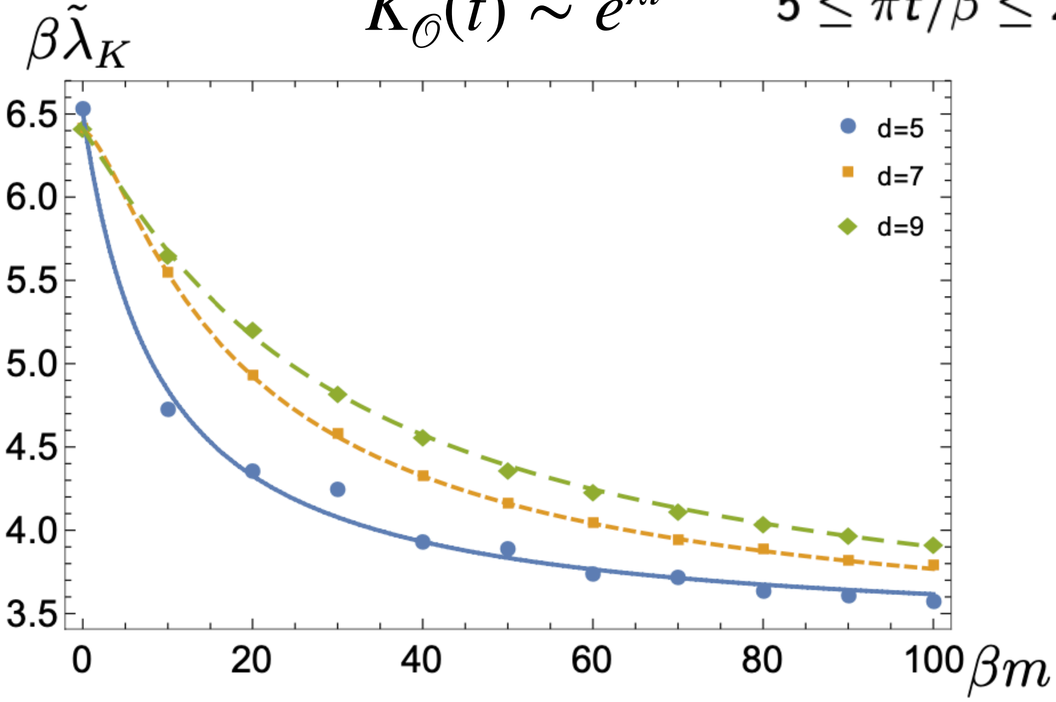
2212.14702: Camargo, Jahnke, KYK, Nishida







$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$ $\quad 5 \leq \pi t/\beta \leq 2.$



$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\rm odd} + \alpha_{\rm even}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\rm odd} - \gamma_{\rm even}|} - \frac{1}{k_3^{(d)}}\right) + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\rm odd} - \gamma_{\rm even}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$

Staggering

$$b_n \sim \alpha_{\rm odd}\, n + \gamma_{\rm odd} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\rm even}\, n + \gamma_{\rm even} \quad (\text{even } n)$$

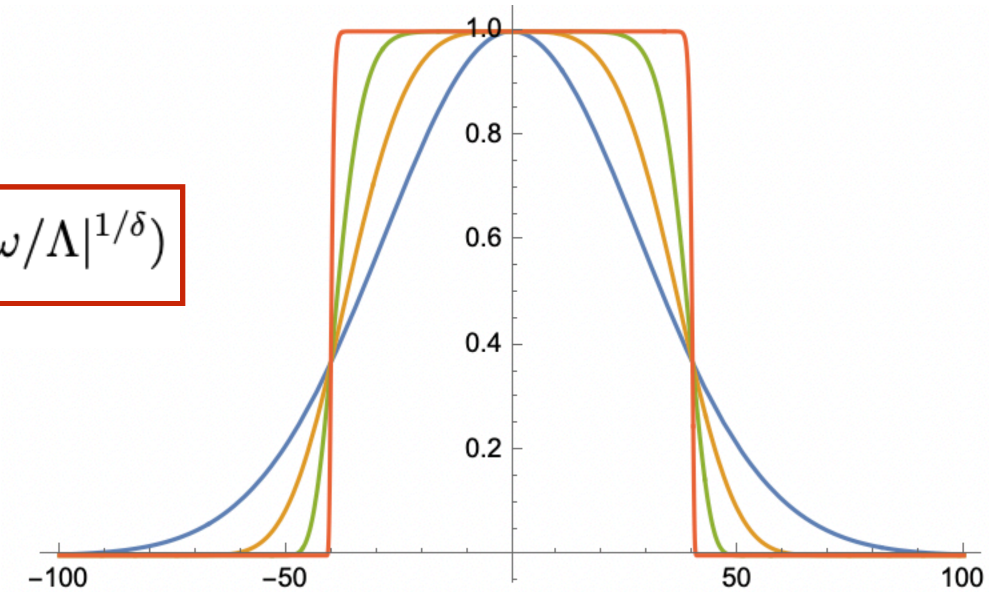
$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

Non-trivial UV-cutoff effect

2212.14702: Camargo, Jahnke, KYK, Nishida

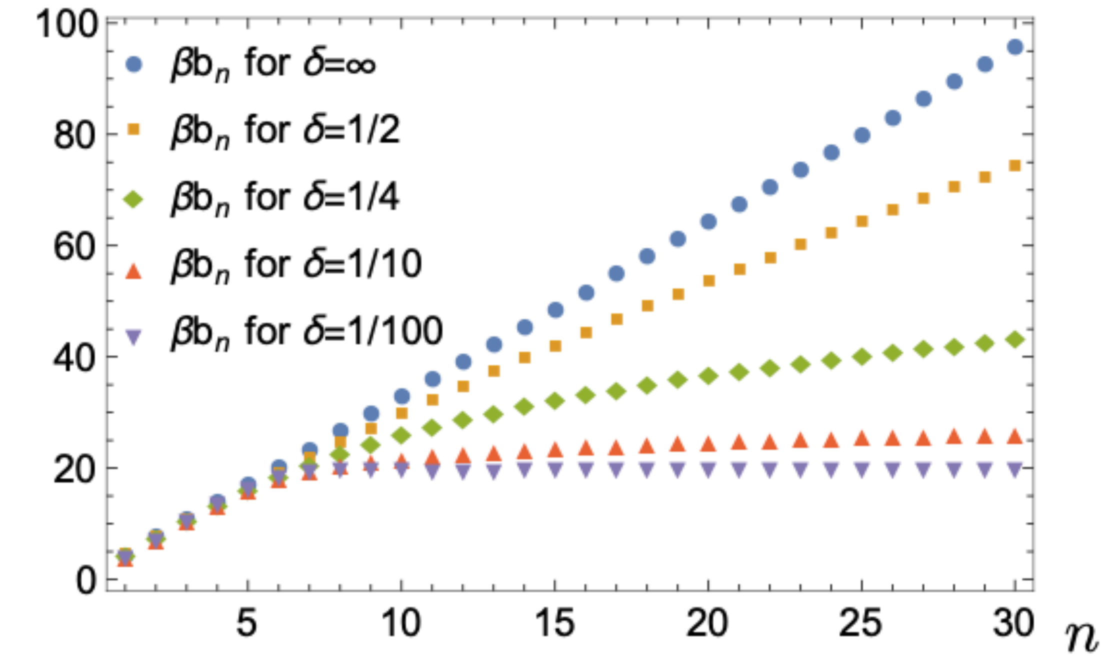
m=0, d=4

$$f^W(\omega) = N(\beta,\Lambda,\delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\exp(-|\omega/\Lambda|^{1/\delta})$$

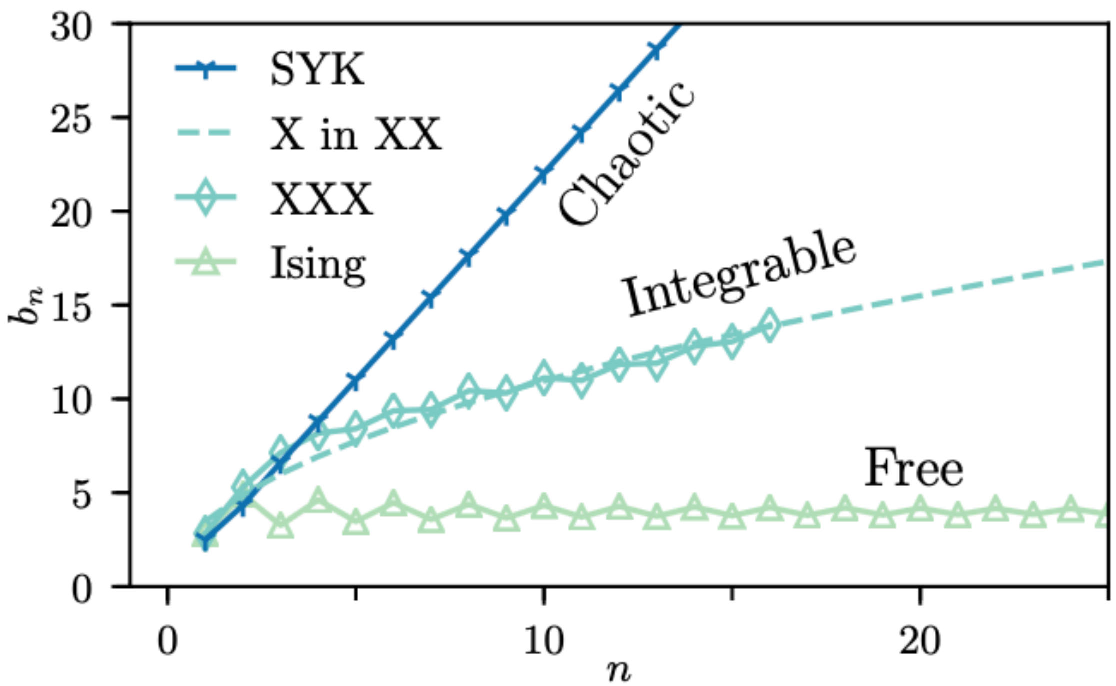


$$b_n \sim n^\delta \Longleftrightarrow f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$

$\delta \leq 1$

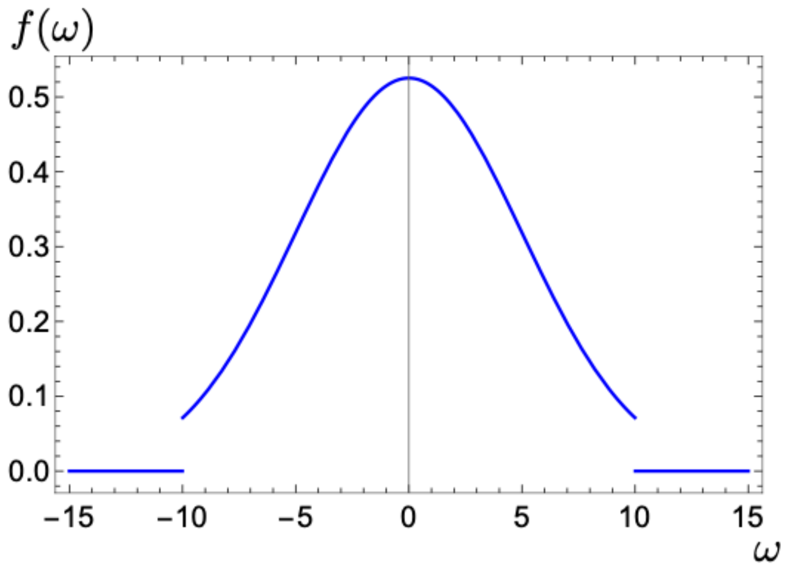


$\beta\Lambda = 40.$

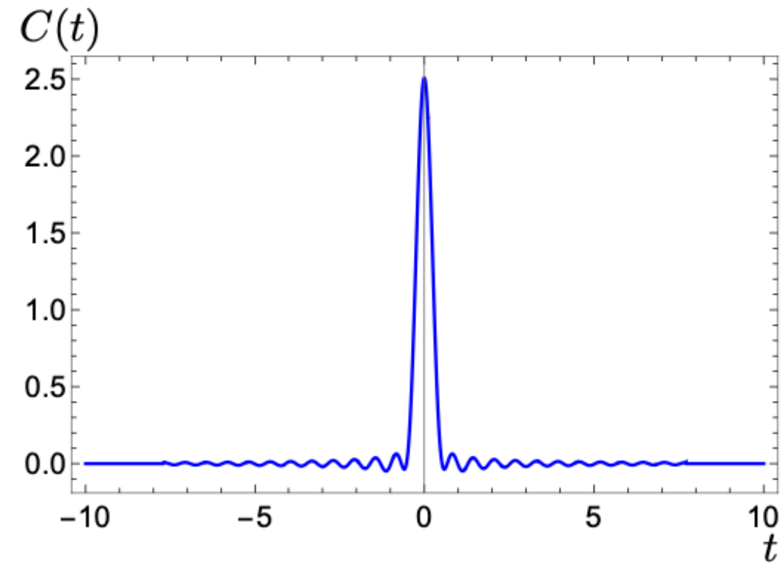


Condition for staggering

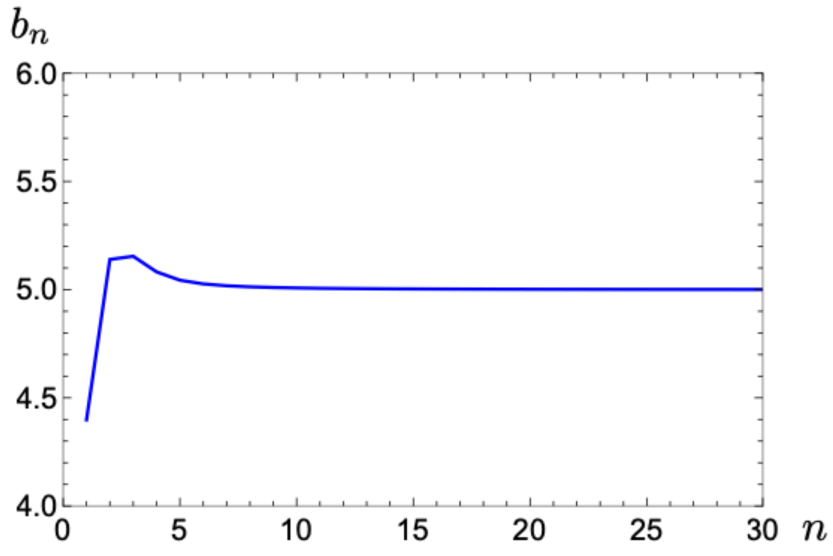
$$f(\omega) = \frac{\sqrt{2\pi}}{\sigma\,\mathrm{Erf}\left(\frac{\Lambda}{\sqrt{2}\sigma}\right)} \begin{cases} e^{-\frac{\omega^2}{2\sigma^2}} & \text{if } |\omega| \leq \Lambda \\ 0 & \text{if } |\omega| > \Lambda \end{cases}$$



(a)



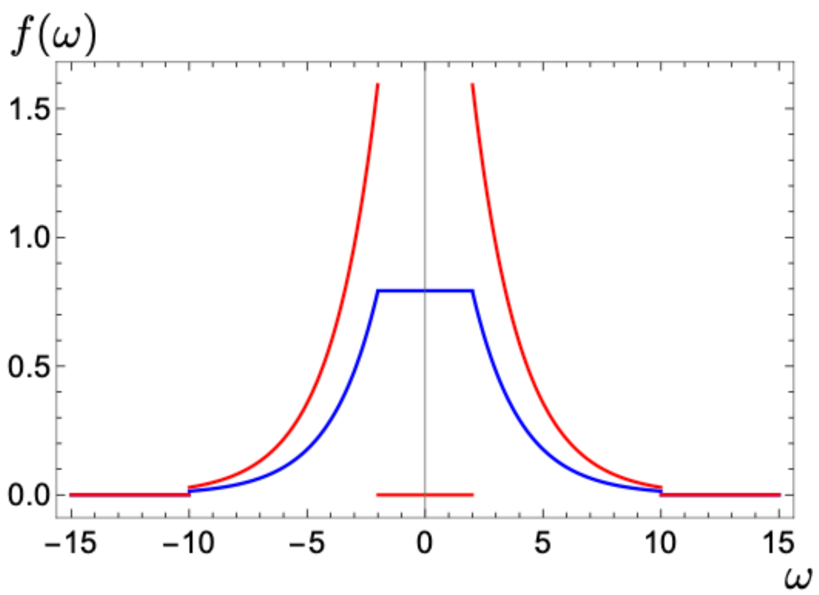
(b)

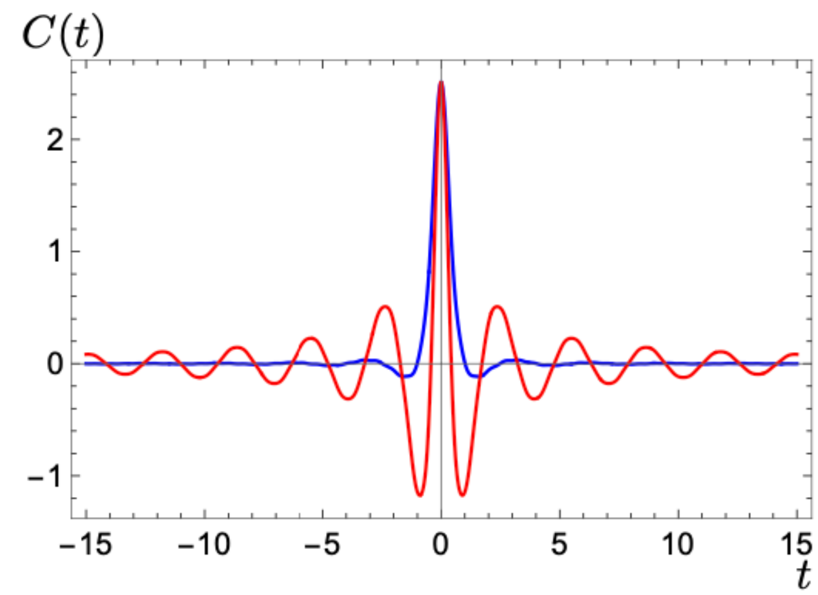


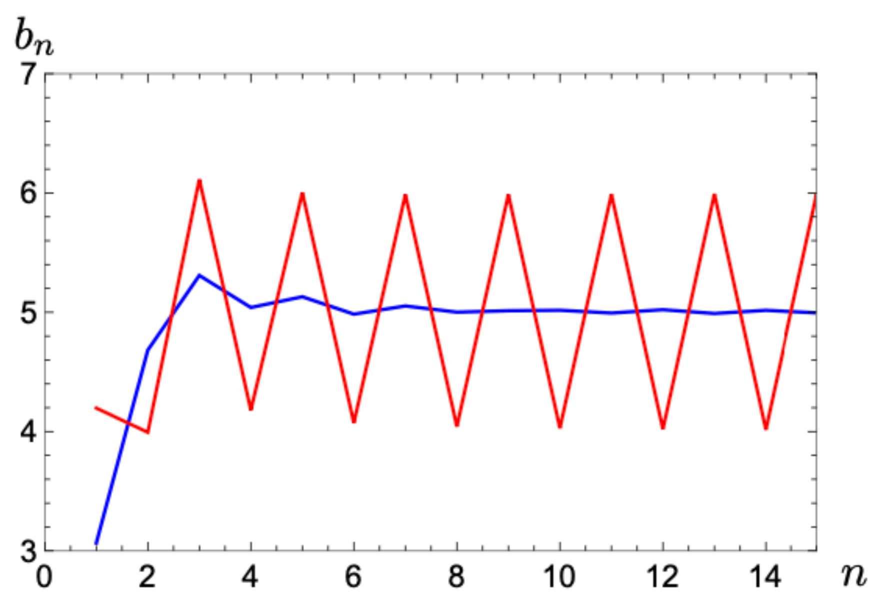
Condition for staggering

2212.14702: Camargo, Jahnke, KYK, Nishida

$$f(\omega) = \frac{\pi a e^{a(\Lambda+m)}}{e^{a\Lambda}(ahm+1) - e^{am}} \begin{cases} h\, e^{-am} & \text{if } |\omega| \leq m \\ e^{-a|\omega|} & \text{if } m < |\omega| < \Lambda \\ 0 & \text{if } |\omega| \geq \Lambda \end{cases}$$

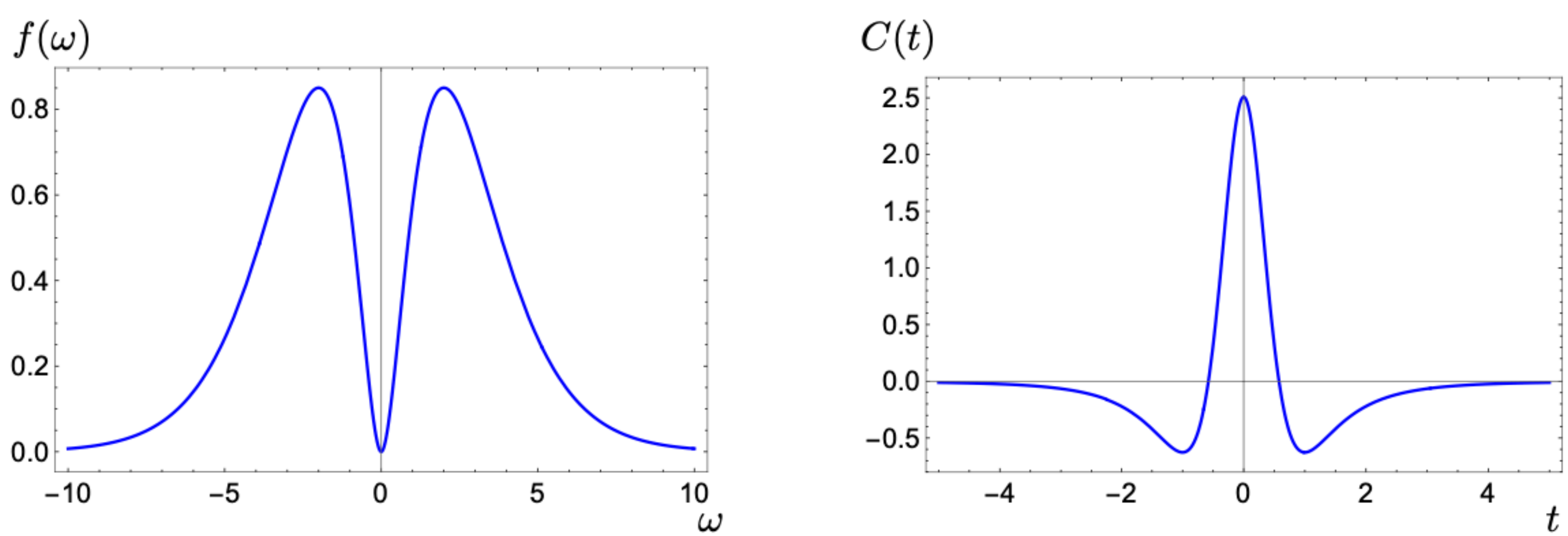




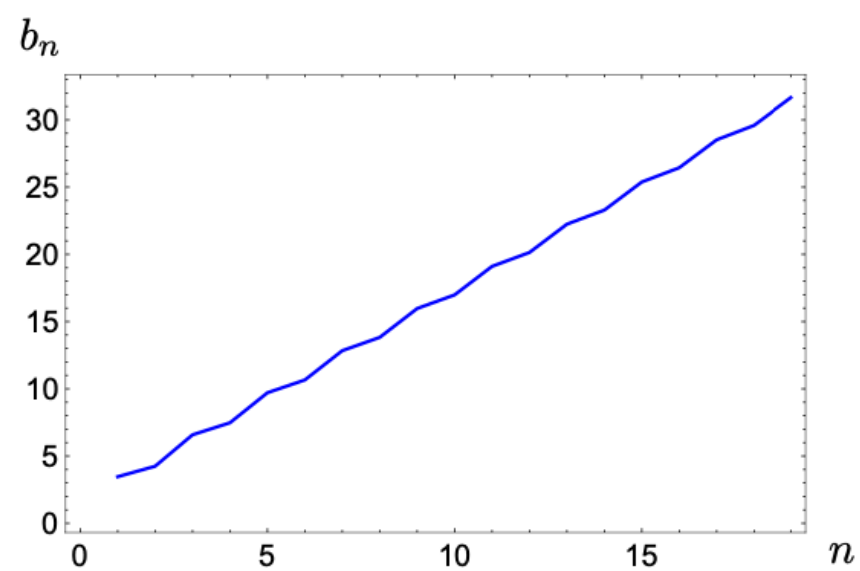


Condition for staggering

$$f(\omega) = N(\omega_0, \delta, \lambda) \left| \frac{\omega}{\omega_0} \right|^{\lambda} e^{-\left|\frac{\omega}{\omega_0}\right|^{\frac{2}{\delta}}}$$



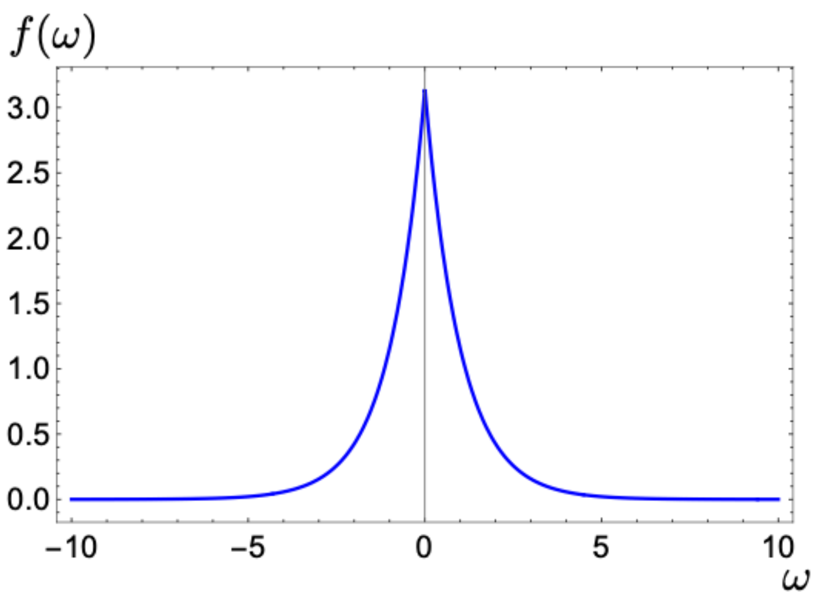
(a) (b)



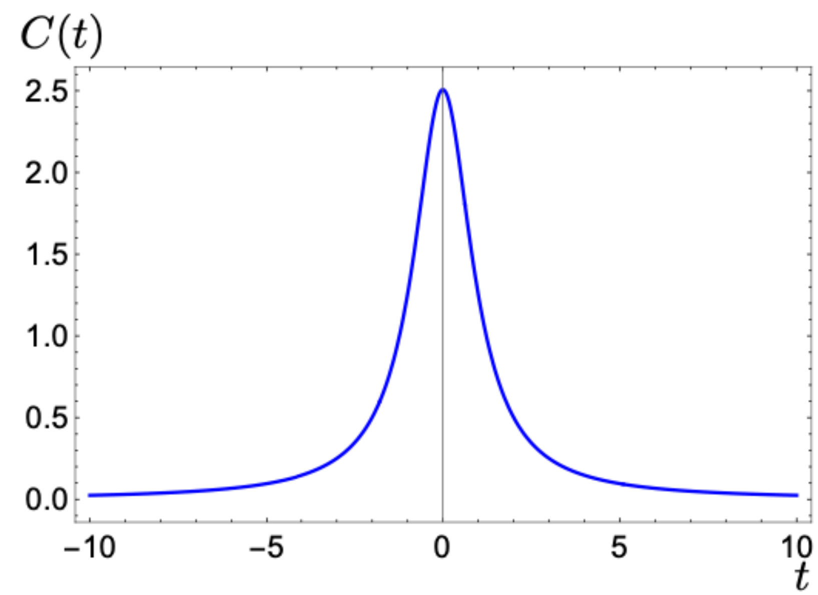
Condition for staggering

2212.14702: Camargo, Jahnke, KYK, Nishida

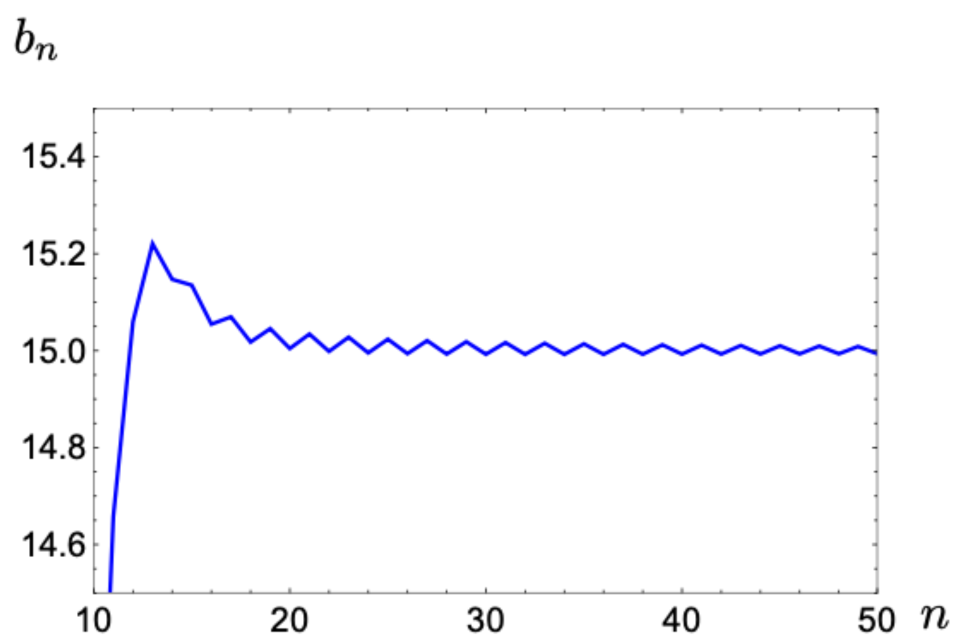
$$f(\omega) = \begin{cases} N(\omega_0, \Lambda) e^{-\left|\frac{\omega}{\omega_0}\right|} & \text{if } |\omega| \leq \Lambda \\ 0 & \text{if } |\omega| > \Lambda \end{cases}$$



(a)



(b)



Summary (method)

- Is it possible to extract the chaos-info from a C(t) or the power spectrum?

Lanczos coefficients

$$\mu_{2n} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

$f^W(\omega) \longrightarrow \mu_{2n}$

$$\Pi^W(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} f^W(\omega)$$

$$b_1^{2n}\cdots b_n^2 = \det\left(\mu_{i+j}\right)_{0\le i,j\le n}$$

$f^W(\omega) \downarrow\; C(t) = \Pi^W(t) = \varphi_0(t)$

$\mu_{2n} \downarrow\; b_n$

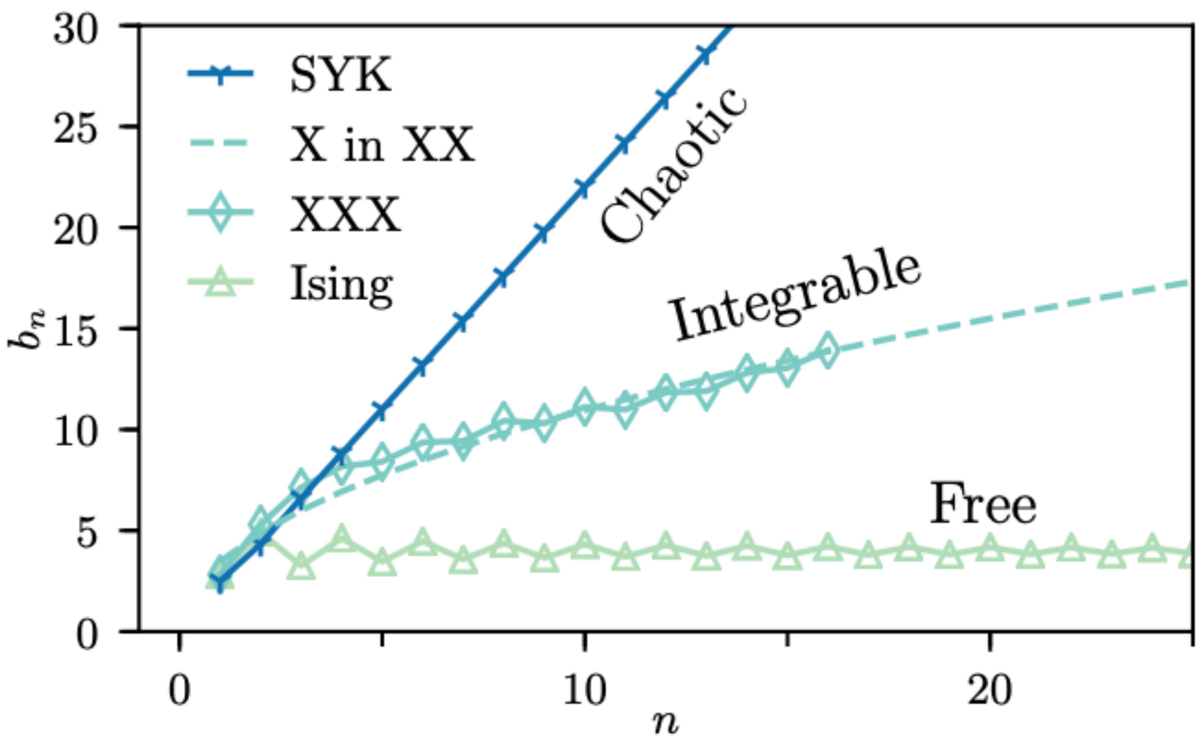
K-complexity

$$\dot\varphi_0(t) = b_0 \overbrace{\varphi_{-1}(t)}^{=0} - b_1\varphi_1(t)$$
$$\dot\varphi_1(t) = b_1\varphi_0(t) - b_2\varphi_2(t)$$
$$\vdots$$
$$\dot\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$K_{\mathcal{O}}(t) = \sum_{n=1}^{n_{\max}} n\,|\varphi_n(t)|^2, \quad n_{\max} = 200.$$

Summary (Lattice systems)

- Is it possible to extract the chaos-info from a C(t) or a power spectrum?
  - Seems to be possible for Lattice systems.



Universal operator growth hypothesis

the slowest possible decay of the power spectrum

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}}$$

Krylov complexity grows exponentially

$$K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

In a chaotic quantum system

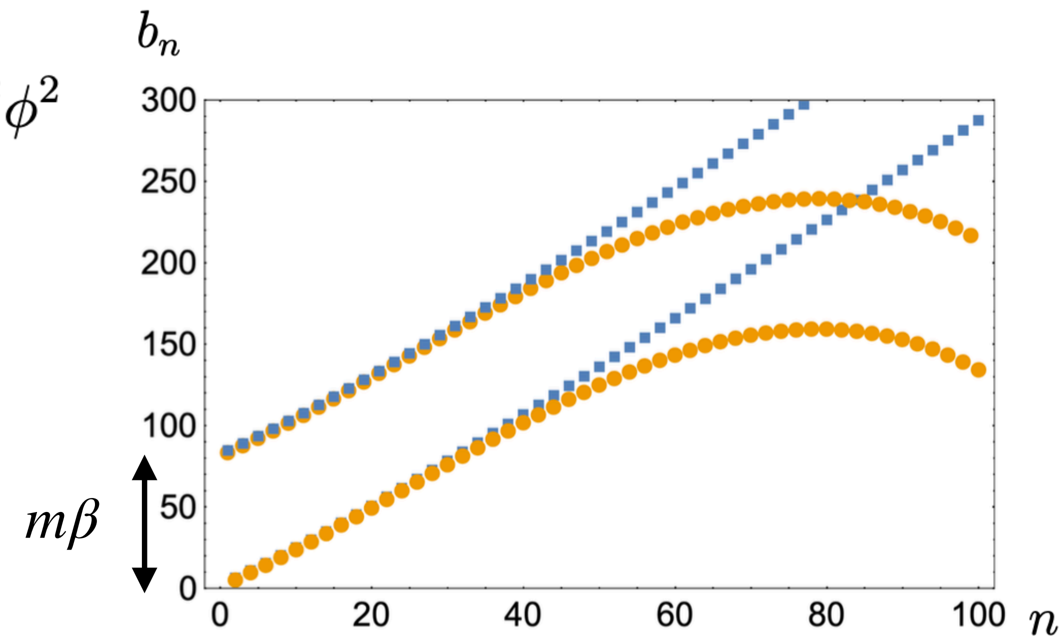
Lanczos coefficients $\{b_n\}$ grow as fast as possible

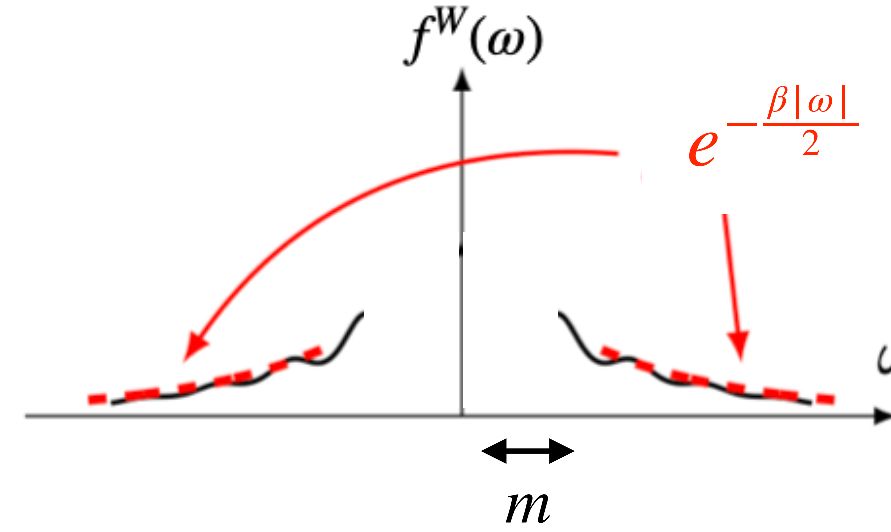
$$b_n \sim \alpha n$$

- Subtleties in saddle point
- Subtleties in QFT

Summary (QFT)

$$\mathcal{L}_E^{\text{free}} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2$$

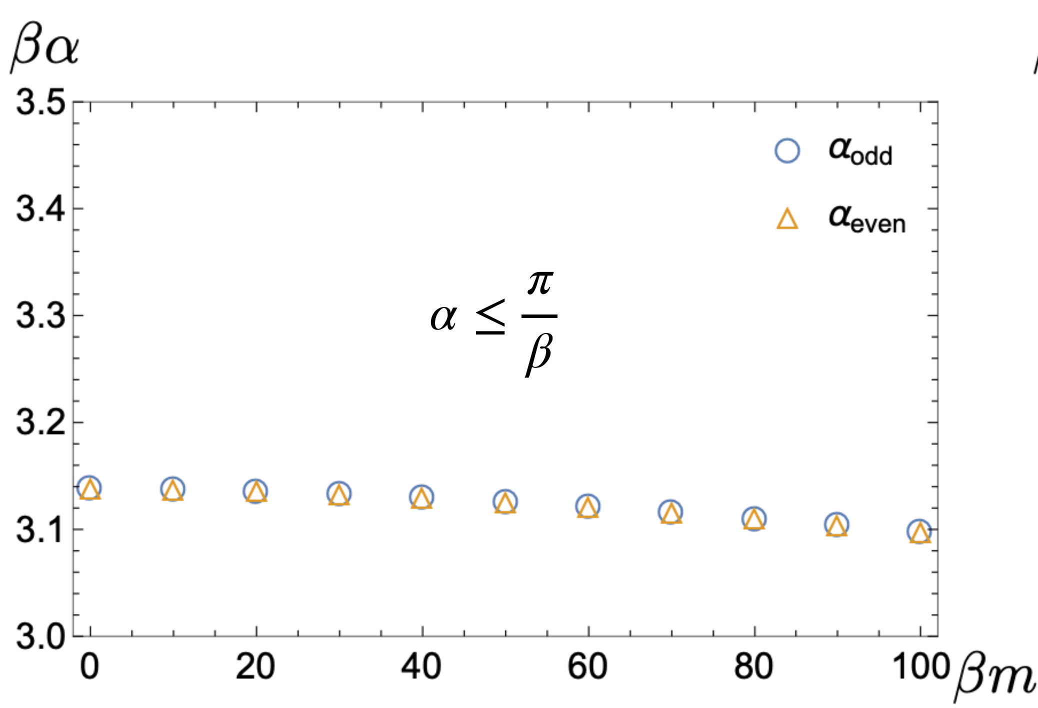




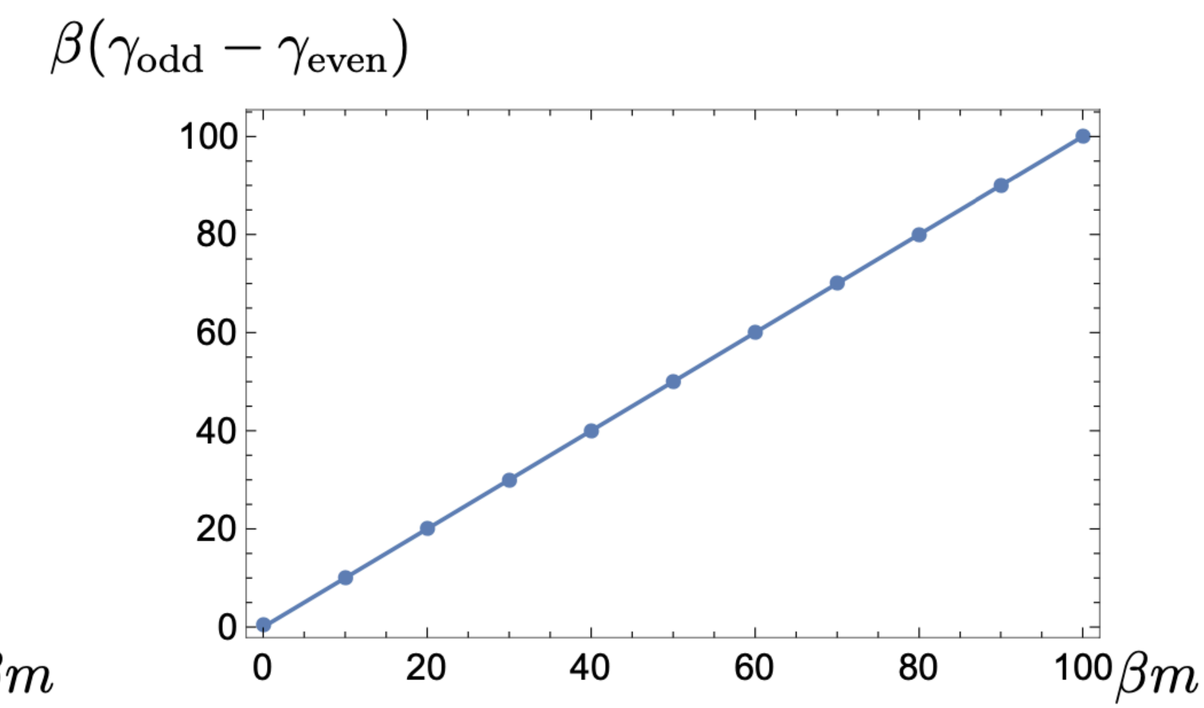
Staggering

$$b_n \sim \alpha_{\text{odd}}\, n + \gamma_{\text{odd}} \quad (\text{odd } n)$$
$$b_n \sim \alpha_{\text{even}}\, n + \gamma_{\text{even}} \quad (\text{even } n)$$

Need to take into account
Low frequency behavior

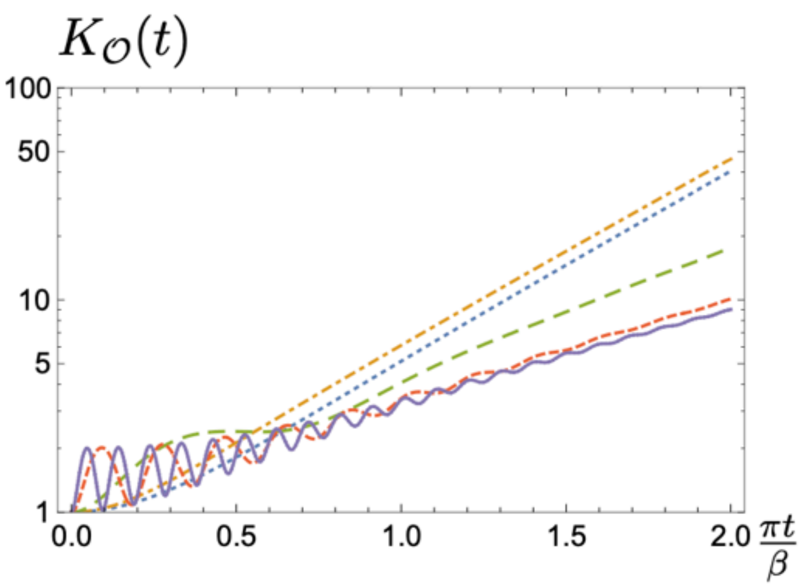


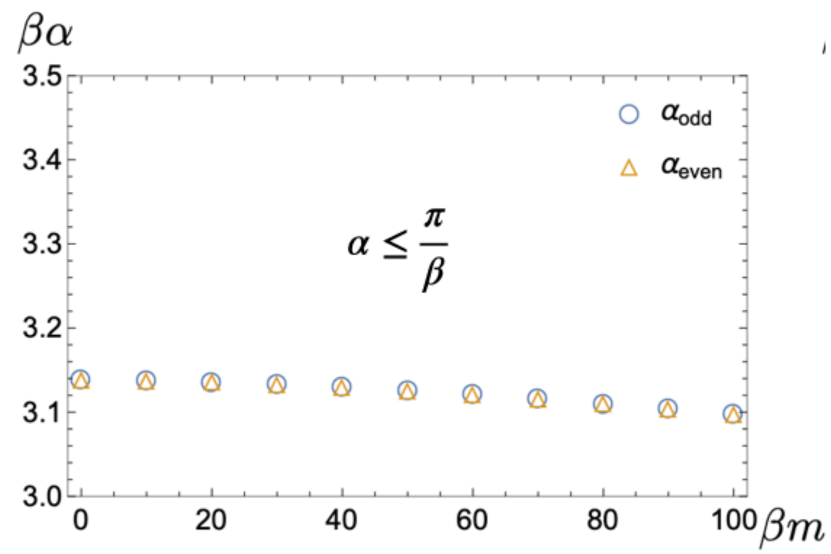
(a) Mass-dependence of α_{odd} and α_{even}

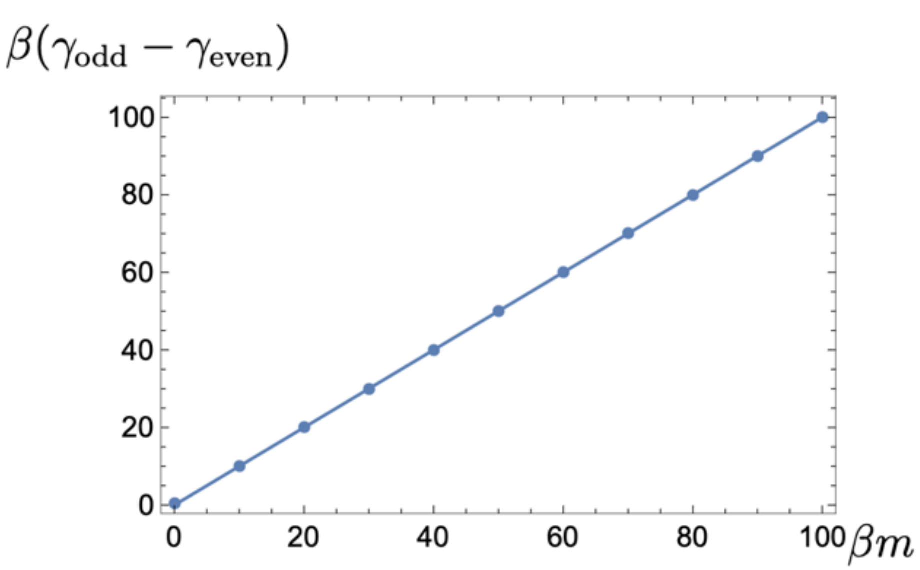


(b) Mass-dependence of $\gamma_{\text{odd}} - \gamma_{\text{even}}$

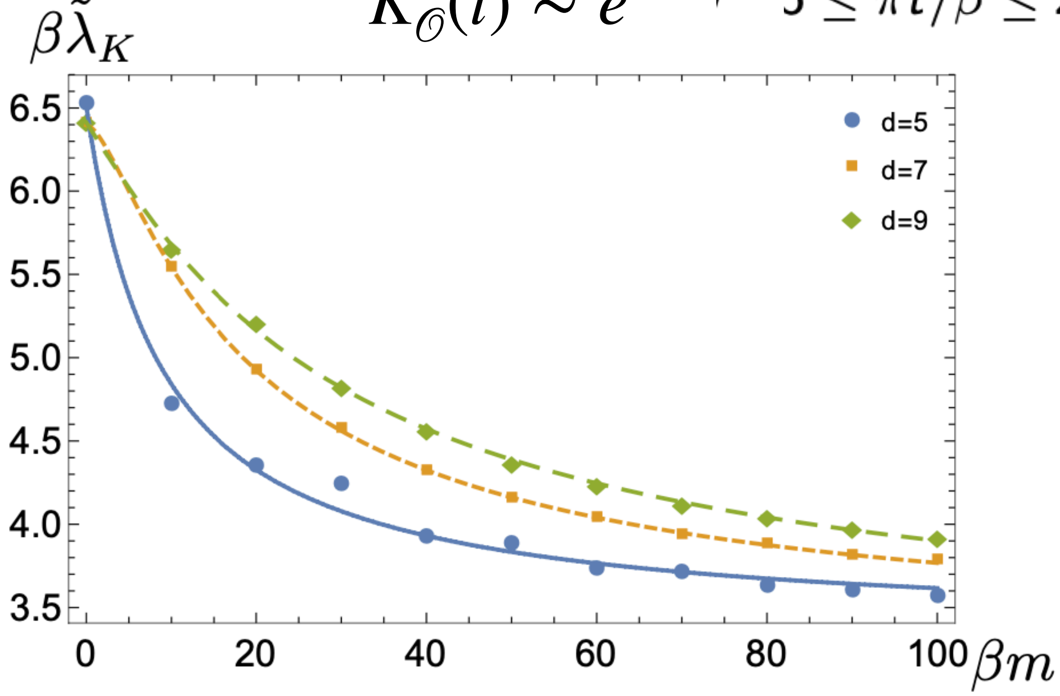
Summary (QFT)







$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda} t}$, $5 \leq \pi t/\beta \leq 2.$



$$\beta\tilde{\lambda}_K^{(d)} = \beta(\alpha_{\text{odd}} + \alpha_{\text{even}}) + k_2^{(d)}\left(\frac{1}{k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|} - \frac{1}{k_3^{(d)}}\right) + k_4^{(d)}\left(\frac{1}{\left(k_3^{(d)} + \beta|\gamma_{\text{odd}} - \gamma_{\text{even}}|\right)^2} - \frac{1}{(k_3^{(d)})^2}\right),$$

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \iff b_n \sim \alpha n \iff K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

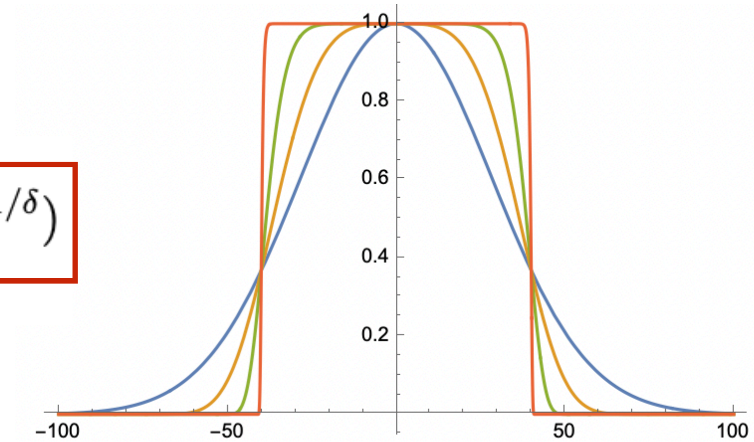
Only if b_n is a smooth function of n, Otherwise

$$f^W(\omega) \sim e^{-\frac{\pi|\omega|}{2\alpha}} \not\Leftrightarrow b_n \not\sim \alpha n \not\Leftrightarrow K_{\mathcal{O}}(t) \sim e^{2\alpha t}$$

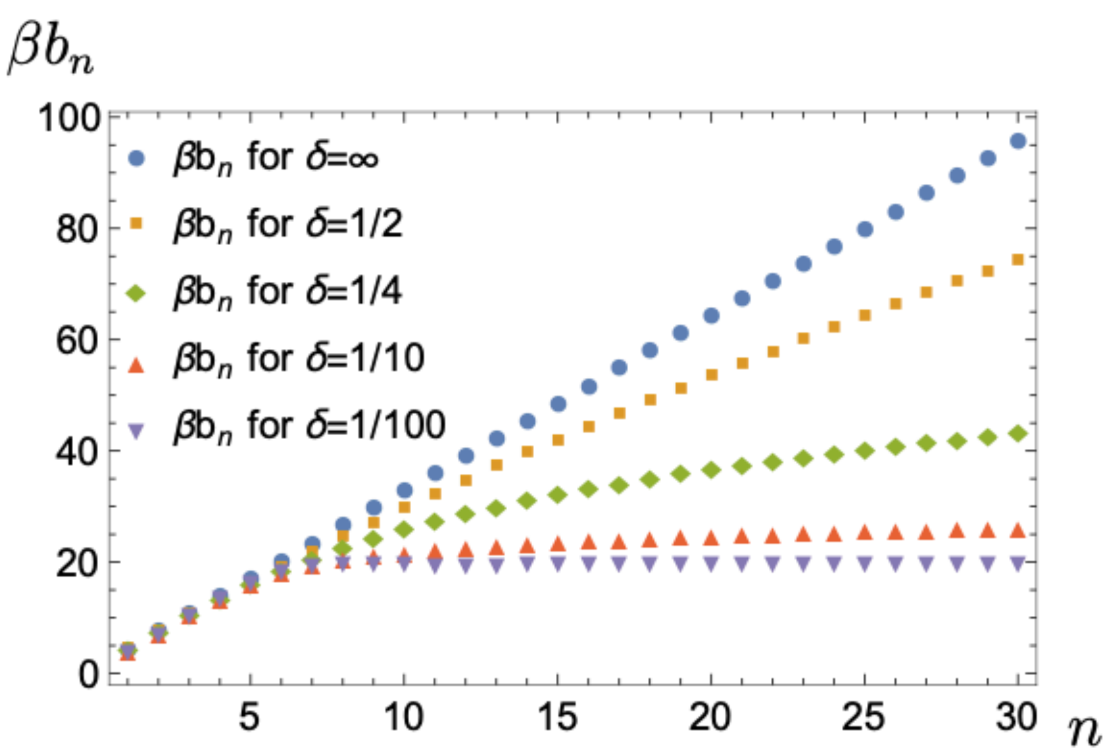
Summary (QFT)

m=0, d=4

$$f^W(\omega) = N(\beta,\Lambda,\delta)\frac{\omega}{\sinh(\frac{\beta\omega}{2})}\boxed{\exp(-|\omega/\Lambda|^{1/\delta})}$$

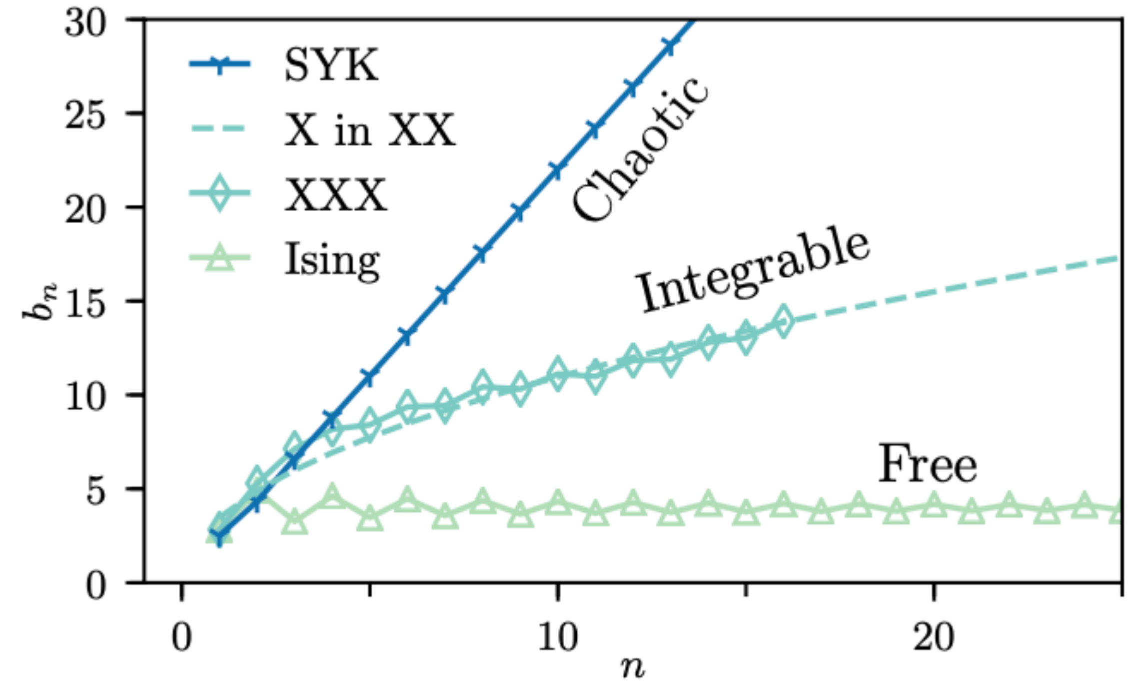


$$b_n \sim n^\delta \iff f^W(\omega) \sim \exp(-|\omega/\omega_0|^{1/\delta})$$



$\beta\Lambda = 40.$

$\delta \leq 1$



- Is it possible to extract the chaos-info from a C(t) or a power spectrum?
- More scales: compact space, interaction, other spins, open systems etc
- Holographic counterpart?
- State (spread) complexity?
- Observations, conjectures, mathematical justification