Project:

SOME ASPECTS OF GAUSSIAN QUANTUM MARKOV SEMIGROUPS

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Fock Space

We work on the **Hilbert space** $\mathcal{H} = \Gamma(\mathbb{C}^d) \simeq \Gamma(\mathbb{C}) \otimes \cdots \otimes \Gamma(\mathbb{C})$.
$(e(\alpha) = e(\alpha_1, \ldots, \alpha_d))_\alpha$ the canonical orthonormal basis.
We consider **annihilation and creation operators** $a_j, a_j^\dagger$

$$a_j e(\alpha_1, \ldots, \alpha_d) = \sqrt{\alpha_j} e(\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_d)$$
$$a_j^\dagger e(\alpha_1, \ldots, \alpha_d) = \sqrt{\alpha_j + 1} e(\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_d).$$

which are **unbounded operators**.

They satisfy $\left[a_j, a_k^\dagger\right] = \delta_{jk} \mathbb{1}$ and $a_j^* = a_j^\dagger$.

Quantum Markov Semigroups

The evolution is usually given on $\mathcal{B}(\mathcal{H})$ via a **QMS** $\mathcal{T} = (\mathcal{T}_t)_t$. Consider the generator of $\mathcal{T}$ in the **GKLS form**

$$\mathcal{L}(x) = \mathrm{i}\,[H, x] + \frac{1}{2}\sum_{\ell=1}^{d}\left(2L_\ell^* x L_\ell - \{L_\ell^* L_\ell, x\}\right), \quad x \in \mathcal{B}(\mathsf{h})$$

We want to consider

$$H = \text{ quadratic polynomial in } a_j, a_j^\dagger,$$
$$L_\ell = \text{ linear polynomial in } a_j, a_j^\dagger.$$

For $\xi = \sum_\alpha \xi_\alpha e(\alpha)$, $x \in \mathcal{B}(\mathcal{H})$ we define

$$\begin{aligned}\pounds(x)[\xi', \xi] &= \mathrm{i}\left\langle H\xi', x\xi\right\rangle - \mathrm{i}\left\langle \xi', xH\xi\right\rangle \\ &\quad + \frac{1}{2}\sum_{\ell}\left[2\left\langle L_\ell\xi', xL_\ell\xi\right\rangle - \left\langle \xi', xL_\ell^* L_\ell\xi\right\rangle - \left\langle L_\ell^* L_\ell\xi', x\xi\right\rangle\right]\end{aligned}$$

Gaussian QMS

$$H = \sum_{j,k=1}^{d} \left[\Omega_{jk} a_j^\dagger a_k + \frac{\kappa_{jk}}{2} a_j^\dagger a_k^\dagger + \frac{\overline{\kappa_{jk}}}{2} a_j a_k \right] + \sum_{j=1}^{d} \left[\frac{\zeta_j}{2} a_j^\dagger + \frac{\overline{\zeta_j}}{2} a_j \right]$$

$$L_\ell = \sum_{j=1}^{d} \overline{v}_\ell a_j + u_\ell a_j^\dagger, \quad \ell = 1, \ldots, m$$

with $\Omega = \Omega^*$, $\kappa = \kappa^T$ and $\ker(U^*) \cap \ker(V^T) = \{0\}$.

Theorem (Agredo, Fagnola, Poletti (2021))

We can construct a unique QMS, $\mathcal{T} = (\mathcal{T}_t)_t$ such that

$$\frac{d}{dt} \left\langle \xi', \mathcal{T}_t(x)\xi \right\rangle \Big|_{t=0} = \pounds(x)[\xi', \xi].$$

We call it ***Gaussian QMS associated with*** *H, L_ℓ.*

Gaussian states

$W(z)$ are the Weyl operators, defined by

$$W(z) := e^{\sum_{j=1}^{d} z_j a_j^\dagger - \bar{z}_j a_j}$$

Definition

ρ is a gaussian state if

$$\hat{\rho}(z) = \mathrm{tr}(\rho W(z)) = \exp\left\{ -\mathrm{i}\,\mathrm{Re}\,\langle \omega, z\rangle - \frac{1}{2}\mathrm{Re}\,\langle z, Sz\rangle \right\}$$

for every $z \in \mathbb{C}^d$. For certain $\omega \in \mathbb{C}^d$ and $S : \mathbb{C}^d \to \mathbb{C}^d$ an invertible, **real linear** operator. We write $\rho = \rho_{\omega,S}$.

$$\hat{\rho}(z) = \exp\left\{ -\mathrm{i}\left\langle \begin{pmatrix} \mathrm{Re}\,\omega \\ \mathrm{Im}\,\omega \end{pmatrix}, \begin{pmatrix} \mathrm{Re}\,z \\ \mathrm{Im}\,z \end{pmatrix} \right\rangle - \frac{1}{2}\left\langle \begin{pmatrix} \mathrm{Re}\,z \\ \mathrm{Im}\,z \end{pmatrix}, S_{\mathbb{R}^{2d}} \begin{pmatrix} \mathrm{Re}\,z \\ \mathrm{Im}\,z \end{pmatrix} \right\rangle \right\}.$$

Why Gaussian Semigroups?

Theorem (Agredo, Fagnola, Poletti (2021))

If $\rho = \rho_{(\omega,S)}$ *then* $\rho_t := \mathcal{T}_{*t}(\rho)$ *is still a gaussian state* $\rho_{(\omega_t,S_t)}$ *with*

$$\omega_t = e^{tZ^T}\omega - \int_0^t e^{sZ^T}\zeta ds$$

$$S_t = e^{tZ^T}Se^{TZ} + \int_0^t e^{sZ^T}Ce^{sZ}ds,$$

where Z *and* C *are the real linear operators*

$$Zz = \left(\frac{1}{2}\overline{(U^*U - V^*V)} + \mathrm{i}\Omega\right)z + \left(\frac{1}{2}\left(U^TV - V^TU\right) + \mathrm{i}\kappa\right)\overline{z}$$

$$Cz = \left(\frac{1}{2}\overline{(U^*U + V^*V)}\right)z + \left(\frac{1}{2}\left(U^TV + V^TU\right)\right)\overline{z}$$

More motivation

Theorem

It holds $\mathcal{T}_t(W(z)) = c_t(z)W(e^{tZ}z)$ *with*

$$c_t(z) = \exp\left\{-\frac{1}{2}\int_0^t \mathrm{Re}\left\langle e^{sZ}z, Ce^{sZ}z\right\rangle ds + \mathrm{i}\int_0^t \mathrm{Re}\left\langle \zeta, e^{sZ}z\right\rangle ds\right\}$$

The converse also holds:

Theorem (Agredo, Fagnola, Poletti (2021))

The following are equivalent:

- $\mathcal{T}$ *is a gaussian QMS associated with* H, L_ℓ;
- $\mathcal{T}_t(W(z)) = c_t(z)W(\mathrm{e}^{tZ}z)$, *for some* C, Z, ζ;
- $\mathcal{T}_*$ *preserves the set of gaussian states.*

List of problems

Problem 1: Irreducibility

A semigroup is irreducible if and only if for every p projection

$$\mathcal{T}_t(p) \geq p \Rightarrow p = 0, p = \mathbb{1}$$

Theorem (Fagnola, Rebolledo)

Let p be a projection and $Rg(p)$ its range. Then $\mathcal{T}_t(p) \geq p$ if and only if

(i) *$Rg(p)$ is invariant for the strongly continuous contraction semigroup e^{tG}*

(ii) *$L_\ell u = pL_\ell u$, for $u \in Dom(G) \cap Rg(p)$*

$$G = -\mathrm{i}H - \frac{1}{2}\sum_{\ell} L_\ell^* L_\ell$$

Decoherence-free Subalgebra

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathcal{H}) \,|\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x), \\ \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*), \, \forall t \geq 0\}.$$

Theorem (Agredo, Fagnola, Poletti (2021))

The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the generalised commutant of the set

$$\{\delta_H^n(L_\ell), \delta_H^n(L_\ell^*) | \ell = 1, \ldots, m, 0 \leq n \leq 2d-1\}$$

where $\delta_H(X) = [H, X]$.

$x \in \mathcal{B}(\mathcal{H})$ is in the generalised commutant of A if

$$xA \subset Ax.$$

$\mathcal{N}(\mathcal{T})$ for gaussian QMSs

Theorem (Agredo, Fagnola, Poletti)

The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by Weyl operators $W(z)$ such that z belonging to real subspaces of $\ker(C)$ that are $Z-$invariant. Moreover, up to unitary equivalence,

$$\mathcal{N}(\mathcal{T}) = L^\infty(\mathbb{R}^{d_c};\mathbb{C})\,\overline{\otimes}\,\mathcal{B}(\Gamma(\mathbb{C}^{d_f}))$$

for a pair of natural numbers $d_c, d_f \leq d$.

EID's definition "à la" Blanchard-Olkiewicz

Definition

There is environment induced decoherence (EID) on the system described by $\mathcal{T}$, if there exist a $\mathcal{T}_t$-invariant von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ and a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{M}$ such that:

EID1 $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$,

EID2 $\mathcal{M}_1$ is a maximal von Neumann subalgebra of $\mathcal{M}$ on which every $\mathcal{T}_t$ acts a $*$-automorphism,

EID3 $w^* - \lim_{t\to\infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$.

- $\mathcal{M}_1$ *decoherence-free algebra*
- $\mathcal{M}_2$ *space of not-detectable observables*.

problem 2: Decoherence

EID holds for a Gaussian Quantum Markov semigroup?
$\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$?