Towards an operator algebraic Breeden–Litzenberger formula

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Motivation for Breeden–Litzenberger (1978)

- Let K be a random variable representing the strike price of an option.
- Let X be a random variable representing the stock value after one year.
- Your payoff is max$\{X - K, 0\}$.

Suppose you know the expected value of your payoff for every value k of the strike price.

Can you then determine the density function of the stock value, $f_X(x)$?

Breeden–Litzenberger (1978) formula

Define $C(k) = E\left[\max\{X - k, 0\}\right]$. Then

$$\frac{d^2}{dk^2} C(k) = f_X(k).$$

Their paper uses calculus of finite differences. (Trading data is discrete.)

The continuous proof uses differentiation under the integral with variable limits.

Continuous proof of Breeden–Litzenberger, part 1

$$
\begin{aligned}
C(k) &= E\left[\mathsf{max}\{X - k, 0\}\right] \\
&= \int_0^\infty \mathsf{max}\{x - k, 0\} f_X(x) dx \\
&= \int_k^\infty (x - k) f_X(x) dx
\end{aligned}
$$

How to differentiate this with respect to k?

Differentiation under the integral, fixed limits

Suppose $g : X \times [a,b] \to \mathbb{C}$, with $-\infty < a < b < \infty$, and that $g(\cdot,t) : X \to \mathbb{C}$ is integrable for each $t \in [a,b]$. Let $G(t) = \int_X g(x,t)d\mu(x)$.

(1) Suppose there exists $h \in L^1(\mu)$ s.t. $|g(x,t)| \leq h(x)$ for all x,t. If $\lim_{t\to t_0} g(x,t) = g(x,t_0)$ for every x, then $\lim_{t\to t_0} G(t) = G(t_0)$; in particular, if $g(x,\cdot)$ is continuous for each x, then $G(t)$ is continuous.

(2) Suppose that $\frac{\partial g}{\partial t}$ exists and there is a $h \in L^1(\mu)$ s.t. $|\frac{\partial}{\partial t} g(x,t)| \leq h(x)$ for all x,t. Then G is differentiable and $G'(t) = \int \frac{\partial}{\partial t} g(x,t)d\mu(x)$.

Proof uses dominated convergence theorem and mean value theorem. (See e.g. Folland *Real Analysis* Theorem 2.27.)

Differentiation under the integral, variable limits

$$\frac{d}{dt}\int_{a(t)}^{b(t)} g(x,t)dx =$$
$$\int_{a(t)}^{b(t)} \frac{\partial}{\partial k} g(x,t)dx + g(b(t),t)\frac{db}{dt} - g(a(t),t)\frac{da}{dt}$$

Proof uses multivariable chain rule and fundamental theorem of calculus to extend the fixed-limits version.

Continuous proof of Breeden–Litzenberger, part 2

$$\begin{aligned}
\frac{d}{dk}C(k) &= \frac{d}{dk}\int_k^\infty (x-k)f_X(x)dx \\
&= \frac{d}{dk}\lim_{b\to\infty}\int_k^b (x-k)f_X(x)dx \\
&= \lim_{b\to\infty}\left(\frac{d}{dk}\int_k^b (x-k)f_X(x)dx\right) \\
&= \lim_{b\to\infty}\left(\int_k^b \frac{\partial}{\partial k}(x-k)f_X(x)dx + ((b)-k)\frac{d(b)}{dk} - ((k)-k)\frac{d(k)}{dk}\right) \\
&= \lim_{b\to\infty}\left(\int_k^b -f_X(x)dx + (b-k)\frac{db}{dk} - 0\right) \\
&= -\int_k^\infty f_X(x)dx + 0 - 0 = -\left(1 - \int_{-\infty}^k f_X(x)dx\right) \\
&= -(1 - F_X(k)) = F_X(k) - 1
\end{aligned}$$

Continuous proof of Breeden–Litzenberger, part 3

Now differentiate again.

$$\frac{d}{dk}\left(F_X(k) - 1\right) = f_X(k)$$

So, yes, you can recover the density function of X by differentiating the expectation of max$\{X - k, 0\}$ twice with respect to k.

$$\frac{d^2}{dk^2} E\left[\text{max}\{X - k, 0\}\right] = f_X(k) \qquad (1)$$

What is the operator algebraic interpretation?

- expectation = trace, as we know.
- random variables X and K = operators in von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$?
- max$\{X - k, 0\}$ = some kind of pairing between $\mathcal{M}$ and $\mathcal{N}$?
- recovering a density function = recovering an element in the predual of $\mathcal{M}$?
- What is "differentiation under the integral?" Possibly related to closable derivations?
- What's the cumulative distribution function?

Thank you!