RSK construction of the KPZ fixed point

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based on joint work with

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Order of current variance and diffusivity in the asymmetric simple exclusion process

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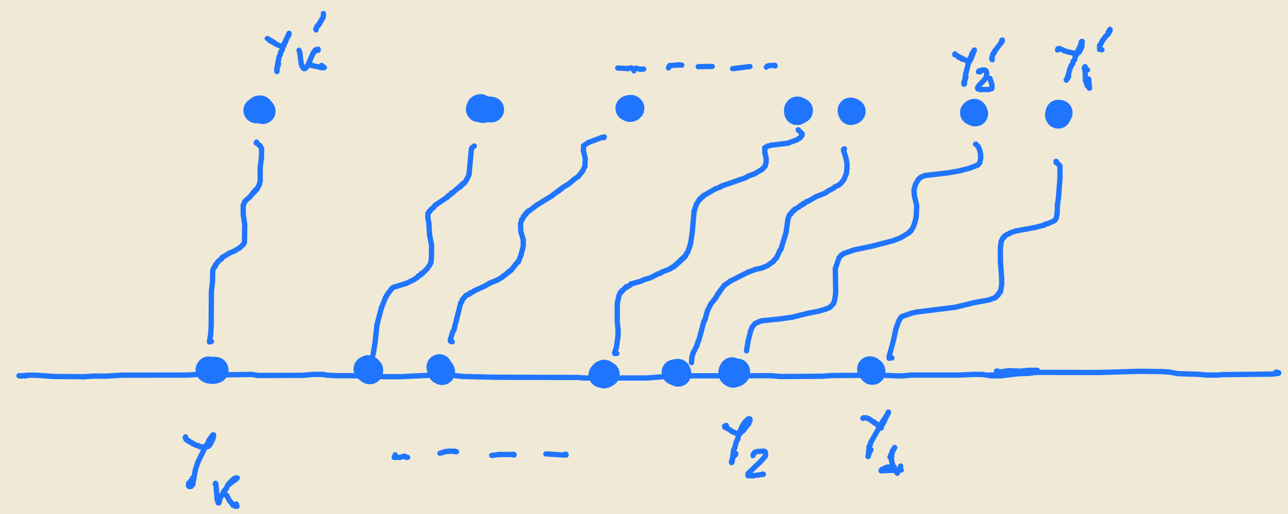
The core formula of KPZ fixed point

(Matetski-Quastel-Remenik)



$$\mathbb{P}\left(\{ y_{k_i}(t) \geq s_i \}_{i=1,\dots} \right) = \det\left(I - \chi_s K \chi_s \right)_{\ell^2(\{1,\dots,n\}\times\mathbb{Z})}$$

with $\chi_{s_i}(k_i, x) := \mathbb{1}_{x<s_i}$

$$K(m,x;n,x') := -Q_{(m,n]}\mathbb{1}_{m>n} + S^{\circ}_{[1,m],(0,t]}\, S^{\mathrm{epi}(y)}_{[1,n],(0,t]}(x,x')$$

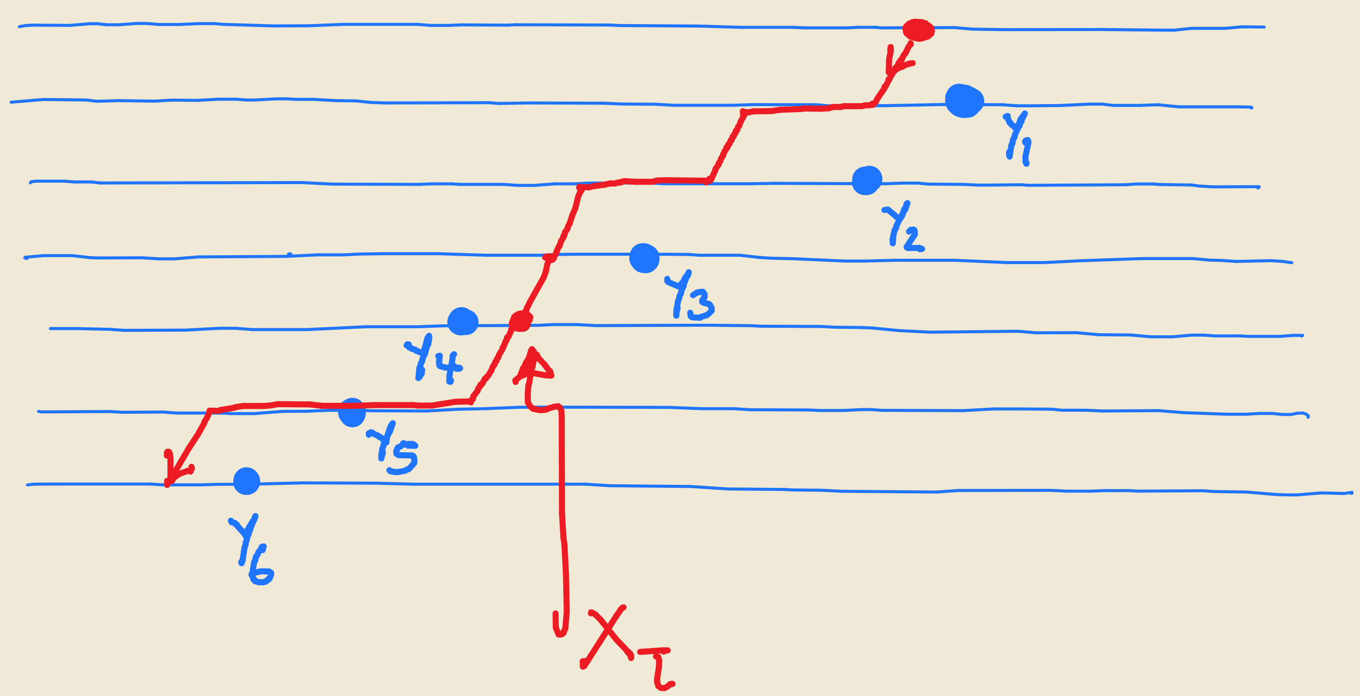
$$S^{\mathrm{epi}(y)}_{[1,n],(0,t]}(x,y) := \mathbb{E}_x\left[\bar{S}_{[\tau+1,n],(0,t]}(X_\tau, y)\, \mathbb{1}_{\tau<n} \right]$$

& $(X_n)_{n\geq 1}$ a geometric random walk with

$$\mathbb{P}\left(X_{i+1} = y \mid X_i = x \right) \propto Q_i(x,y) := q_i^{y-x} \mathbb{1}_{y<x}$$

$$Q_{(m,n]} := Q_{m+1} \circ \cdots \circ Q_n$$

$$\tau := \min \{ n : X_n > Y_{n+1} \}$$



The details of TASEP

$$\mathbb{P}_t\left(\underset{k}{\circ}\curvearrowright \right) = \frac{p_t q_k}{1+p_t q_k}$$

with $q_k > 1$ & $p_t q_k < 1$

Update rule : sequential from first to last particle

TASEP with inhomogeneous rates :

Hydrodynamics : Krug - Seppäläinen '99

Emrah 16'

Emrah - Janjigian - Seppäläinen '21

Integrable : Johansson '00

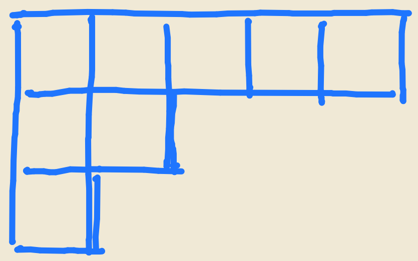
Borodin - Pesche '08

& many more

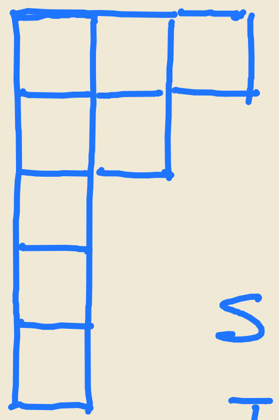
Robinson – Schensted – Knuth

row insertion, column insertion, dual-row, dual-column

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ & & \vdots & & & \\ & & & & & \end{bmatrix} \xleftrightarrow{\text{dual-column}}$$

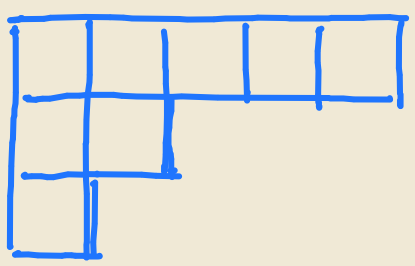


shape $\lambda = (5, 2, 1)$

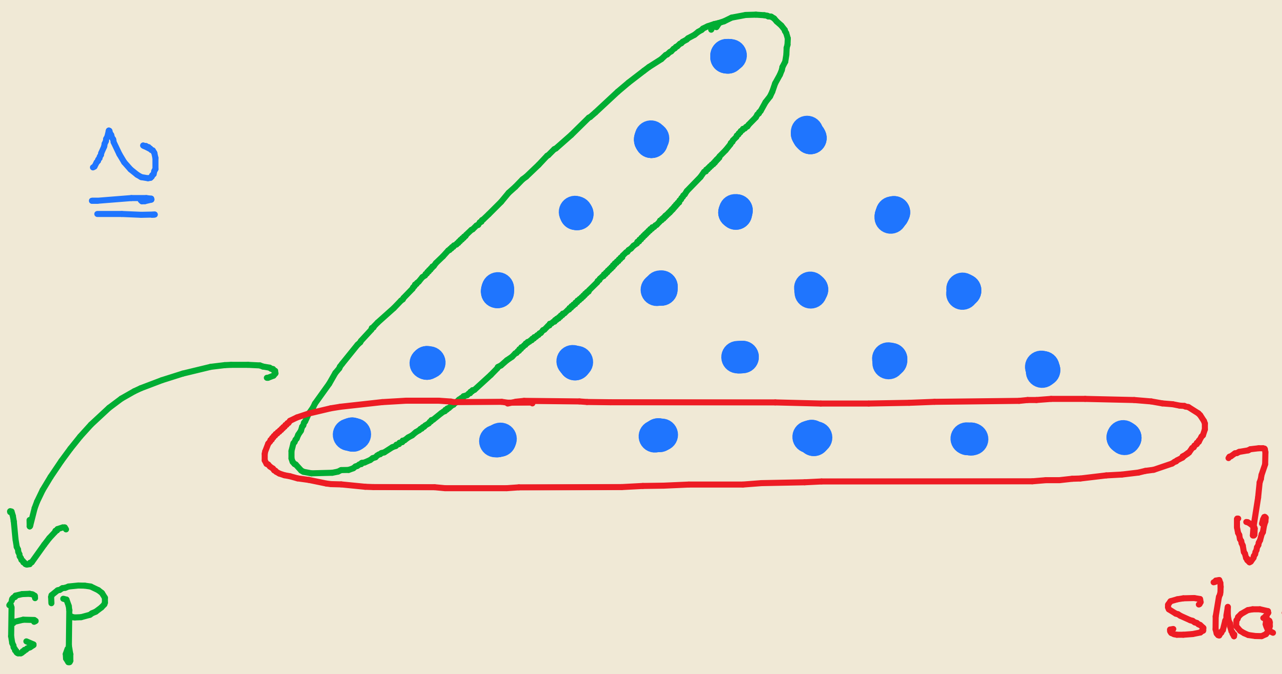


Shape $\lambda^T = (3, 2, 1, 1, 1)$

Gelfand – Tsetlin



$\cong$



Probabilities

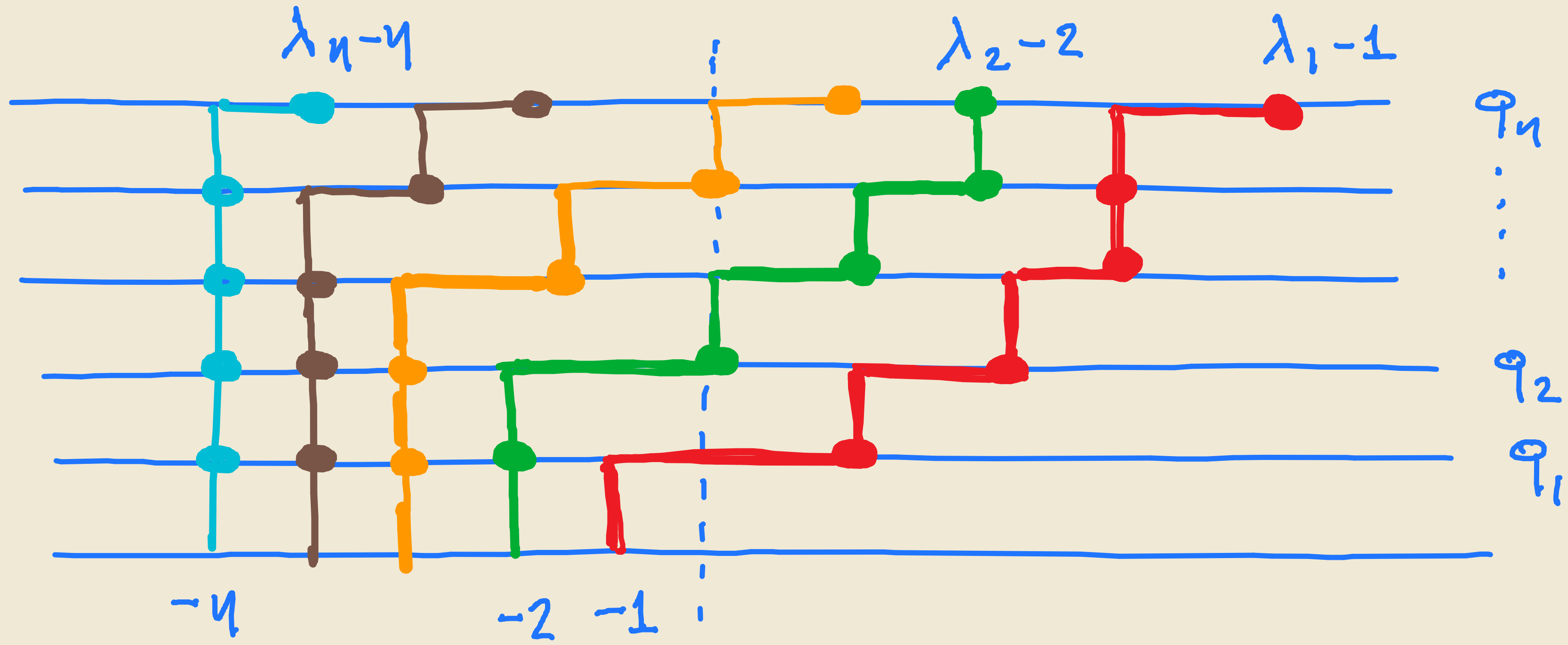
If $(W_{ij})_{i\geq 1,\, j=1,\dots,k}$ independent $\in \{0,1\}$

$$\mathbb{P}(W_{ij}=1) = \frac{p_i q_j}{1+p_i q_j}$$

Then $$\mathbb{P}(\mathrm{sh}\,P = \mathrm{sh}\,Q^T = \lambda) = \frac{1}{\prod_{i,j}(1+p_i q_j)}\, s_\lambda(q)\, s_{\lambda^T}(p)$$

Schur functions & paths

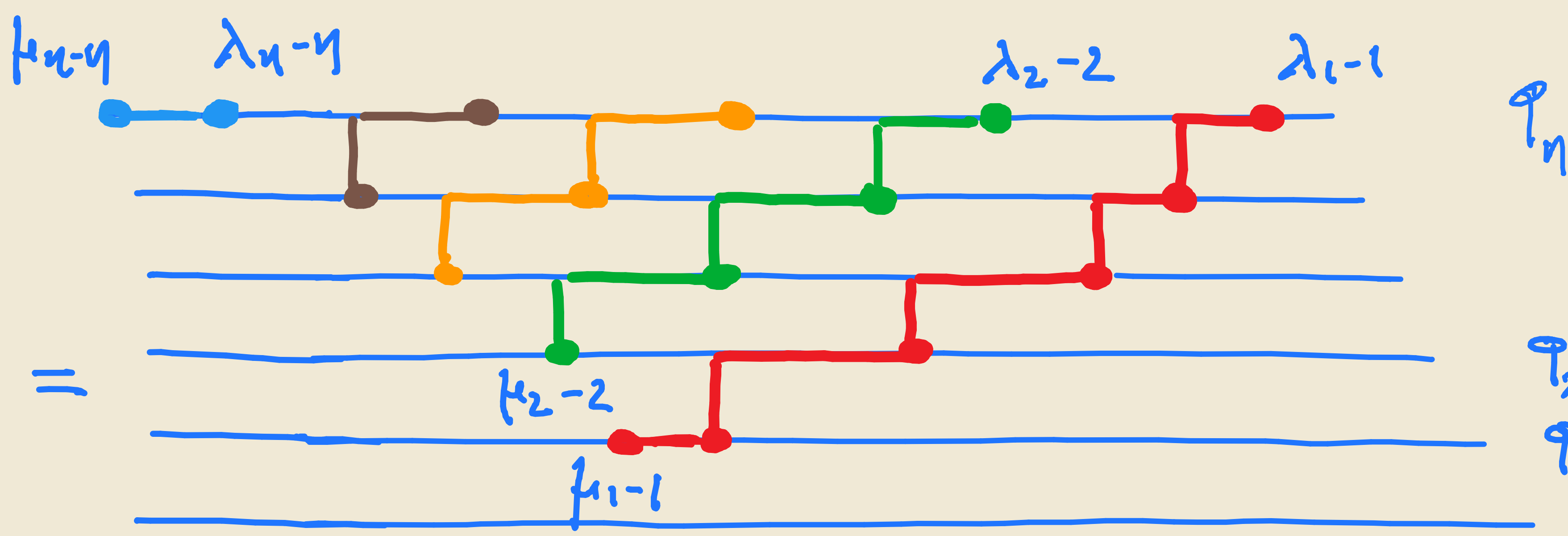
$S_\lambda(q_1,\dots,q_n) =$



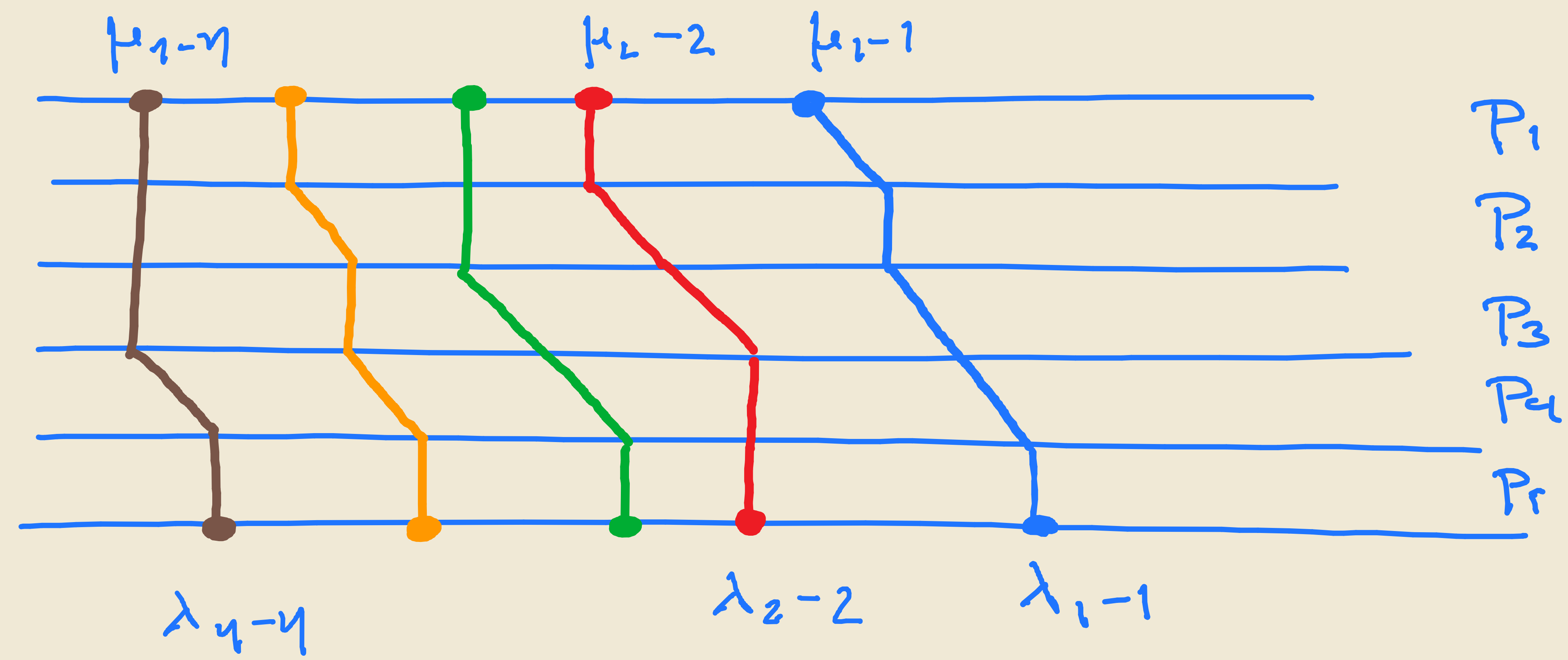
$$= \det\left(\widehat{Q}_1 \circ \widehat{Q}_2 \circ \cdots \circ \widehat{Q}_n\,(-i, \lambda_j - j)\right)_{i,j \le n}$$

with $\quad \widehat{Q}_i(x,y) = q_i^{\,y-x}\ \mathbb{1}_{y \geq x}$

$S_{\lambda/\mu}(q_1,\dots,q_n) =$

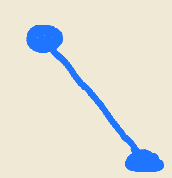


$$s_{\lambda^T/\mu^T}(p_1,\dots,p_n) =$$

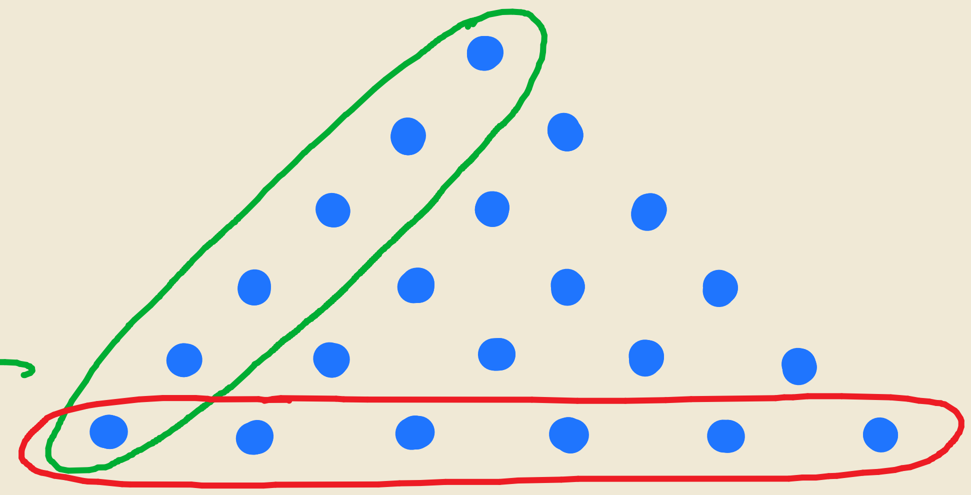


Weights

$R_i($  $) = 1$

$R_i($  $) = p_i$

Intertwining & TASEP transition



TASEP
transition kernel : $P(Y,Y')$

shape λ
transition kernel $R(\mu,\lambda)$

intertwiner $\Lambda(\lambda, Y)$

$$\Lambda P = R \Lambda$$

$$\Rightarrow \quad P = \Lambda^{-1} R \Lambda$$

cf. Dieker–Warren '08

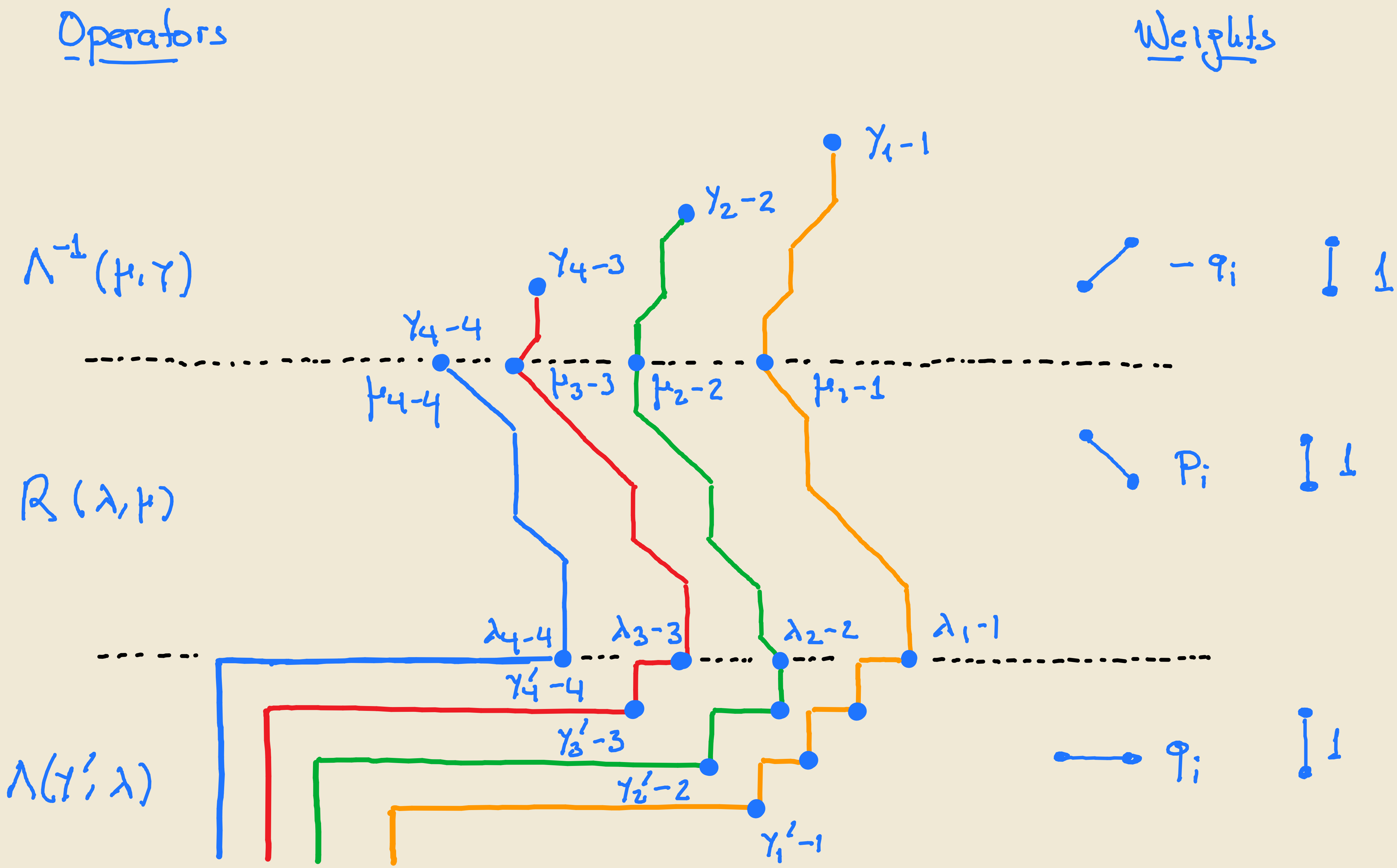
Λ : kernel of Schur functions fixing shape & left edge

$$\Lambda^{-1}(Y,\mu) \overset{\text{DW08}}{=\!=} \det\left(e_{Y_j-\mu_i-j+i}(0,\dots,0,q_{j+1},\dots,q_N)\right)_{i,j}$$

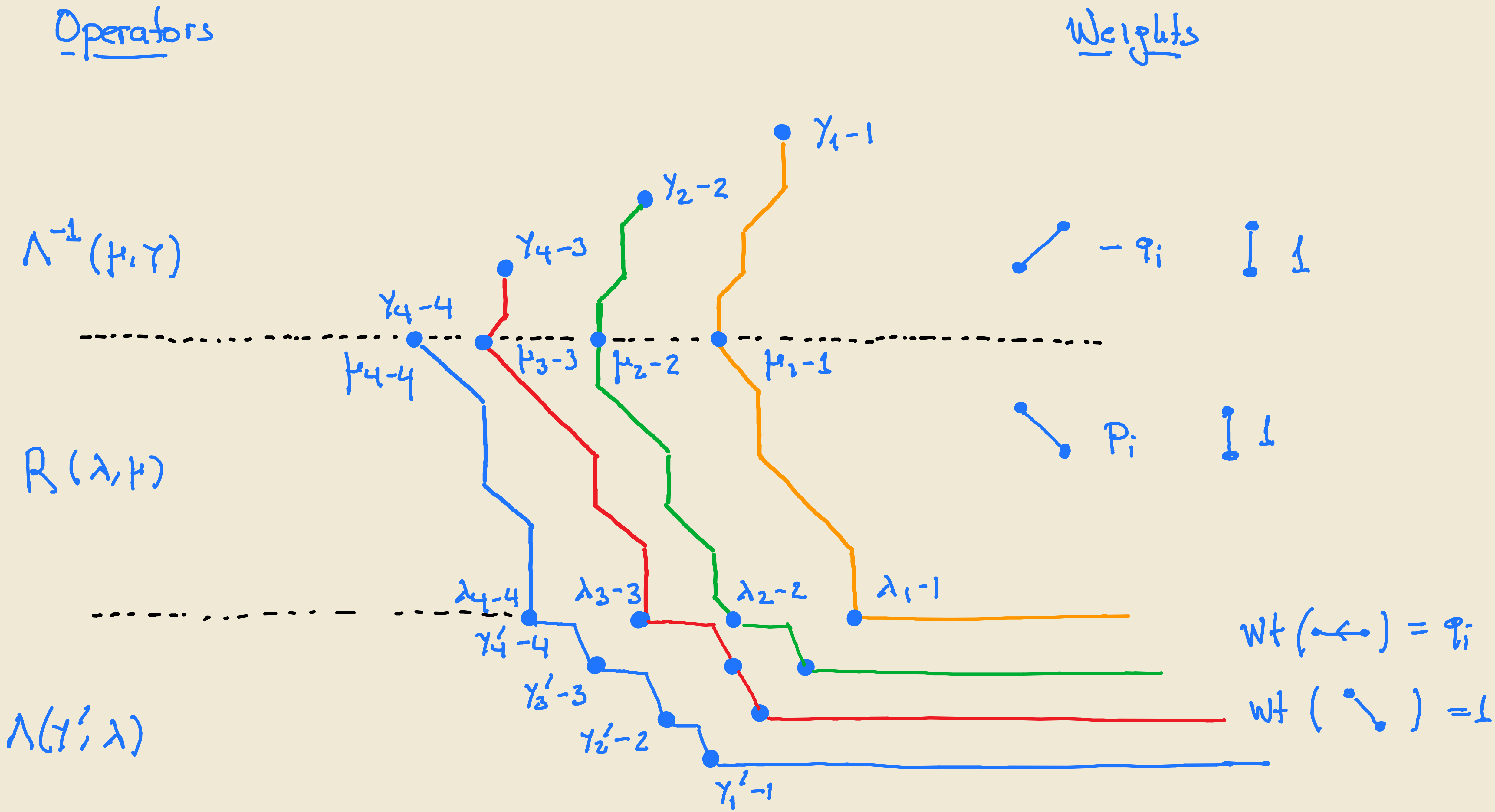
$$= \det\left((-1)^{N-j} Q^{-1}_{(j,N]}(\mu_i - i, Y_j - j)\right)_{i,j}$$

with $Q_i^{-1}(x,y) = -1\,{}^{y}_{x} + q_i\, 1\,{}^{y=x+1}_{x}$

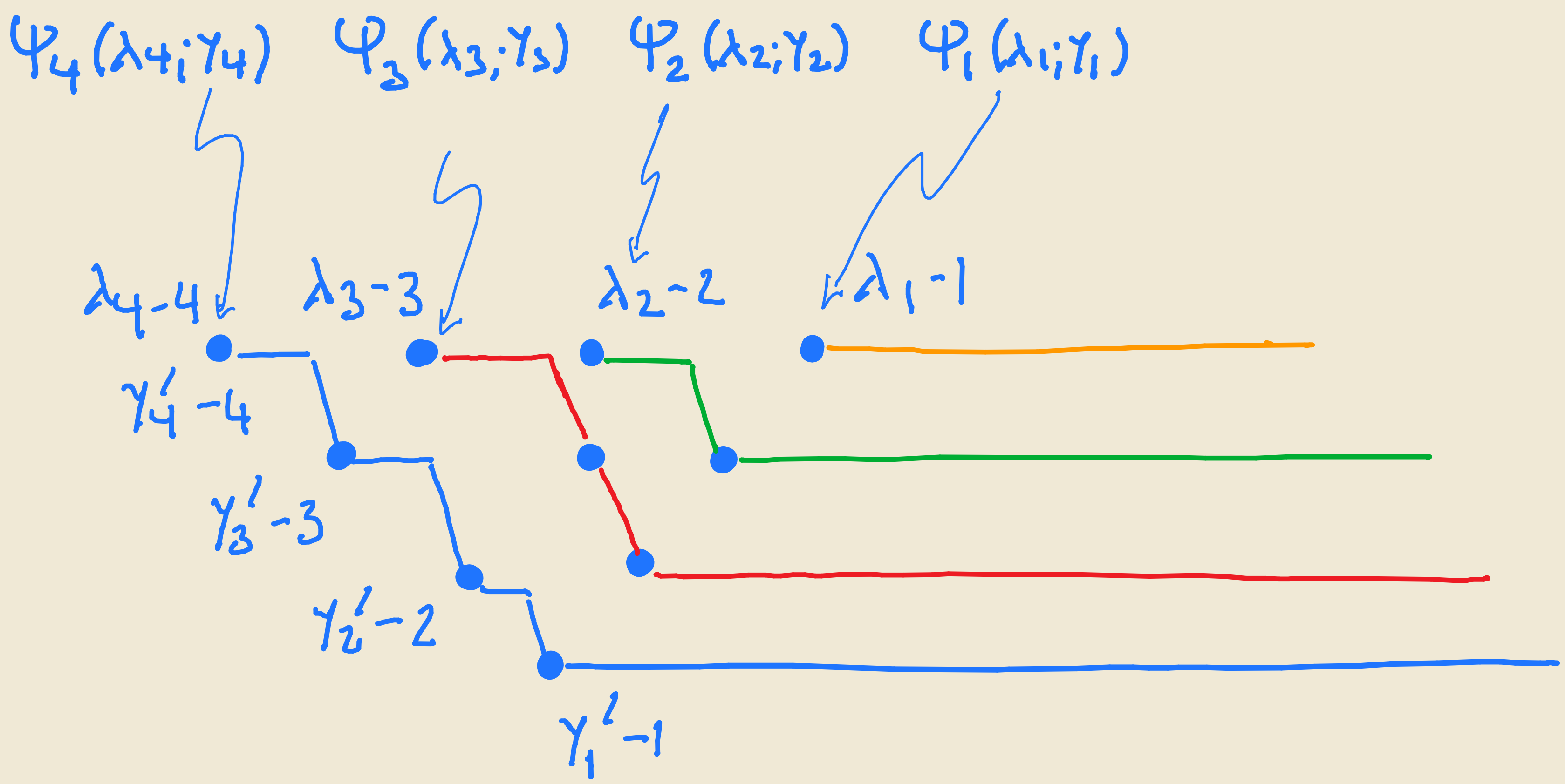
The path picture of intertwining



More appropriate path representation

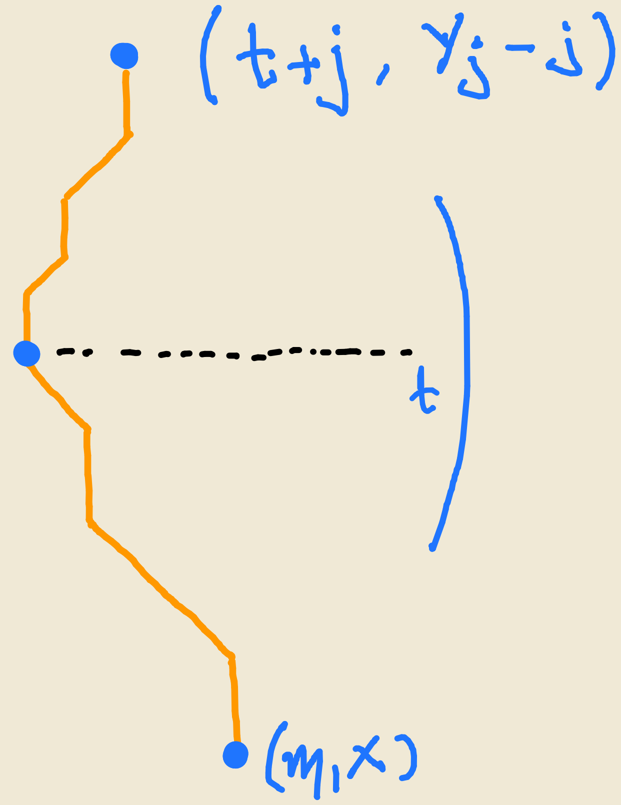


Restricted determinantal process



with

$\Psi_i^m(x) = \mathrm{wt}$ (



)

$$\text{Total weight} \propto \prod_{k=1}^{N} \det\left(Q_k(x_{i-1}^{k-1}, x_j^{k})\right)_{i,j \le k} \cdot \det\left(\Psi_i^N(x_j^N)\right)_{i,j \le N}$$

TASEP transition = restriction left marginal

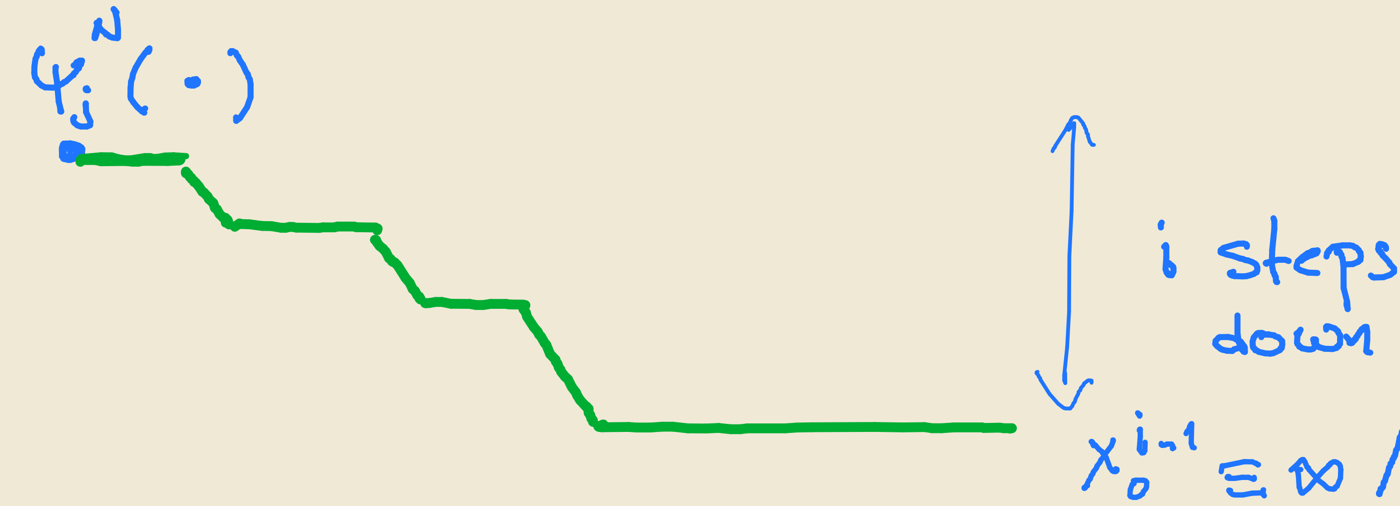
$$= \det(I + K)$$

$$K(m,x;n,y) = -Q_{(m,n]}(x,y)\, 1_{n>m} \; +$$

$$+ \sum_{i=1}^{N} \Psi_i^{m}(x) \cdot \underbrace{\sum_{j=1}^{N} M^{-1}_{i,j}\, Q_{[j,n]}(x_0^{j-1}, y)}_{\Phi_j^{n}(y)}$$

biorthogonal ensemble

with

$$M_{ij} = \mathrm{wt}\Big($$  $$\Big)$$

$$= \sum_{y} Q_i \circ Q_{i+1} \circ \cdots \circ Q_N(\infty, z)\, \varphi_j^N(z)$$

Key Observation : M_{ij} is upper-triangular

$$M_{ij} = \sum_{z} Q_{i} \circ \cdots \circ Q_{N} \, (\infty, z) \cdot \varphi_{j}^{N}(z)$$

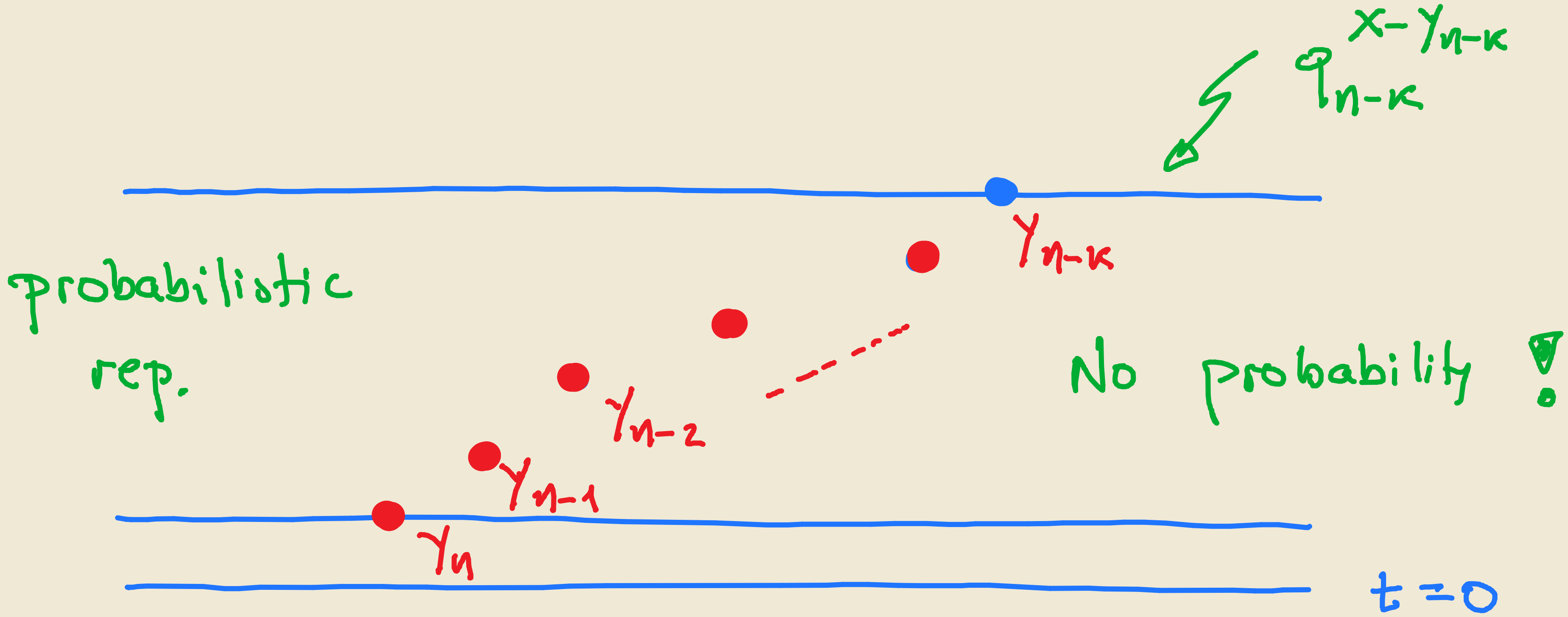
$$= \sum_{z} Q_{i} \circ \cdots \circ Q_{N} \, (\infty, z) \cdot R_{(r,t]} \circ Q_{N}^{-1} \circ \cdots \circ Q_{j+1}^{-1} \, (z, y_j - j)$$

$$= R_{(r,t]} \circ Q_{i} \circ \cdots \circ Q_{N} \circ Q_{N}^{-1} \circ \cdots Q_{j+1}^{-1} \, (\infty, y_j - j)$$

$$\underset{\text{if } i>j}{=} R_{(r,t]} \circ Q_{j+1}^{-1} \circ \cdots \circ Q_{i-1}^{-1} \, (\infty, y_j - j)$$

$$= 0$$

The boundary value problem



If

$$h_k^n(\ell+1, x) = h_k^n(\ell, \cdot) \circ Q_{n-\ell}^{-1}(x) \qquad x \in \mathbb{Z}, \; \ell < k$$

$$h_k^n(\ell, y_{n-\ell}) = 0 \qquad \ell < k$$

$$h_k^n(k, x) = \varphi_{n-k}^{x-y_{n-k}} \qquad x \in \mathbb{Z}$$

then

$$\Phi_{n-k}^{n}(x) = h_k^n(0, \cdot) \circ \left(R_{(r,t]}\right)^{-1}(x)$$

Thanks