Homogenization of the Invariant Measure for Nondivergence Form Elliptic Equations

Jessica Lin

McGill University

Random Growth Models and KPZ Universality

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Collaborators

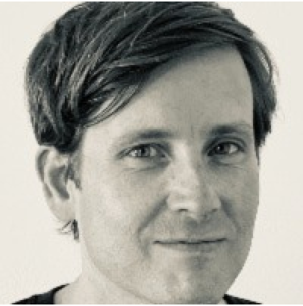

Figure: Scott Armstrong (NYU/Courant)

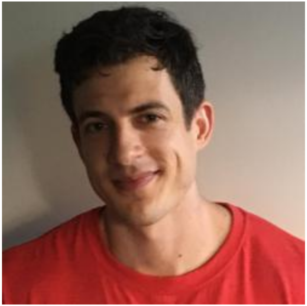

Figure: Ben Fehrman (Oxford, soon to be LSU)

Nondivergence Form Equations in Random Environments

We consider

$$-\text{tr}\left(\mathbf{A}(x)D^2u\right) = 0,$$

and the parabolic counterpart

$$\partial_t u - \text{tr}\left(\mathbf{A}(x)D^2u\right) = 0,$$

where the symmetric, matrix-valued function $\mathbf{A}$ is an element of the set of admissible coefficient fields Ω:

$$\Omega := \left\{\mathbf{A}(\cdot) \, : \, \lambda Id \leq \mathbf{A}(\cdot) \leq \Lambda Id, [\mathbf{A}]_{C^{\mathbf{0},\alpha_\mathbf{0}}(\mathbb{R}^d)} \leq K_0\right\}.$$

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We will make this random by putting an appropriate probability measure $\mathbb{P}$ on this set.

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Define $\mathcal{F} := \mathcal{F}(\mathbb{R}^d)$, and then consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Furthermore, we define a measurable translation operator $\{\tau_y\}_{y \in \mathbb{R}^d} : \Omega \to \Omega$, according to

$$(\tau_y \mathbf{A})(x) := \mathbf{A}(x + y)$$

Assumption on the Environment

(P1) $\mathbb{P}$ has $\mathbb{Z}^d$–stationary statistics: that is, for every $z \in \mathbb{Z}^d$ and $E \in \mathcal{F}$,

$$\mathbb{P}[E] = \mathbb{P}[\tau_z E].$$

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(P2) $\mathbb{P}$ has a finite range of dependence: that is, for all Borel subsets U, V of $\mathbb{R}^d$ such that $dist(U, V) \geq 1$,

$$\mathcal{F}(U) \text{ and } \mathcal{F}(V) \text{ are } \mathbb{P}\text{–independent.}$$

(This is a stronger version of ergodicity)

Homogenization of the PDE

For $\mathbf{A}(\cdot) \in \Omega$, and for each $\varepsilon > 0$, consider

$$\begin{cases} -\operatorname{tr}\left(\mathbf{A}\left(\frac{x}{\varepsilon}\right) D^2 u^\varepsilon\right) = 0 & \text{in} \quad U, \\ u^\varepsilon = g & \text{on} \quad \partial U. \end{cases}$$

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Homogenization Statement: There exists a deterministic matrix $\overline{\mathbf{A}}$ such that for $\mathbb{P}$-a.e $\mathbf{A}$, $u^\varepsilon \to u$ uniformly in U, where u solves

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Remark: Equivalently, by rescaling we can study

$$-\operatorname{tr}\left(\mathbf{A}(x) D^2 u_{\varepsilon}\right) = 0 \quad \text{and} \quad -\operatorname{tr}\left(\overline{\mathbf{A}} D^2 \bar{u}_{\varepsilon}\right) = 0 \quad \text{in } \varepsilon^{-1} U.$$

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Main Challenges:

- Identifying $\overline{A}$
- Proving the convergence of $u^\varepsilon \to u$.

The PDE Approach: Correctors

Typical ansatz of homogenization,

$$u^{\varepsilon}(x) \approx u(x) + \varepsilon^2 \phi\left(\frac{x}{\varepsilon}\right) + \dots$$

where ϕ is the 2nd-order corrector.

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$$\begin{cases} -\mathrm{tr}\left(\mathbf{A}(x)(D^2 u + D^2\phi)\right) = -\mathrm{tr}(\overline{\mathbf{A}} D^2 u) \quad \text{in} \quad \mathbb{R}^d, \\ \lim_{\varepsilon\to 0} \left|\left|\varepsilon^2 \phi\right|\right|_{L^\infty} = 0 \quad \mathbb{P}\text{-a.s.} \end{cases}$$

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The typical approach for such equations has been to study an appropriate *approximate* corrector which allows us to both identify $\overline{\mathbf{A}}$ and also prove the convergence. The construction of $\overline{\mathbf{A}}$ is *variational*.

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References: Kozlov and Jikov, Kozlov, Oleinik (Linear); Caffarelli, Souganidis, and Wang (Fully Nonlinear)

Quantitative Homogenization

Given that we are interested in the $\mathbb{P}$-a.s. convergence of $u^\varepsilon \to u$, we now ask, for which f, g do we have

$$\mathbb{P}\left[\sup_{x\in U}|u^\varepsilon(x) - u(x)| \geq f(\varepsilon)\right] \leq g(\varepsilon)?$$

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Armstrong and Smart (Fully Nonlinear): There exists $\alpha \in (0,1)$ and $C > 0$ such that

$$\mathbb{P}\left[\sup_{x \in U} |u^\varepsilon(x) - u(x)| \geq \varepsilon^\alpha\right] \leq C \exp(-\varepsilon^{-d^-}).$$

(For every $p \in (0, d)$,

$$\mathbb{P}\left[\sup_{x \in U} |u^\varepsilon(x) - u(x)| \geq \varepsilon^\alpha\right] \leq C \exp(-\varepsilon^{-p}).)$$

The First Stochastic Homogenization Result (Papanicolaou and Varadhan '82)

Consider $(X_t)_{t\geq 0}$, a stochastic process evolving according to the SDE

$$dX_t = \sigma(X_t)\, dW_t$$

where $\{W_t\}_{t>0}$ is a Brownian motion and $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is a given Hölder continuous function satisfying the uniform nondegeneracy condition

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Then for $\mathbf{A} := \frac{1}{2}\sigma\sigma^t$, the infinitesimal generator of this process is given by

$$\varphi \mapsto \operatorname{tr}(\mathbf{A} D^2\varphi).$$

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Quenched Invariance Principle: For $\mathbb{P}$-a.e. $\mathbf{A}$, the rescaled process $X_t^\varepsilon := \varepsilon X_{t/\varepsilon^2}$ converges in law (under $\mathbf{P}^{\mathbf{A}}$) as $\varepsilon \to 0$ to a Brownian motion with covariance $(2\overline{\mathbf{A}})^{\frac{1}{2}}$.

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Thus, for any $\theta \in \mathbb{R}^d$,

$$E^{\mathbf{P}^{\mathbf{A}}}\left[\exp\left(i\left\langle \theta, \frac{X_t}{\sqrt{t}}\right\rangle + \frac{1}{2}\left\langle \theta, \frac{1}{t}\left(\int_0^t \mathbf{A}_{ij}(X_s)\,ds\right)\theta\right\rangle\right)\right] = E^{\mathbf{P}^{\mathbf{A}}}\left[\exp(0)\right] = 1.$$

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If we could identify $\overline{\mathbf{A}}$ such that for $\mathbb{P}$-a.e. $\mathbf{A}$, and $\mathbf{P}^{\mathbf{A}}$ almost surely (equivalently $\mathbb{P}\otimes\mathbf{P}^{\mathbf{A}}$-almost-surely),

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbf{A}_{ij}(X_s)ds = \overline{\mathbf{A}}_{ij},$$

then

$$\lim_{t\to\infty} E^{P_{\mathbf{A}}}\left[\exp\left(i\left\langle\theta, \frac{X_t}{\sqrt{t}}\right\rangle\right)\right] = \exp\left(-\left\langle\theta, \frac{1}{2}\overline{\mathbf{A}}_{ij}\theta\right\rangle\right).$$

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Invariant Measures

Establishing an ergodic theorem amounts to finding an ergodic *invariant measure* μ, which means $\mathbb{P} \otimes \mathbf{P}^{\mathbf{A}}$-almost surely,

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The invariant measure $d\mu$ should be mutually absolutely continuous with respect to $d\mathbb{P}$. In particular, we seek $m = m(\mathbf{A})$ such that $d\mu = m\, d\mathbb{P}$. In this case, we would have that

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References (Quantitative): Guo, Peterson, Tran (BRWRE); Guo and Tran (BRWRE)

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- Typical approaches using PDEs do not identify $\overline{\mathbf{A}}$ via this invariant measure.
- Since this model is nonreversible, the invariant measure does not have an explicit representation formula.
- What is the invariant measure m in the PDE setting?
- How can we improve our understanding of these two methods, individually and globally, to promote more collaborative approaches on this topic?

Transition Probabilities and the Parabolic Green Function

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In the language of PDEs, the density of the transition probability is exactly the parabolic Green Function. For each $\mathbf{A} \in \Omega$, we consider $P(t, x, y)$ solving, for each $y \in \mathbb{R}^d$,

$$\begin{cases} \partial_t P(\cdot, \cdot, y) - \operatorname{tr}(\mathbf{A} D^2 P(\cdot, \cdot, y)) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ P(0, \cdot, y) = \delta(\cdot - y) & \text{on } \mathbb{R}^d. \end{cases}$$

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Similarly, we have $\overline{P}(t, x - y)$ the parabolic Green Function of the homogenized equation,

$$\begin{cases} \partial_t \overline{P}(\cdot, \cdot - y) - \operatorname{tr}(\overline{\mathbf{A}} D^2 \overline{P}(\cdot, \cdot - y)) = 0 & \text{in } (0, \infty) \times \mathbb{R}^d, \\ \overline{P}(0, \cdot, y) = \delta(\cdot - y) & \text{on } \mathbb{R}^d, \end{cases}$$

where we know $\overline{P}$ is a Gaussian.

Mass Preservation

The homogenized equation (constant coefficient) preserves mass, so

$$\int_{\mathbb{R}^d} \overline{P}(t, x-y)\, dx = 1 \quad \text{for all } t \geq 0, \text{ for all } y \in \mathbb{R}^d.$$

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Do we witness "effective mass conservation"?

Theorem (Armstrong, Fehrman, L., 2022)

Let v solve

$$\begin{cases} \partial_t v - tr(\mathbf{A} D^2 v) = 0 & in\ (0,\infty) \times \mathbb{R}^d, \\ v(0,x) = v_0 & on\ \mathbb{R}^d, \end{cases}$$

$$|v_0(x)| \leq MR^{-d} \exp\left(-\frac{|x|^2}{R^2}\right),$$

There exists $\gamma \in (0,1)$, a random variable $\mathcal{Y}$, with $\mathbb{E}[\exp(\mathcal{Y}^{d^-})] \leq C$ and a random constant $c[v_0]$, such that, for every $R \geq \mathcal{Y}$, for every $t \geq R^2$,

$$\left| v(t,x) - c[v_0]\overline{P}(t,x) \right| \leq CM \left(\frac{t}{R^2}\right)^{-\gamma} t^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{Ct}\right),$$

where

$$c[v_0] = \lim_{t\to\infty} \int_{\mathbb{R}^d} v(t,x)\, dx.$$

Theorem (contd.)

$$\left|v(t,x) - c[v_0]\overline{P}(t,x)\right| \leq CM \left(\frac{t}{R^2}\right)^{-\gamma} t^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{Ct}\right),$$

If moreover, for some $\sigma \in (0,1]$, $v_0 \in C^{0,\sigma}(B_R)$, *then for* $c[v_0] = \lim_{t\to\infty} \int v(t,x)\,dx$,

$$\left|c[v_0] - \int_{\mathbb{R}^d} v_0(x)\,dx\right| \leq CM(1 + M^{-1}R^{d+\sigma}[v_0]_{C^{0,\sigma}(B_R)})R^{-\gamma}.$$

Theorem (contd.)

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If moreover, for some $\sigma \in (0,1]$, $v_0 \in C^{0,\sigma}(B_R)$, *then for* $c[v_0] = \lim_{t\to\infty} \int v(t,x)\,dx$,

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Remark: Reminiscent of the classical fact that under certain hypotheses, any solution of the heat equation asymptotically converges to the parabolic Green function weighted by the initial mass.

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Remark: The results become deterministic for $R \geq \mathcal{Y}$ where $\mathbb{E}[\exp(\mathcal{Y}^p)] \leq C$ for $p \in (0,d)$.

Homogenization for the Parabolic Green Function

Theorem (Armstrong, Fehrman, L., 2022)

There exists a random variable $\mathcal{Y}$ with $\mathbb{E}[\exp(\mathcal{Y}^{d^-})] \le C$, such that for every $y \in \square_0$ (the unit cube), there is a positive random constant $m(y)$ such that, for every $t \ge \mathcal{Y}^2$ and $x \in \mathbb{R}^d$,

$$\left|P(t,x,y) - m(y)\overline{P}(t,x-y)\right| \le Cm(y)\left(\frac{t}{\mathcal{Y}^2}\right)^{-\gamma} t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right),$$

where $m(y) := \lim_{t\to\infty} \int_{\mathbb{R}^d} P(t,x,y)\,dx$.
By stationarity, we can construct $m(y)$ for any $y \in \mathbb{R}^d$.

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What is this $m(y)$?

Unravelling the $m(y)$

The function $y \mapsto m(y)$ turns out to be a $\mathbb{Z}^d$–stationary *invariant measure* with $\mathbb{E}[\fint_{\Box_0} m(y)\, dy] = 1$.

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By an *invariant measure* m in an open subset $U \subseteq \mathbb{R}^d$, we mean a solution of the adjoint equation, which is formally written in coordinates as

$$-\sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}\left(\mathbf{A}_{ij} m\right) = 0 \quad \text{in } U.$$

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The equation is interpreted in the weak sense: precisely, a Radon measure μ is an invariant measure in U if

$$\int_U \operatorname{tr}\left(\mathbf{A}(y) D^2\varphi(y)\right) d\mu(y) = 0, \quad \forall \varphi \in C_c^\infty(U)$$

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and we identify $d\mu(y) = m(y)\,dy$. This $m(y)$, lifted into the probability space, is the exact same unique invariant measure as constructed in Papanicolaou and Varadhan!

Quantifying Weak Convergence

Theorem (Armstrong, Fehrman, L., 2022)

There exists a random variable $\mathcal{Y}$ with $\mathbb{E}[\exp(\mathcal{Y}^{d^-})] \leq C$, such that for every $R \geq \mathcal{Y}$, for m as defined before,

$$\left| \fint_{R\square_0} m(x)\,dx - 1 \right| + \left| \fint_{R\square_0} m(x)\mathbf{A}(x)\,dx - \overline{\mathbf{A}} \right| \leq CR^{-\gamma}.$$

Observe that the function $\overline{m}(x) \equiv 1$ is precisely the invariant measure to

$$-\sum_{i,j=1}^{d} \partial_{x_i}\partial_{x_j}\left(\overline{\mathbf{A}}_{ij}\overline{m}\right) = 0,$$

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subject to $\mathbb{E}[\fint_{\square_0} \overline{m}(x)\,dx] = 1$, so the first part is a *homogenization result for the invariant measure*.

The second part gives us a way of *computing* the coefficents in nondivergence form homogenization, which was priorly not known from the "nonlinear" approach to homogenization.

Optimal Stochastic Integrability

The prior estimates also demonstrate that all of our results exhibit *optimal stochastic integrability*. By Chebyshev, we have that

$$\mathbb{P}\left[\left|\fint_{R\square_0} m(x)\mathbf{A}(x)\,dx - \overline{\mathbf{A}}\right| > CR^{-\gamma}\right] \leq C\exp(-R^p)$$

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In a "random checkerboard" with white squares ($\mathbf{A} = \mathrm{Id}$) and black squares ($\mathbf{A} = 2\mathrm{Id}$), then the homogenized coefficient $\mathrm{Id} < \overline{\mathbf{A}} < 2\mathrm{Id}$. However, the probability of deviating from $\overline{\mathbf{A}}$ of size at least $\frac{1}{2}$ must be no smaller than the probability of seeing all white squares or all black squares. This probability is like $\left(\frac{1}{2}\right)^{R^d} = \exp(-cR^d)$.

Consequence 1: Heat Kernel Estimates

Our result implies that for $t \geq \mathcal{Y}^2$,

$$cm(y)t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{ct}\right) \leq P(t,x,y) \leq Cm(y)t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{Ct}\right),$$

and $\mathcal{Y}^{-q} \leq \inf_{y \in B_{\mathcal{Y}}} m(y) \leq \sup_{y \in B_{\mathcal{Y}}} m(y) \leq \mathcal{Y}^{(d-1-\delta)}$, for some q universal.

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Similar results by Escauriaza (PDE methods), Mustapha (Discrete, Probability Methods), Guo and Tran (Discrete, Probability Methods); also Deuschel and Guo in discrete time-dependent setting.

$$\left[c \frac{m(y)}{\int_{B_{\sqrt{t}}(y)} m(z)\,dz} \exp\left(-\frac{|x-y|^2}{ct}\right) \right.$$
$$\left. \leq P(t,x,y) \leq C \frac{m(y)}{\int_{B_{\sqrt{t}}(y)} m(z)\,dz} \exp\left(-\frac{|x-y|^2}{Ct}\right)\right]$$

Consequence 2: Quantitative Ergodicity.

Our results imply a rate of convergence on the ergodic theorem for the environment process. There exists $\gamma \in (0, 1/2)$ and a random variable $\mathcal{Y}$ such that $\mathbb{E}[\exp(\mathcal{Y}^{d^-})] \leq C$, and for all $T \geq \mathcal{Y}^2$,

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This is a quenched estimate; it depends on the environment only through $\mathcal{Y}$, while sharply bounding the trajectories.

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- We prove the first quantitative homogenization result for the parabolic Green function (local limit theorem), using PDE methods.

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Summary

- We prove the first quantitative homogenization result for the parabolic Green function (local limit theorem), using PDE methods.
- From there, we construct the unique ergodic invariant measure from Papanicolaou and Varadhan in a quenched fashion.
- We obtain several consequences: heat kernel bounds and quenched quantitative ergodicity.
- Further questions: obtaining optimal convergence rates (central limit theorem) for the correctors.

Thank you very much for your attention.
Happy Birthday Timo!

A few words about the proof of the First Theorem

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- This lets us prove the estimate up until some large, but finite time. In this time, since our solution is close to the solution of the heat equation, our solution has spread out in a very precise way. This implies that the solution at the terminal time can be tented by a Gaussian on a larger lengthscale.

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- We then bootstrap this argument to a larger lengthscale. We keep very close track of errors that we make in every step, and this allows us to conclude that

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} v(t,x)\, dx \quad \text{exists.}$$

The Second Theorem: Quantitative Homogenization of the Parabolic Green Function

Recall that since $m(y) := \lim_{t\to\infty} \int_{\mathbb{R}^d} P(t,x,y)\,dx$, we simply apply the prior Theorem to $P(t,x,y)$ for each $y \in \mathbb{R}^d$. We just need a small PDE argument to argue that

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Proof of Invariance

Recall that by the first Theorem, if v solves the heterogeneous equation with $v(0,x) = v_0 = \phi \in C_c^\infty$, then by an application of the first and second Theorems,

$$c[v_0] = \lim_{t\to\infty} \int_{\mathbb{R}^d} v(t,x)\,dx = \lim_{t\to\infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P(t,x,y) v_0(y)\,dy\,dx$$
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Moreover, we know that $c[v_0] = c[v(t,\cdot)]$ for any $t > 0$, and this is *invariance*.
This implies that

$$0 = \partial_t \int_{\mathbb{R}^d} m(y)v(t,y)\,dy = \int_{\mathbb{R}^d} m(y)\partial_t v(t,y)\,dy = \int_{\mathbb{R}^d} m(y)\mathrm{tr}(\mathbf{A}D^2 v(t,y))\,dy$$

Sending $t \to 0$, we get

$$\int_{\mathbb{R}^d} m(y)\mathrm{tr}(\mathbf{A}D^2\phi(y)) = 0.$$