# Universality of 2D Yang–Mills

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"Invariant measure and universality of the 2D Yang–Mills Langevin dynamic"
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## Background

**Classical physics:** principle of least action: $\min_\phi \mathcal{S}(\phi)$

**Quantum physics:** functional integral w.r.t. $\exp(-\mathcal{S}(\phi))D\phi$

**Stochastic quantization:** $\partial_t \phi = -\nabla \mathcal{S}(\phi) + \xi$

In this talk $\mathcal{S}(A)$ will be Yang–Mills action, and $A$ will be a Lie algebra valued 1-form.

## Outline of talk

1. Introduce Yang–Mills model, i.e. define $\mathcal{S}(A)$
2. Introduce a class of lattice Yang–Mills models.
3. Recall previous construction $\partial_t A = -\nabla\mathcal{S}(A) + \xi$
   [Chandra,Chevyrev,Hairer,S.]
4. Prove that the dynamics of the lattice models converge to the continuum dynamic (Convergence step + Identify limit)
5. Invariant measure.

## Yang–Mills model

Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra.

$$A = A_1 dx_1 + \cdots + A_d dx_d \ , \qquad A_i(x) \in \mathfrak{g}$$

Yang–Mills action:

$$\mathcal{S}(A) = \int \|F_A\|^2 dx \qquad \text{with} \qquad F_A^{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

Gauge symmetry: $\forall G$-valued function $g$,

$$A \mapsto gAg^{-1} - (dg)g^{-1} \quad \text{leaves } \mathcal{S}(A) \text{ invariant.}$$

Rmk: Quantization $\exp(-\mathcal{S}(A))DA$ is completely formal.

## Lattice Yang–Mills models

(well-defined) lattice models which preserve gauge symmetry.

On $d$-dimensional lattice, we have $U_{xy} \in G$ for each edge $(x, y)$. (convention: $U_{xy} = U_{yx}^{-1}$)

$$\exp(-\mathcal{S}(U)) \prod_{(x,y)} dU_{xy} \qquad \text{where } dU_{xy} \text{ is Haar measure on } G$$

$$\mathcal{S}(U) = \sum_p s(U_{xy} U_{yz} U_{zw} U_{wx}) \qquad \text{with } p = (x, y, z, w)$$

$$s : G \to \mathbf{R} \qquad s(gug^{-1}) = s(u)$$

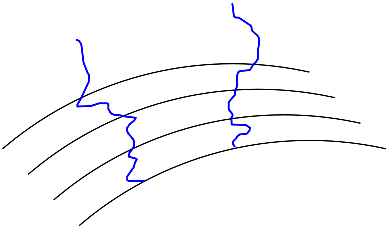
Gauge invariance under $U_{xy} \mapsto g_x U_{xy} g_y^{-1}$, $\forall G$-valued function $g$

Examples: (1) $s(u) = \operatorname{Re}\operatorname{Tr}(\mathrm{id} - u)$ "Wilson"
(2) $\exp(-s(u))$ is heat kernel on $G$ "Villain" (3) $s(u) = |\mathrm{id}, u|_G^2$ "Manton"

Previous work by Chandra, Cheryvev, Hairer and S. (2020,2022)

$\partial_t A = -\nabla \mathcal{S}(A) +$ "DeTurck term" $+ \xi$ on $\mathbf{T}^2$ and $\mathbf{T}^3$

$$\partial_t A_i = \Delta A_i + \big[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\big] + CA_i + \xi_i$$



[CCHS]: There is a finite shift of $C$ such that $A(t)$ has **gauge covariance** property, and thus induces a Markov process $X$ on the space of gauge orbits $\{A\}/\{\text{gauge}\}$.

The above work [CCHS] raised two questions:

(1) University. Is the Markov process $X$ the universal limit of the dynamics of all those discrete models?

(2) Invariant measure. In 80s-90s, 2D YM measure constructed by Driver, Gross, King, Levy, Sengupta (in the sense of random holonomies). Is it really invariant measure under $X$?

[Chevyrev-S. '23] Yes to both questions on $\mathbf{T}^2$.

- Proof of (2) relies on (1).
- Corollary:
  universality of dynamic $\Rightarrow$ universality of 2D YM measure.
  (proof uses uniqueness of invariant measure i.e. ergodicity)

## Universality (from discrete to continuum)

Discrete: $U : E \to G$ is $G$-valued. Continuum: $A$ is $\mathfrak{g}$-valued.

$$\partial_t U = -\nabla \mathcal{S}(U) + \dot{\mathfrak{B}}$$

$\mathfrak{B}$ assigns each edge a $G$-valued BM. This dynamic looks "far" from the limiting SPDE for $A$.

Our strategy is to use $\exp : E^{\mathfrak{g}} \to E^G$ to pull back $\mathcal{S}(U)$ and the Riemannian metric on $E^G$ and $\mathfrak{B}$ on $E^G$ to $E^{\mathfrak{g}}$.

After calculations, $A := \log U$ will satisfies a discrete equation which looks "closer" to the SPDE in continuum.

Recall the SPDE in continuum in [CCHS]

$$\partial_t A_i = \Delta A_i + \left[A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]\right] + CA_i + \xi_i \qquad (1)$$

The discrete equation has the form ($\varepsilon$ is lattice spacing)

$$\partial_t A_i^\varepsilon = \Delta A_i^\varepsilon + \left[A_j^\varepsilon, 2\partial_j A_i^\varepsilon - \partial_i A_j^\varepsilon + [A_j^\varepsilon, A_i^\varepsilon]\right] + \xi_i^\varepsilon + \text{“errors”} \qquad (2)$$

Many (≈50?) error terms, for instance

$$\varepsilon[A_j, [\partial A_j, A_i]] \to cA_i$$

(similar with [Hairer-Quastel'15] on KPZ)

**Proposition.** As $\varepsilon \to 0$, solution of (2) converges to a limit, which is solution of (1) with 'some' $\bar{C}$.

Question: is the limit the same as [CCHS]?
We argue that there is a unique $C$ s.t. (1) is gauge covariant.

## **Identify the limit** (topological argument)

Abelian example $G = U(1)$: $A = A_1 dx_1 + A_2 dx_2$

$$\partial_t A_i = \Delta A_i + C A_i + \xi_i \qquad \text{on } \mathbf{R}_+ \times \mathbf{T}^2 \qquad (1)$$

Gauge transformation: $g \circ A = A - dg\, g^{-1}$ where $g$ is $U(1)$ valued.
Wilson loop observable: $\exp(\int_\ell A)$ for a loop $\ell$.
It's gauge invariant, because $\int_\ell dg\, g^{-1} \in 2\pi i \mathbf{Z}$

Claim: Eq (1) is gauge covariant if and only if $C = 0$.

<u>$C = 0$ case:</u> Assume $\bar{A}(0) = g_0 \circ A(0)$
Then $\bar{A}(t) = g(t) \circ A(t)$ where $g(t)$ solves:

$$\partial_t g\, g^{-1} = d^*(dg\, g^{-1}) \qquad g(0) = g_0$$

$C \neq 0$ case:

Consider $A(0) = 2\pi i\, dx_1$ and $\bar{A}(0) = 0$
They are gauge equivalent $A(0) = \bar{A}(0) - de^{-2\pi i x_1}\, e^{2\pi i x_1}$

$$A(t) = \int_0^t P(t-s, x-y)\xi(dsdy) + e^{tC}A(0)$$
$$\bar{A}(t) = \int_0^t P(t-s, x-y)\xi(dsdy)$$

where $P = (\partial_t - \Delta - C)^{-1}$.
Take $\ell(s) = (s, 0) \subset \mathbf{T}^2$. We have $\mathbf{E}\exp(\int_\ell \bar{A}(t)) \neq \mathbf{E}\exp(\int_\ell A(t))$.

This is because

$$\exp(\int_\ell e^{tC} A(0)) = \exp(e^{tC} 2\pi i) \neq 1$$

## Identify the limit

**Non-abelian case (general $G$) more complicated...**

Euler estimates: for small $t$, nonlinear effect is of next order comparing to the discrepancy created in the previous page

Roughly speaking we will look for a curve $\zeta : [0,1] \to \mathfrak{g}$ with $\zeta(0) = 0 \neq \zeta(1)$ such that its lift $L : [0,1] \to G$ is given by

$$dL\, L^{-1} = d\zeta \qquad L(0) = L(1) = id$$

This is done using sub-Riemannian geometry (Chow-Rashevsky).

## Invariant measure

Theorem. There is a unique prob measure $\mu$ on the orbit space, whose holonomies agree with earlier construction (Levy '90s)
$\mu$ is the unique invariant measure of $X$ constructed by [CCHS].

1. On lattice $\mu_\varepsilon$ is explicitly invariant.
2. Pass to limit

   J.Bourgain'96 *"Invariant measures for the 2D-defocusing nonlinear Schrödinger equation"*

$$\mathbf{P}\Big(\sup_{t\in[0,\delta]}\|X_\varepsilon(t)\|\geq h\Big)$$
$$\leq \mathbf{P}\Big(\sup_{t\in[0,\delta]}\|X_\varepsilon(t)\|\geq h \,\Big|\, \|X_\varepsilon(0)\|\leq L\Big)+\mathbf{P}\Big(\|X_\varepsilon(0)\|>L\Big)$$

$X_\varepsilon(0)\sim\mu_\varepsilon$, moments bound + Markov inequality

## Moments bound (Gauge fixing and "rough Uhlenbeck estimates")

K.Uhlenbeck'82: *"Connections with $L^p$ bounds on curvature"*

- Assuming $A$ is small, one can bound $A$ by curvature $F_A$ in Coulomb gauge
- Piece together local bounds by "continuity argument".
- This paper influenced many deep results in differential geometry later on.

1. At large scales, we fix an axial gauge to have good probability properties
2. At intermediate scales where $A$ becomes reasonably small, we fix Coulomb gauge (all the way down to the smallest scales) to have sharp regularity properties.

(Sharpening earlier work by [Chevyrev'19])

References:

1. H. Shen, Stochastic quantization of an Abelian gauge theory, *Comm. Math. Phys. (2021) 384(3), 1445-1512*
2. A. Chandra, I. Chevyrev, M. Hairer and H. Shen, Langevin dynamic for the 2D Yang–Mills measure, *Publ. Math. IHES. 2022, 1-147.*
3. A. Chandra, I. Chevyrev, M. Hairer and H. Shen, Stochastic quantisation of Yang–Mills–Higgs in 3D, *arXiv:2201.03487*
4. H. Shen, S. Smith and R. Zhu, A new derivation of the finite N master loop equation for lattice Yang–Mills, *arXiv:2202.00880*
5. H. Shen, R. Zhu and X. Zhu, A stochastic analysis approach to lattice Yang–Mills at strong coupling. *Comm. Math. Phys, 2022: 1-47.*
6. I.Chevyrev and H. Shen, Invariant measure and universality of the 2D Yang–Mills Langevin dynamic. *arXiv:2302.12160*

Thank you and happy birthday Timo!