Kinetic Theory for Hamilton-Jacobi Equation

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Motivation

Hamilton-Jacobi PDE

In many models of interest we encounter an interface that separates different phases and is evolving with time. The interface at a location x and time t changes with a rate that depends on (x, t), and the inclination of the interface at that location. If the interface is represented by a graph of a function $u : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$, then a natural model for its evolution is a Hamilton-Jacobi PDE:

$$u_t = H(x, t, u_x), \qquad u(x, 0) = g(x).$$

We may also study $\rho = u_x$ (almost equivalently)

$$\rho_t = (H(x, t, \rho))_x.$$

(In discrete setting some of the variables x, t or u are discrete; examples SEP, HAD, etc.)

H is often random (hence u is random), and we are interested in various scaling limits of solutions.

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Examples

$H(x,t,p) = H_0(p) - V(x,t)$ where $H_0(p)$ is convex, and formally

$$V(x,t) = \sum_{i \in I} 1\!\!1(x = x_i)\delta_{s_i}(t),$$

where $\omega = \{(x_i, s_i) : i \in I\}$ is a Poisson point process.
When $H_0(p) = \frac{1}{2}p^2$, and $d = 1$, this HJE was studied by Bakhtin, Cator, Khanin (2014) (existence of invariant measures).
When $H_0(p) = |p|$, the model is equivalent to Polynuclear Growth, and is exactly solvable.

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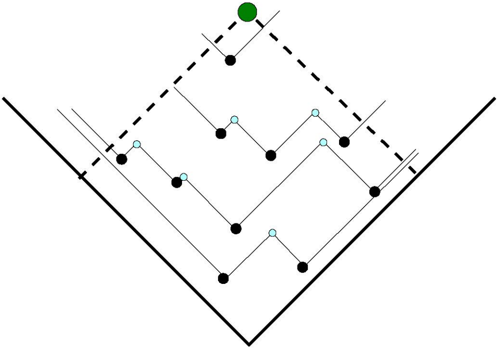

Level sets of $u(x,t) = 1, 2, 3.4$ when $u(x,0) = -\infty \mathbb{1}(x \neq 0)$.

A Natural Question/Strategy

Write Φ_t for the the flow of HJE (in other words $u(\cdot, t) = (\Phi_t g)(\cdot)$).

Select g (or ∇g) according to a (reasonable) probability measure μ^0. Let us write μ^t for the law of $u(\cdot, t)$ (or $\rho(\cdot, t)$) at time t: $\mu^t = \Phi_t^* \mu^0$.

Question: Can we find a nice/tractable/explicit evolution equation for μ^t?

More Realistic Question: Can we find a family $\mathcal{M}$ of measures that is invariant under Φ_t^*? Describe Φ_t^* on $\mathcal{M}$.

This talk: We describe an invariant family $\mathcal{M} = \{\nu(f) : f \text{ kernel}\}$ with $\Phi_t^* \nu(f) = \nu(\Psi_t(f))$, and we describe the evolution $\Psi_t(f)$ when either $H(x, t, p)$, $d = 1$, or $H(x, t, p) = H(p)$ and g is piecewise linear convex function. [Kaspar-FR (2016,2019) after a conjecture of Menon-Srinivasan (2010), FR-Ouaki (2022, 2023), FR (2023)]

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Assumption: General H, $d = 1$

Given $z = (y, s) \in \mathbb{R}^{d+1}$, by a fundamental solution $W(\cdot; z) : \mathbb{R} \times (s, \infty) \to \mathbb{R}$ associated with z we mean

$$W(x, t; z) = \sup \int_s^t L(\xi(\theta), \theta, \dot{\xi}(\theta))\, d\theta,$$

where the supremum is over

$$\xi \in C^1([s, t]; \mathbb{R}^d),\ \xi(s) = y,\ \xi(t) = x.$$

and L is the Legendre transform of H in the p-variable:

$$L(x, t, v) = \inf_p \left(p \cdot v + H(x, t, p)\right), \quad H(x, t, p) = \sup_v \left(L(x, t, v) - p \cdot v\right).$$

We also set $M(x, t; z) = W_x(x, t; z)$ for the x-derivative of W.

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A solution u, subject to an initial condition $u(x,s) = u^0(x)$, has a representation

$$u(x,t) = \sup_y \left(u^0(y) + W(x,t;y,s)\right), \quad t \geq s.$$

We search for a solution of the form

$$u(x,t) = \sup_{y \in I} \left(g(y) + W(x,t;y,s)\right), \quad t \geq s,$$

with I a discrete set. Alternatively

$$\rho(x,t) = W_x(x,t;y(x,t),s) = M(x,t;y(x,t),s),$$

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Assumption: A Theorem (General H, $d = 1$)

If $\rho(x, t_0) = M(x, t; y^0(x), s$, for some $t_0 > s$, and for a Markov jump process y^0 associated with $g(x, s, y_-, y_+)$, then for $t > t_0$, we have $\rho(x, t) = M(x, t; y(x, t))$, where $y(\cdot, t)$ is a Markov jump process associated with $g(x, t, y_-, y_+)$. Assume that the kernel $g(x, t, y_-, y_+)$ satisfies the following (kinetic) equation:

$$g_t - (\hat{v}g)_x = Q(g) = Q^+(g) - Q^-(g) = Q^+(g) - gL(g),$$

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Observe that u is convex in (x, t).
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for a discrete set P. There would be a minimal set $P(t)$ such that

$$u(x,t) = \sup_{\rho \in P(t)}(x \cdot \rho - h(\rho) + tH(\rho)),$$

$$s < t \implies P(t) \subseteq P(s).$$

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$$g(x) = \sup_{\rho \in P} (x \cdot \rho - h(\rho)) \implies u(x,t) = \sup_{\rho \in P} (x \cdot \rho - h(\rho) + tH(\rho)),$$

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Piecewise linear convex g

P discrete, $h : P \to \mathbb{R}$,

$$g(x) = \sup_{\rho \in P}(x \cdot \rho - h(\rho)).$$

There exists a tessellation $\{C(\rho) : \rho \in P\}$, $C(\rho)$ convex polytope/polyhydron such that

$$g(x) = \sum_{\rho \in P} \mathbb{1}(x \in C(\rho))(x \cdot \rho - h(\rho))$$

$$\nabla g(x) = \sum_{\rho \in P} \mathbb{1}(x \in C(\rho))\rho.$$

Similarly, there exists a tessellation $\{\hat{C}(x) : x \in \hat{P}\}$, $\hat{C}(x)$ convex polytope/polyhydron such that

$$g^*(\rho) = \sum_{x \in \hat{P}} \mathbb{1}(\rho \in \hat{C}(x))(x \cdot \rho - g(x))$$

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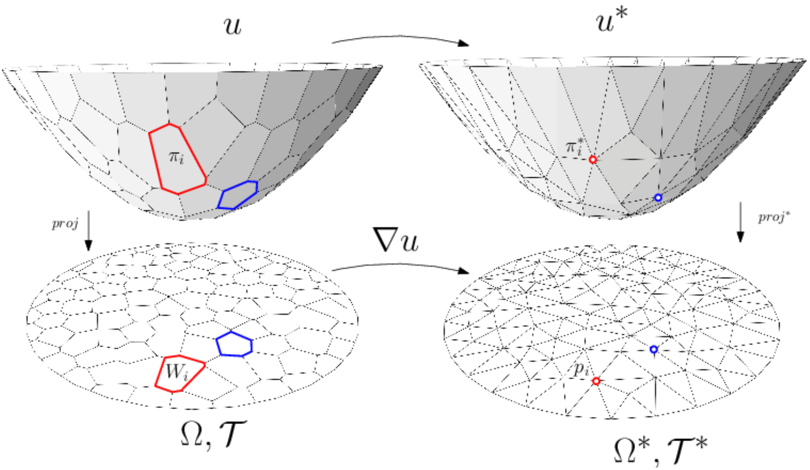
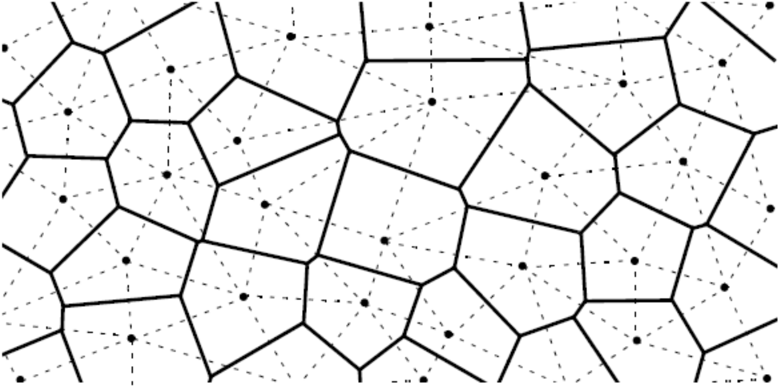


u
u^*
π_i
π_i^*
$proj$
$proj^*$
∇u
W_i
p_i
$\Omega, \mathcal{T}$
$\Omega^*, \mathcal{T}^*$



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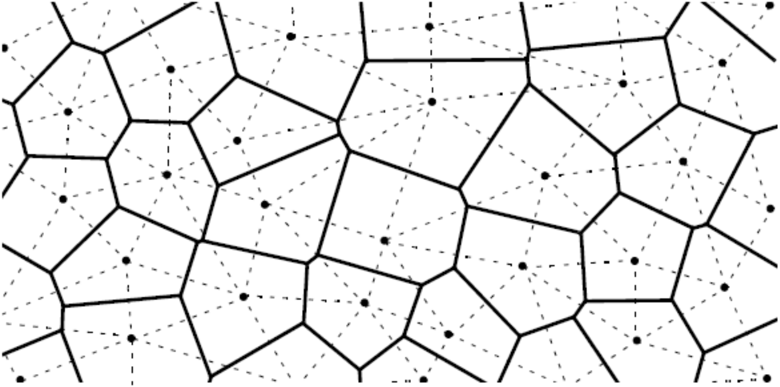
$h(\rho) = |\rho|^2/2$: Voronoi tessellation



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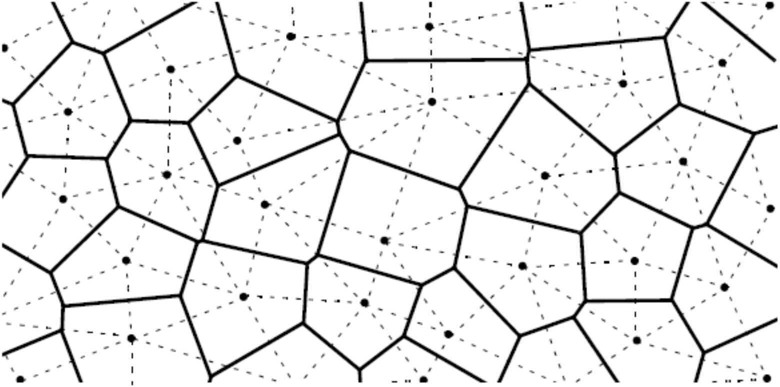

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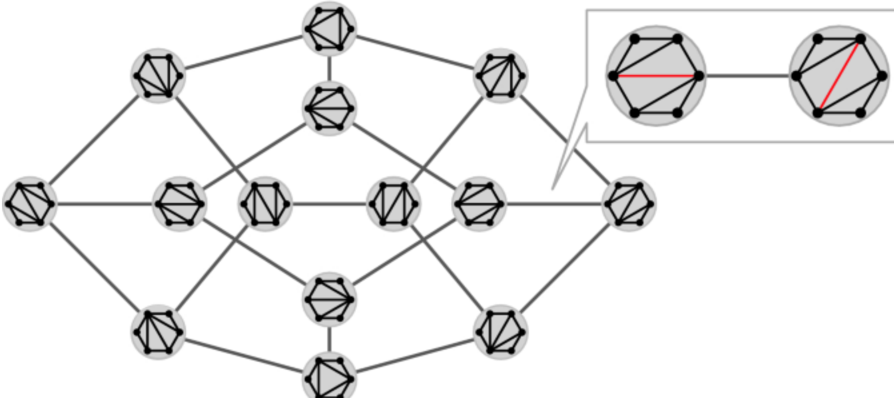
Gelfand-Kapranov-Zelevinsky:

1. The vertices $\sigma_{\mathbf{T}}$ of $\Sigma(P)$ correspond to regular/coherent triangulations $\mathbf{T}$.

2. When there is an edge between $\sigma_{\mathbf{T}}$ and $\sigma_{\mathbf{T}'}$?
When $\sigma_{\mathbf{T}}$ and $\sigma_{\mathbf{T}'}$ differ on a subtriagulation: The discrepancy $\sigma_{\mathbf{S}}$ and $\sigma_{\mathbf{S}'}$ are the two possible triangulations of a circuit.



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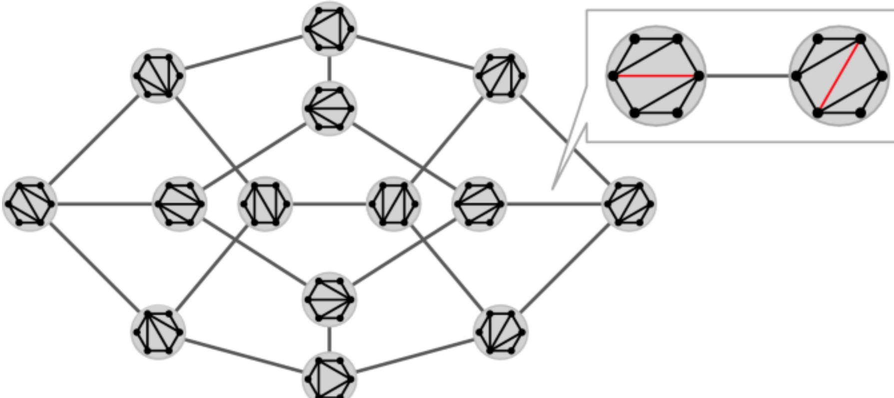

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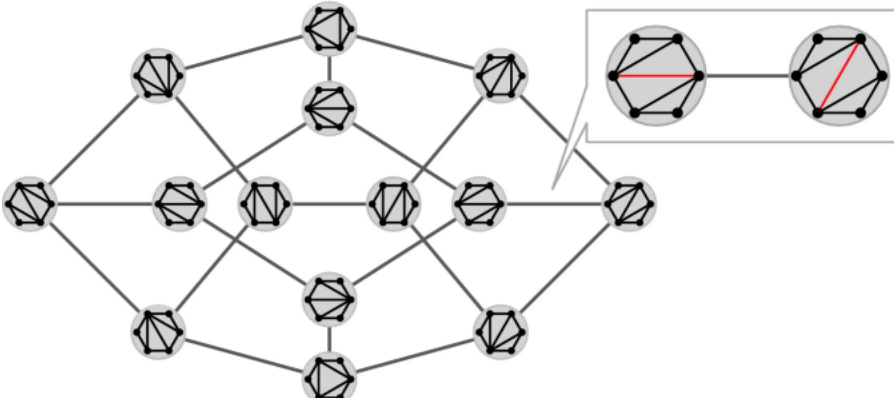

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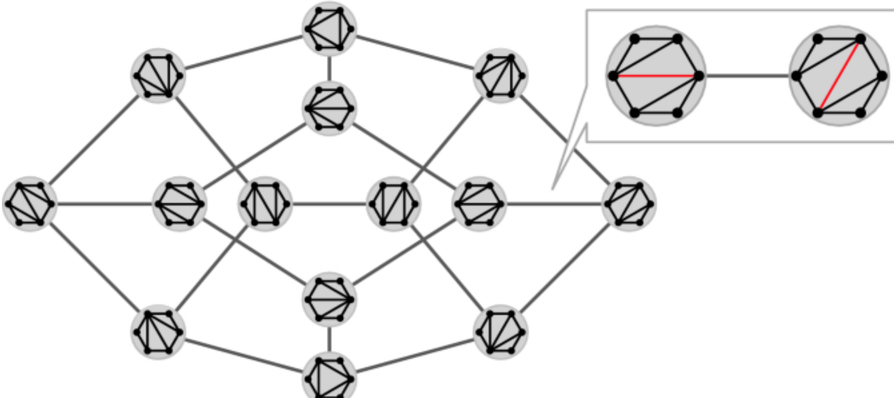

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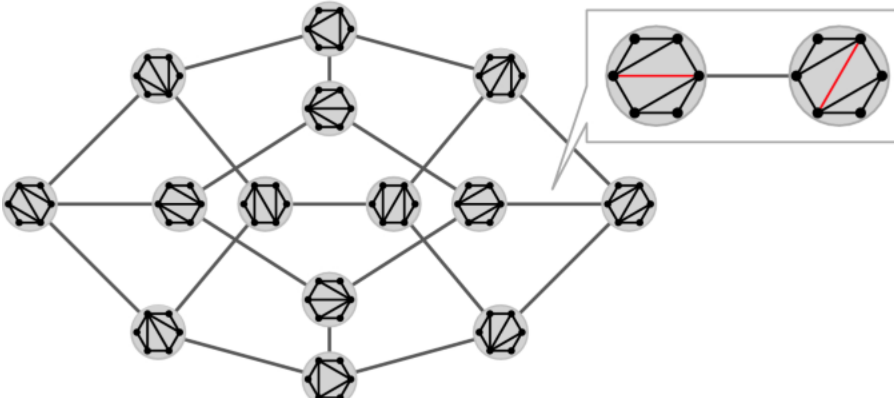

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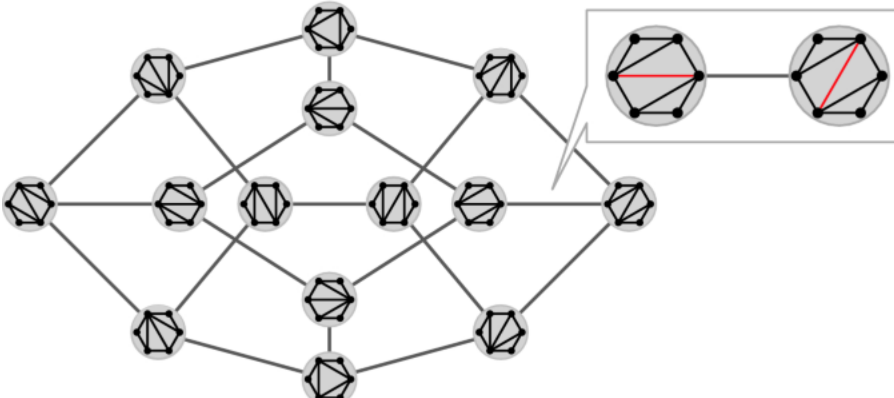

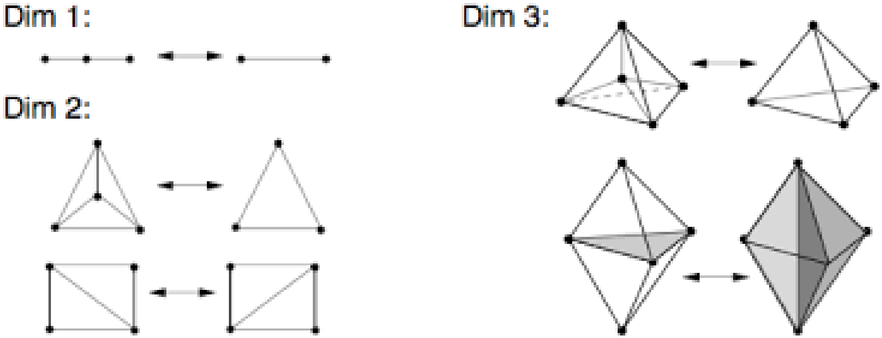
Dim 1:
Dim 2:
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$d = 2$:

(i) Either diagonals are swapped,

(ii) or three triangles are replaced with one triangle.

In the context of Hamilton-Jacobi equation (i) means the occurrence of a collision between two vertices of the corresponding Laguerre tessellation.

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Hamilton-Jacobi Dynamics

We wish to understand the dynamics of $t \mapsto \mathbf{X}_t$ and $t \mapsto \mathbf{T}_t$.
Without loss of generality we may assume that P is finite.
(Speed of propagation is finite.)
Main Theorem: There are times

$$t_0 = 0 < t_1 < \cdots < t_k < t_{k+1} = \infty,$$

such that
1. In (t_i, t_{i+1}), we have a free motion.
2. At transition

$$t_i- \to t_i+,$$

we either have a coagulation or collision.
3. For $t > t_k$, the triangulation associated with h^t is very special (stable). We call it anti-H triangulation.
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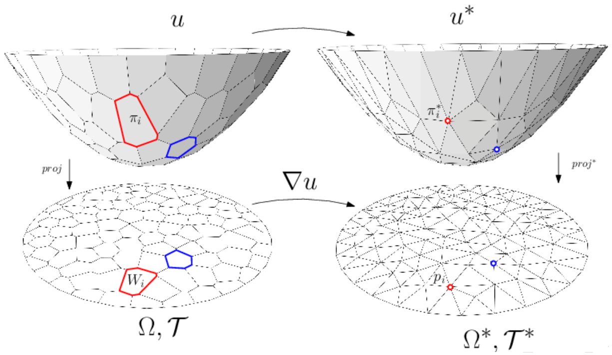
During a free motion interval:

u^*: The triangulation (domains of linearity of u^*) $\mathbf{T}_t$ stays put, but the slopes of the graph of u^* change linearly with a velocity that will be described shortly.

u: The slopes of the graph stay put. The vertices of $\mathbf{X}_t$ travel according to their velocities. If t, t' are two times in the interval, then the corresponding faces in $\mathbf{X}_t$ and $\mathbf{X}_{t'}$ are parallel. Angles do not change.



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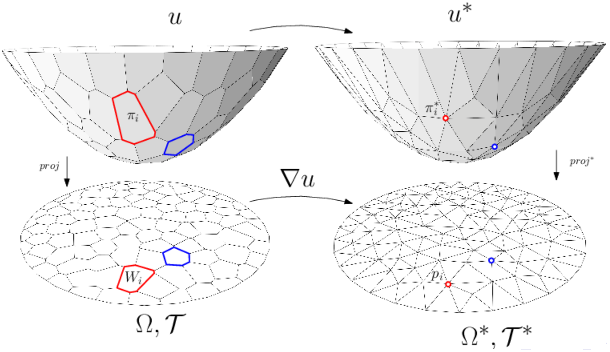

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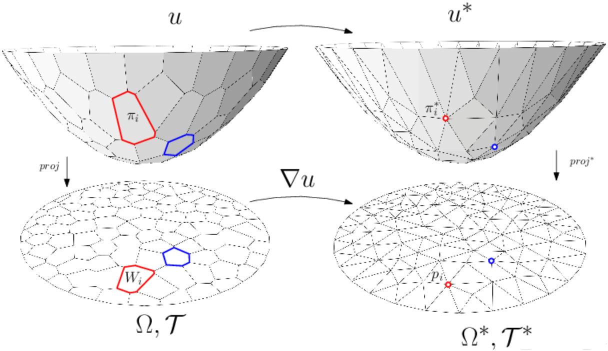

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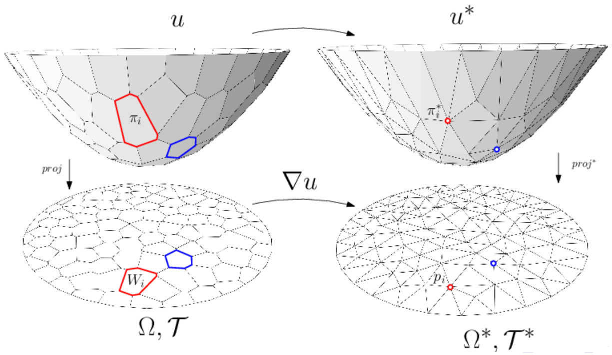

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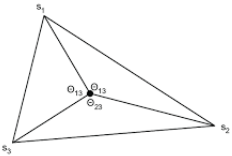

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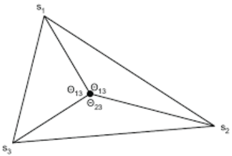

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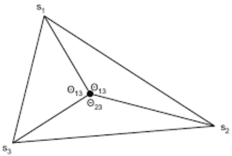

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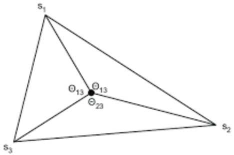

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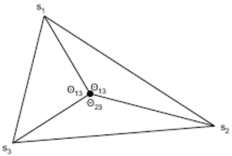

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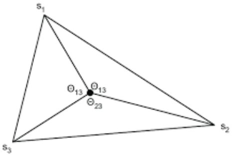

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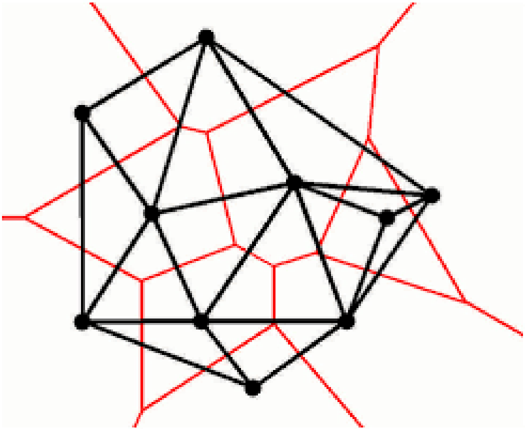
Hamilton-Jacobi Dynamics: Coagulation

The red triangle shrinks: Triangles in $\mathbf{X}_t$ can only shrink (not true for other type of cells).



Hamilton-Jacobi Dynamics: Collision

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u: Before t_i there are two vertices that travel according to their velocities and move towards each other.

At t_i, these vertices collide and gain new velocities.

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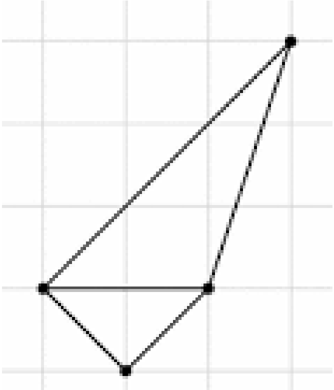

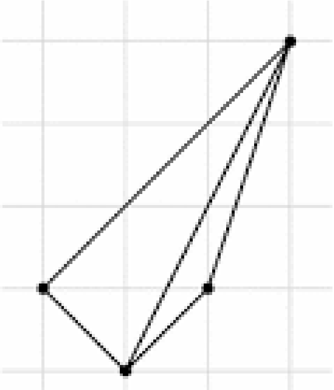

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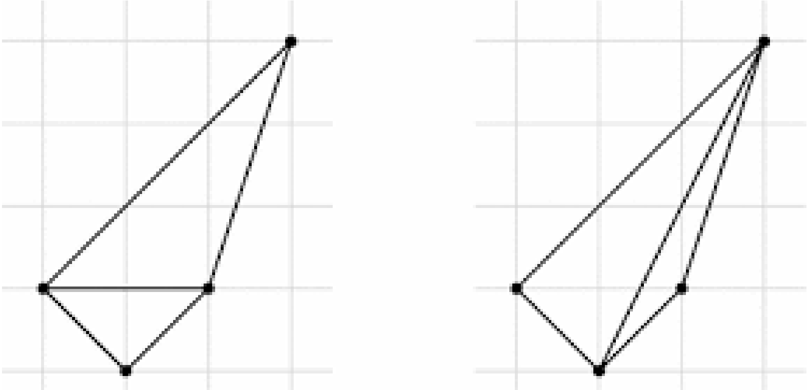

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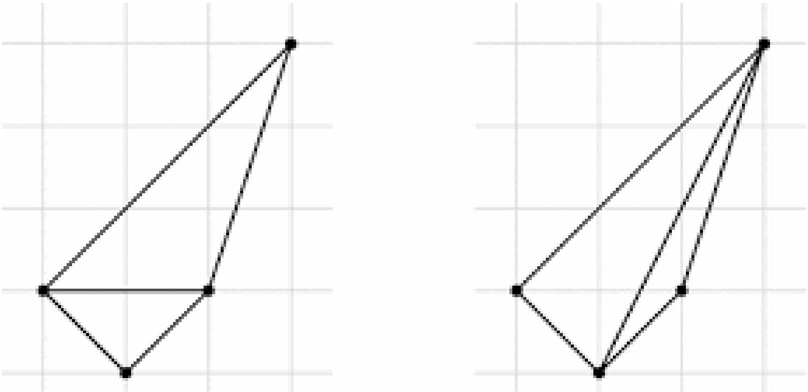

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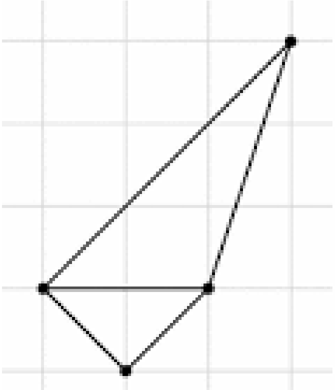
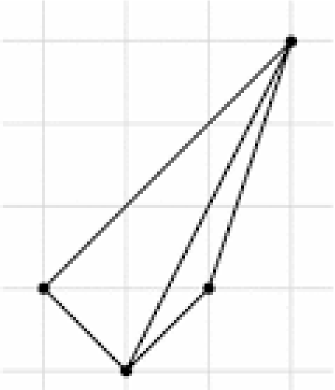
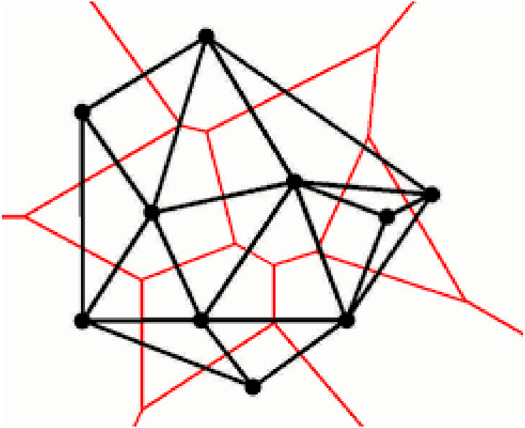




Hamilton-Jacobi Dynamics: Collision

Two red vertices may get closer or move away from each other.



Hamilton-Jacobi Dynamics: Velocities

Remarks

1. $X(\rho) \cap X(\rho')$ is a common face of $X(\rho)$ and $X(\rho')$.
The vector $\rho - \rho' \perp X(\rho) \cap X(\rho')$ (In dimension one this is known as Rankine-Hugoniot Formula).
It points from $X(\rho')$ side to $X(\rho)$ side (this is entropy condition/viscosity criteria).

2. If T is a triangle/simplex in the triangulation, then it is associated with a vertex $x(T) = x^t(T)$ that is uniquely determined from solving

$$x^t(T) \cdot (\rho - \rho') = h^t(\rho) - h^t(\rho'), \qquad \rho, \rho' \in T.$$

3. The velocity of $x^t(T)$ is $-v(T)$, where $v(T)$ is the unique solution of the linear system

$$v(T) \cdot (\rho - \rho') = H(\rho) - H(\rho'), \qquad \rho, \rho' \in T.$$

Moral: v is a vertex in the tessellation $\mathbf{X}(H)$.

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Hamilton-Jacobi Dynamics: Circuits

If R is a circuit, then there exists a function $c : R \to (0, \infty)$ and a decomposition $R = R^- \cup R^+$ such that

$$\sum_{m \in R^{\pm}} c(m) = 1,$$

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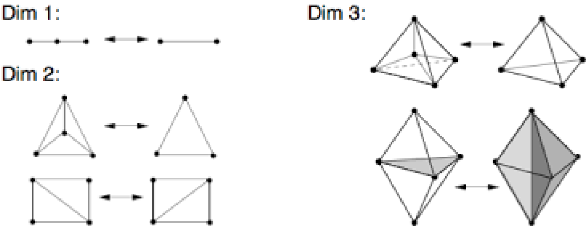

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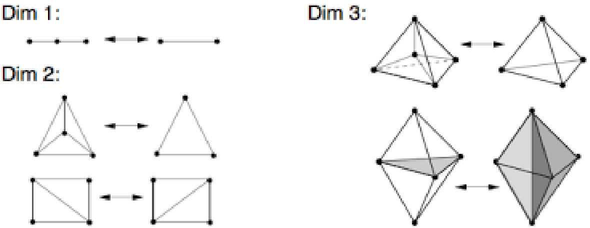

Hamilton-Jacobi Dynamics: Positive Edges

There are two triangulations:

$$\mathbf{T}^{\pm}(R) = \{R \setminus \{m\} : \; m \in R^{\mp}\}.$$

Choose $\pm$ so that

$$\hat{H}(R) = \sum_{m \in R^+} c(m)H(m) - \sum_{m \in R^-} c(m)H(m) \geq 0.$$

In this way the restriction of H to R is associated with the triangulation $\mathbf{T}^-(R)$.
If two triangulations $\mathbf{T}$ and $\mathbf{T}'$ are vertices of an edge of the secondary polytope, then they differ only on a circuit R.
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1. The time of a coagulation of a shrinking $f : R \to \mathbb{R}$:

$$\tau = \frac{\hat{f}(R)}{\hat{H}(R)}.$$

2. If $f : R \to \mathbb{R}$, and $\hat{f}(R) < 0$, then the triangulation induced by f is $\mathbf{T}^{+}(R)$ and there will be no collision.
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Measures on Laguerre Tessellations

Goal: We wish to construct a family $\{\nu(f)\}$ of probability measures on $\mathcal{C}$ for a given kernel $f(x, \rho_-, \rho_+)$; $x \in \mathbb{R}^d$, $\rho^\pm \in \mathbb{R}^d$. Here $f(x, \rho^-, \rho^+)$ is a rate at which ρ^- switches to ρ^+ at x. The measure $\nu = \nu(f)$ is a Gibbs-like measure.
Remark: Assume $d = 2$. Let $C(\rho^-)$ and $C(\rho^+)$ be two adjacent cells. Choose $\tau(\rho^-, \rho^+)$ a vector in the direction of the common edge.

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The vector $\rho^+ - \rho^-$ points from the ρ^--side to the ρ^+-side (convexity).

Gibbs Measure/Rough Description

1. Build a random tessellation inside a set, say a box.
2. Vary the size of the box. Verify the consistency.

How do we build our tessellation in a box?

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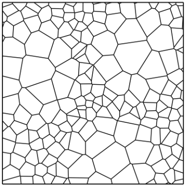
Important Point A choice of $\tau(\rho^{-}, \rho^{+})$ (normal to $\rho^{+} - \rho^{-}$) gives an orientation to edges.



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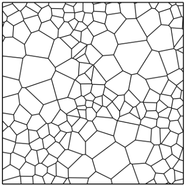

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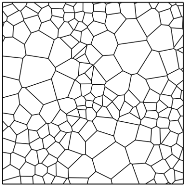

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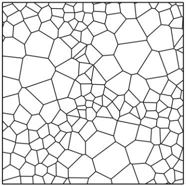

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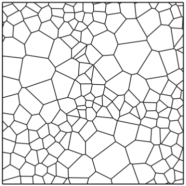

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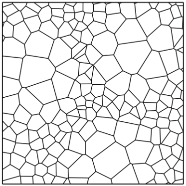

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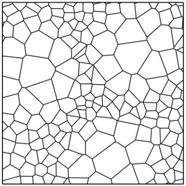

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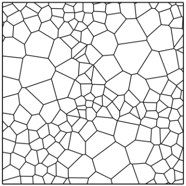

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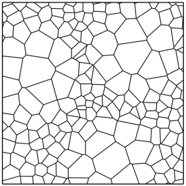

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(Interior Construction) Inside the box, create more vertices: An edge may branch off to two edges.
(Boundary Condition, More Details) Move counter-clockwise with speed one, and change from ρ^- to ρ^+ at point x with rate

$$[\tau(\rho^-, \rho^+) \cdot n(x)]^+ f(x, \rho^-, \rho^+),$$

where $n(x)$ is the inward unit normal at x.
How do we resolve the intersection of edges inside the box? This can be achieved if we assume

$$f(x, \rho_-, \rho_+) > 0 \implies \tau(\rho_-, \rho_+) \text{ points upward}$$

Start from jump points on the boundary, and move them inside the box with unit speed in x_2 direction. These points are only at the bottom or the sides of the box.Think of x_2 as time.

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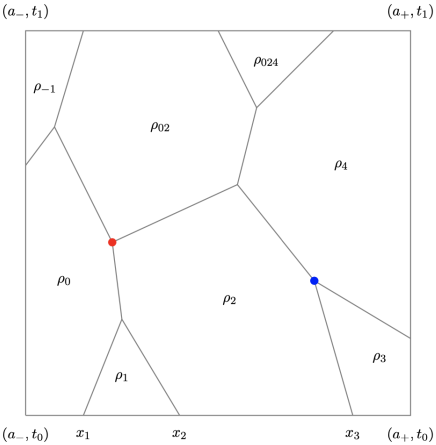

FIGURE 1. The blue dot represents the coagulation of the particles with labels (ρ_2, ρ_3) and (ρ_3, ρ_4) into the particle with label (ρ_2, ρ_4). The red dot represents the fragmentation of the particle with label (ρ_0, ρ_2) into two particles of respective labels (ρ_0, ρ_{02}) and (ρ_{02}, ρ_2).

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(Coalesence) Before collision of edges:
Edge 1: separating $C(\rho^-)$ from $C(\rho^*)$
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Gibbs Measure (Consistency)

These measures are consistent if f satisfies a kinetic equation (FR and Ouaki (2022)).
Set

$$\alpha(\rho_-, \rho_+) = (\rho_+^2 - \rho_-^2)/(\rho_+^1 - \rho_-^1),$$

for the slope of $\rho_+ - \rho_-$, so that we can choose $\tau = (-\alpha, 1)$. Put

$$F = \tau f = (-\alpha f, f), \qquad F^\perp = (f, \alpha f).$$

Kinetic Equation:

$$div(F(\rho_-, \rho_+)) = (F^\perp * F)(\rho_-, \rho_+) - F^\perp \cdot (A(F)(\rho_+) - A(F)(\rho_-)),$$

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$$div(F(\rho_-, \rho_+)) = (F^\perp * F)(\rho_-, \rho_+) - F^\perp \cdot (A(F)(\rho_+) - A(F)(\rho_-)),$$

where

$$A(F)(\rho) = \int F(\rho, m) dm.$$

Gibbs Measure (Consistency)

These measures are consistent if f satisfies a kinetic equation (FR and Ouaki (2022)).
Set

$$\alpha(\rho_-, \rho_+) = (\rho_+^2 - \rho_-^2)/(\rho_+^1 - \rho_-^1),$$

for the slope of $\rho_+ - \rho_-$, so that we can choose $\tau = (-\alpha, 1)$. Put

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Back to HJE

So far we have a family (ν_f : f solves the kinetic equation) of probability measures on $\mathcal{C}$.

Claim This family is invariant under HJ flow in some cases (for example when $H(p_1, p_2) = H_1(p_1) + H_2(p_2)$). The initial $f(x, \rho_, \rho_+)$ evolves to $f(x, t, \rho_, \rho_+)$, which solves another kinetic-like PDE of similar flavor.

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