Instability in the KPZ equation

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Joint work with Sean Groathouse, Firas Rassoul-Agha, and Timo Seppäläinen

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Happy Birthday Timo!

The KPZ equation and SHE

- The KPZ equation is

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}(\partial_x h(t,x))^2 + \beta W(t,x), \quad h(s,x) = h_s(x).$$

- We consider the Cole-Hopf solution $h = \log Z_\beta$, where Z_β solves the SHE

$$\partial_t Z_\beta(t,x) = \frac{1}{2}\partial_{xx} Z_\beta(t,x) + \beta Z_\beta(t,x) W(t,x), \quad Z_\beta(s,x) = e^{h_s(x)}.$$

- We will often work with the four-parameter field $\{Z_\beta(t,y|x,s) : t > s, x, y \in \mathbb{R}\}$ of narrow wedge solutions studied by Alberts, Khanin, Quastel (2012) and Alberts, Janjigian, Rassoul-Agha, Seppäläinen (2022).

The Busemann process

- Janjigian, Rassoul-Agha, and Seppäläinen (2022) constructed the Busemann process $\{b_\beta^{\lambda\pm}(s,x,t,y) : s,x,t,y,\lambda \in \mathbb{R}\}$ defined for all directions simultaneously.
- They define

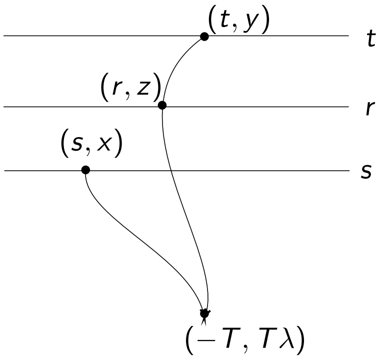
$$\Lambda_\beta^\omega = \{\lambda \in \mathbb{R} : b_\beta^{\lambda-}(s,x,t,y) \neq b_\beta^{\lambda+}(s,x,t,y) \\ \text{for some } (s,x,t,y) \in \mathbb{R}^4\}.$$

Busemann functions for the SHE

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

Busemann functions are global solutions: for $s \in \mathbb{R}$, and $t > r$,

$$e^{b_\beta^{\lambda\pm}(s,x,t,y)} = \int_{\mathbb{R}} e^{b_\beta^{\lambda\pm}(s,x,r,z)} Z_\beta(t,y\,|\,r,z)\,dz.$$



The One Force–One Solution principle

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

- For fixed $\lambda \notin \Lambda_\beta$, whenever $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies appropriate growth conditions, the following limit holds uniformly on compact sets of $(s,x,t,y) \in \mathbb{R}^4$:

$$\lim_{r \to -\infty} \frac{\int_{\mathbb{R}} e^{f(r,z)} Z_\beta(t,y|r,z)\,dz}{\int_{\mathbb{R}} e^{f(r,z)} Z_\beta(s,x|r,z)\,dz} = e^{b_\beta^\lambda(s,x,t,y)}.$$

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- One way for f to satisy these conditions is if $f(r, z) = g(z)$ with $\lim_{|z| \to \infty} \frac{g(z)}{z} = \lambda$.
- We call this the One Force–One Solution principle (1F1S)
- The 1F1S principle fails exactly when $\lambda \in \Lambda_\beta$.

History of the 1F1S principle

- 1F1S principles have been established in a fixed direction for the stochastic Burgers equation and the Burgers equation with random forcing: Sinai (1997), Kifer (1997), Khanin-Mazel-Sinai (2000), Hoang-Khanin (2003), Dirr-Souganidis (2005), Gomes, et. al (2005), Bakhtin-Khanin (2010), Bakhtin (2013), Bakhtin-Cator-Khanin (2014), Bakhtin (2016), Bakhtin-Li (2016), Drivats et. al (2022).

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- Busemann functions in FPP/LPP/polymer models: Hoffman (2005), Hoffman (2008), Damron-Hanson (2014), Ahlberg-Hoffman (2016), Hanson (2018), Georgiou-Rassoul-Agha-Seppäläinen (2017), Janjigian-Rassoul-Agha (2020), Janjigian-Rassoul-Agha (2020), Seppäläinen (2020), Seppääinen-S. (2021), Seppäläinen-S. (2023), Rahman-Virǵ (2021), Busani-Seppäläinen-S. (2022), Ganguly-Zhang (2022).

Discontinuities of the Busemann process

Theorem (Janjigian, Rassoul-Agha, Seppäläinen, 2022)

For each $\lambda \in \mathbb{R}$, $\mathbb{P}(\lambda \in \Lambda_\beta^\omega) = 0$. Furthermore, exactly one of the following is true:

- $\mathbb{P}(\Lambda_\beta^\omega = \varnothing) = 1$
- $\mathbb{P}(\Lambda_\beta^\omega \text{ is countably infinite and dense in } \mathbb{R}) = 1$.

- In exponential and geometric LPP (Janjigian, Rassoul-Agha, Seppäläinen and Georgiou, Rassoul-Agha, Seppäläinen), Brownian LPP (Seppäläinen, S.), and the directed landscape (Busani, Seppäläinen, S.), discontinuities exist.

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Theorem (Groathouse, Rassoul-Agha, Seppäläinen, S., 2023+)

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Describing the Busemann process

- It is enough to describe the distribution of the Busemann process for a fixed time level:
$$\{b_\beta^{\lambda\pm}(s,0,s,x) : x,\lambda \in \mathbb{R}\}.$$
In this case, each $x \mapsto b_\beta^{\lambda\pm}(s,0,s,x)$ is a two-sided Brownian motion with diffusivity β and drift λ.

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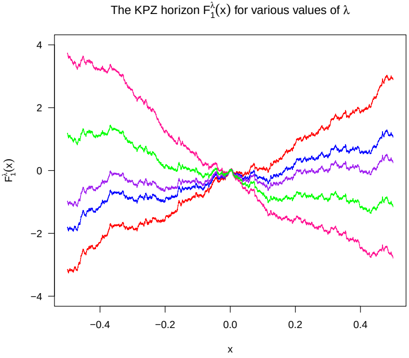

Coupling of Brownian motions with drift

- We construct a process $\{F_\beta^\lambda\}_{\lambda \in \mathbb{R}}$ whose finite-dimensional marginals are described as follows: for $\lambda_1 < \lambda_2 < \cdots < \lambda_k$,

$$(F_\beta^{\lambda_1}, F_\beta^{\lambda_2}, \dots, F_\beta^{\lambda_k}) \overset{d}{=} (Y^1, D_\beta^{(2)}(Y^2, Y^1), \dots, D_\beta^{(k)}(Y^k, \dots, Y^1)),$$

where $Y^1, \dots, Y^k$ are independent Brownian motions with drifts $\lambda_1, \dots, \lambda_k$,

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where $Y^1, \dots, Y^k$ are independent Brownian motions with drifts $\lambda_1, \dots, \lambda_k$, and

$$\begin{aligned} D_\beta^{(2)}(Y^2, Y^1)(y) = Y^1(y) &+ \frac{1}{\beta} \log \int_{-\infty}^{y} e^{\beta(Y^2(x) - Y^1(x))}\, dx \\ &- \frac{1}{\beta} \log \int_{-\infty}^{0} e^{\beta(Y^2(x) - Y^1(x))}\, dx, \end{aligned}$$

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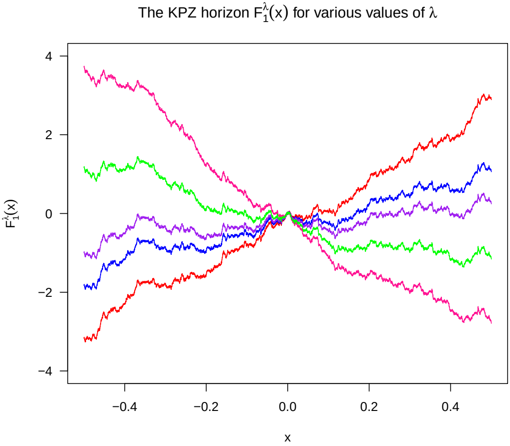
$$D_\beta^{(k)} = D_\beta^{(2)}(D_\beta^{(k-1)}(Y^k, \ldots, Y^2), Y^1).$$

The KPZ horizon

- We call the process $\{F_\beta^\lambda\}_{\lambda \in \mathbb{R}}$ the KPZ horizon.
- The fact that $D_\beta^{(2)}(Y^2, Y^1)$ is a BM with the same drift as Y^2 is a result of O'Connell and Yor (2001).



The KPZ horizon and the Busemann process

Lemma (GRASS 2023+)

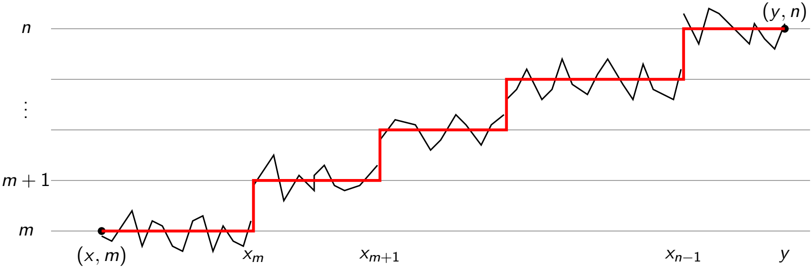
The KPZH describes the Busemann process for the SHE. That is,

$$\{b_\beta^{(\beta\lambda)+}(0,0,0,\bullet)\}_{\lambda\in\mathbb{R}} \stackrel{d}{=} \{\beta F_\beta^\lambda\}_{\lambda\in\mathbb{R}}.$$

The O'Connell-Yor polymer

- For i.i.d Brownian motions $\{B_r\}_{r\in\mathbb{Z}}$, $m \le n$, and $x \le y$, define

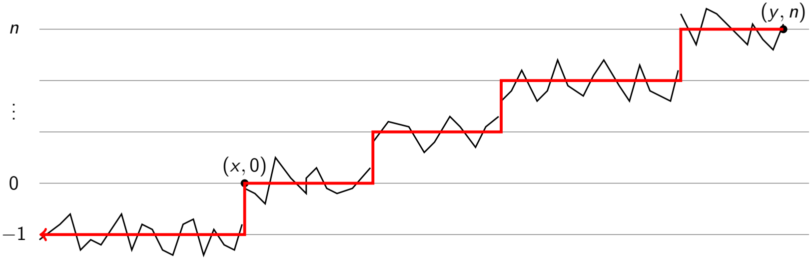
$$Z_\beta^{\mathrm{sd}}(n, y \,|\, m, x) = \int_{x=x_{m-1}<x_m<\cdots<x_{n-1}<x_n=y} \exp\left\{\beta \sum_{r=m}^{n} B_r(x_{r-1}, x_r)\right\} dx_m \cdots dx_{n-1}$$



- For an appropriate random or deterministic initial function $f : \mathbb{R} \to \mathbb{R}$, define, for $n \geq 0$ and $y \in \mathbb{R}$,

$$Z_\beta^{\mathrm{sd}}(n, y | f) = \int_{-\infty}^{y} f(x)\; Z_\beta^{\mathrm{sd}}(n, y | 0, x)\, dx$$

$$\overline{Z_\beta^{\mathrm{sd}}}(n, y | f) = \frac{Z_\beta^{\mathrm{sd}}(n, y | f)}{Z_\beta^{\mathrm{sd}}(n, 0 | f)}.$$



Invariance of the KPZ horizon for O'Connell-Yor polymer

Lemma (GRASS 2023+)

Let $\{F_\beta^\lambda\}_{\lambda \in \mathbb{R}}$ be the KPZH. Then, for $0 < \lambda_1 < \cdots < \lambda_k$ and each $n \geq 0$,

$$\left(\overline{Z_\beta^{\mathrm{sd}}}(n,\bullet|\, e^{\beta F_\beta^{\lambda_1}}), \ldots, \overline{Z_\beta^{\mathrm{sd}}}(n,\bullet|\, e^{\beta F_\beta^{\lambda_k}})\right) \overset{d}{=} \left(e^{\beta F_\beta^{\lambda_1}}, \ldots, e^{\beta F_\beta^{\lambda_k}}\right)$$

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- The recentered versions of Z_β^{sd} satisfy

$$\log \overline{Z_\beta^{\mathrm{sd}}}(n,\bullet|f) = D_\beta^{(2)}(\log \overline{Z_\beta^{\mathrm{sd}}}(n-1,\bullet|f), B_n).$$

- The proof of the theorem follows by an intertwining argument originating from the work of Ferrari and Martin (2007) and adapted in Fan and Seppäläinen (2020), and Seppäläinen and S. (2021).

Scaling invariance of the KPZH

- For $\beta > 0$, the KPZH satisfies the following scaling invariance:

$$\left\{\exp\left(N^{-1/4}\beta F_{N^{-1/4}\beta}^{(\lambda+1/2)N^{-1/4}+N^{1/4}}(y\sqrt{N}) - (\sqrt{N}+1/2)y\right) : y \in \mathbb{R}\right\}_{\lambda\in\mathbb{R}}$$

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- Combined with the convergence of the O'Connell-Yor polymer to the SHE (Nica, 2016) and some additional details, we obtain

Lemma (GRASS 2023+)

For $\lambda_1 < \cdots < \lambda_k$,

$$\left\{\frac{\int_{\mathbb{R}} e^{\beta F_\beta^{\lambda_i}(x)} Z_\beta(t,\bullet|0,x)}{\int_{\mathbb{R}} e^{\beta F_\beta^{\lambda_i}(x)} Z_\beta(t,0|0,x)}\right\}_{1\le i\le k} \stackrel{d}{=} \left\{e^{\beta F_\beta^{\lambda_i}}\right\}_{1\le i\le k}$$

Consequence for Busemann process

Corollary (GRASS 2023+)

The KPZH describes the Busemann process for the SHE. That is,

$$\{b_\beta^{(\beta\lambda)+}(0,0,0,\bullet)\}_{\lambda\in\mathbb{R}} \overset{d}{=} \{\beta F_\beta^\lambda\}_{\lambda\in\mathbb{R}}.$$

Consequences for the Busemann process

- Fix $y > 0$. Then, the process $\lambda \mapsto F_\beta^\lambda(y)$ is strictly increasing and has stationary increments.

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Lemma (GRASS 2023+)

Let $\lambda \mapsto X(\lambda)$ be an increment-stationary, nondecreasing, almost surely continuous process with $\mathbb{E}[X(1) - X(0)] < \infty$. Then, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} n\mathbb{P}(X(n^{-1}) - X(0) > \varepsilon) = 0.$$

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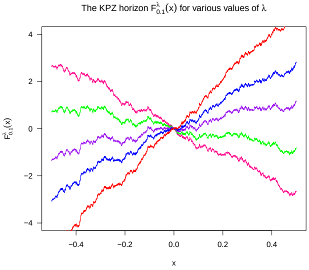
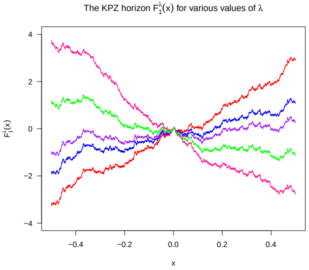
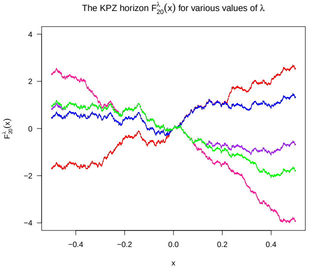
- After a computation, the KPZH satisfies, for $y, \varepsilon > 0$

$$\liminf_{\lambda \searrow 0} \lambda^{-1}\mathbb{P}(F_\beta^\lambda(y) - F_\beta^0(y) > \varepsilon) > 0.$$

Limits of the KPZH in the β parameter







Limits of the KPZH in the β parameter

Theorem (GRASS, 2023+)

As $\beta \to \infty$, we have the distributional convergence

$$(F_\beta^{\lambda_1}, \dots, F_\beta^{\lambda_k}) \Longrightarrow (G_{\lambda_1}, \dots, G_{\lambda_k}),$$

where $\{G_\lambda\}_{\lambda \in \mathbb{R}}$ is the stationary horizon (coupled invariant measures for the KPZ fixed point).

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where B is a standard Brownian motion.

1:2:3 scaling to the KPZ fixed point

- By the scaling relations of the KPZH, for any $\beta > 0$,

$$\{F^{\lambda_i}_{2^{-1/3}T^{1/3}}\}_{1\le i\le k}$$
$$\overset{d}{=} \left\{\beta 2^{1/3}T^{-1/3}F_{\beta}^{\alpha+\beta 2^{1/3}T^{-1/3}\lambda_i}\left(\frac{2^{1/3}T^{2/3}}{\beta^2}\bullet\right) - \frac{2^{2/3}T^{1/3}\alpha}{\beta}\bullet\right\}_{1\le i\le k}.$$

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- Combined with recent results of Wu (2023), the 1:2:3 scaling of the coupled stationary KPZ equation converges to the coupled stationary KPZ fixed point (as a process in space-time).

Thank you

Acknowledgements: During the course of this work, I was partially supported by T. Seppäläinen under National Science Foundation grants DMS-1854619 and DMS-2152362