Geodesic networks in the directed landscape

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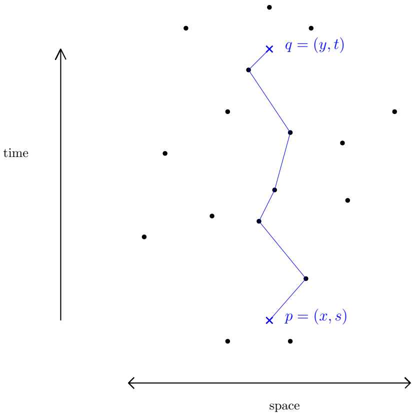
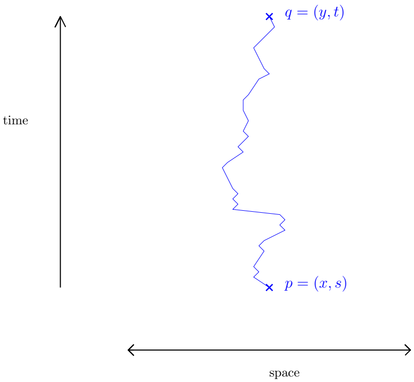
The directed landscape



(a) Poisson LPP



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- $\mathcal{L}$ is a directed metric: $\mathcal{L}(p, q) \geq \mathcal{L}(p, r) + \mathcal{L}(r, q)$
- $\mathcal{L}(x, s; y, t)$ is Hölder-$1/2^-$ (locally Brownian) in x, y but only Hölder-$1/3^-$ in s, t

Paths in $\mathcal{L}$

- Paths are now arbitrary continuous functions $\pi : [s, t] \to \mathbb{R}$.
- We must define length by subdivision. For a function $\pi : [s, t] \to \mathbb{R}$, let

$$|\pi|_{\mathcal{L}} = \inf_{k \in \mathbb{N}} \inf_{s = r_0 < \cdots < r_k = t} \sum_{i=1}^{k} \mathcal{L}(\pi(r_{i-1}), r_i; \pi(r_i), r_i)$$

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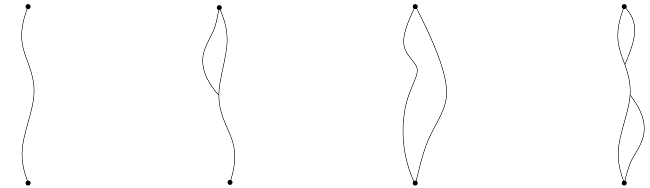
- For any fixed pair p, q, a.s. there is a unique $\mathcal{L}$-geodesic from p to q.
- Not true for all p, q!! **What happens at these exceptional pairs?**

A few possibilities



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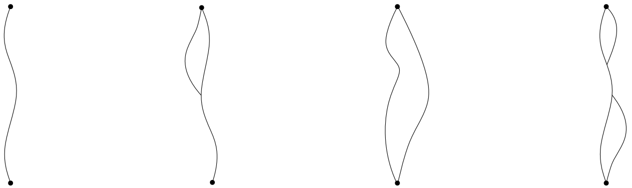

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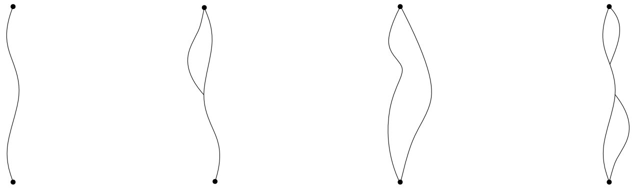

- We call the set of all geodesics from p to q the **geodesic network** from p to q.
- A natural goal is to try to classify the different geodesic networks that will show up in the directed landscape.

Theorem (D.)

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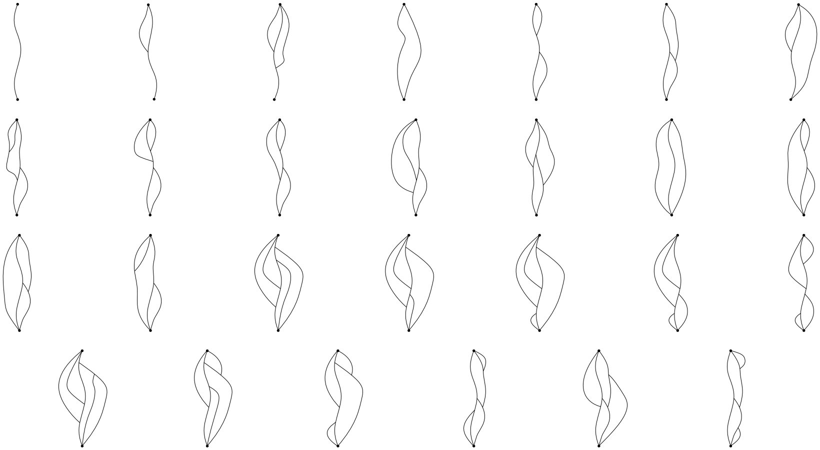
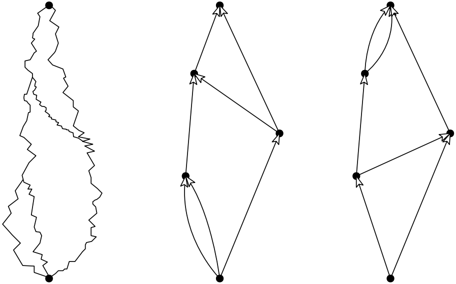


The notion of isomorphism for geodesic networks

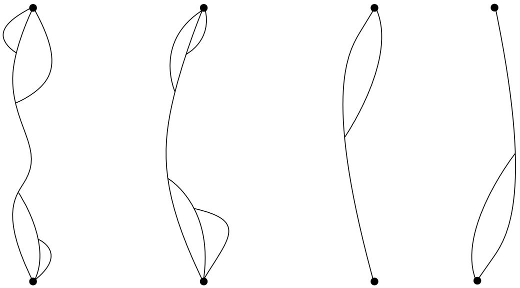
- We associate to each geodesic network a directed graph
- Two networks are isomorphic if the corresponding directed graphs G, G' are either isomorphic, or else G is isomorphic to the transpose of G'



A network, its graph, and its transpose

The notion of isomorphism for geodesic networks

- We associate to each geodesic network a directed graph: the network graph
- Two networks are isomorphic if their network graphs G, G' are either isomorphic, or else G' is isomorphic to the transpose G^T



Isomorphic networks

Rules for geodesic networks

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- Define the 1:2:3 distance on $\mathbb{R}^4_\uparrow$

$$d_{1:2:3}((x,s;y,t),(x',s';y',t')) = |t-t'|^{1/3} + |s-s'|^{1/3} + |x-x'|^{1/2} + |y-y'|^{1/2}.$$

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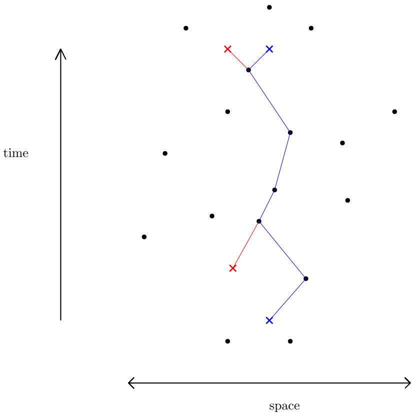
Theorem (D.)

For a graph $G=(V,E)$ satisfying the five rules above, let $N_\mathcal{L}(G)$ denote the set of points in $\mathbb{R}^4_\uparrow$ whose network graph is isomorphic to G. Then:

$$\dim_{1:2:3}(N_\mathcal{L}(G)) = 12 - \frac{|V| + \deg^2(p) + \deg^2(q)}{2}.$$

If the right-hand side above equals 0, *then $N_\mathcal{L}(G)$ is countable.*

Coalescent Geometry in $\mathcal{L}$



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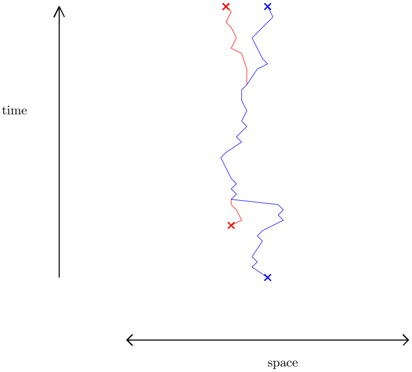

(b) The directed landscape

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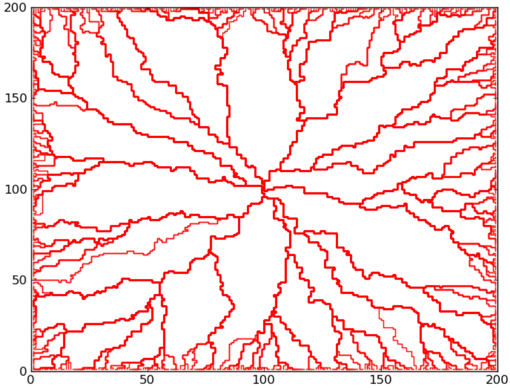
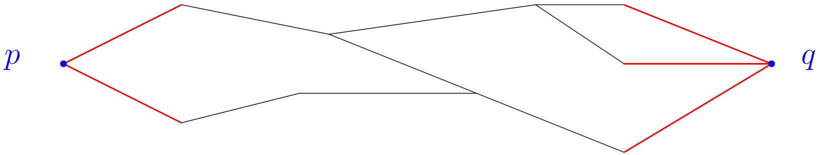
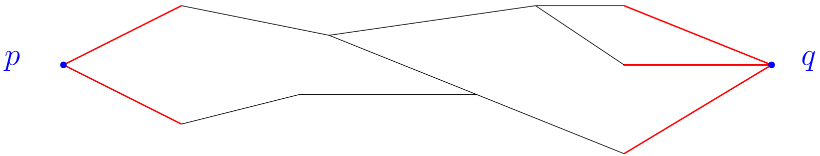


Dissection of a geodesic network



- The ends of the network are special, but the interior is generic
- Rarity of a particular network should be based on the rarity of the endpoint configurations: **geodesic stars**

Dissection of a geodesic network G



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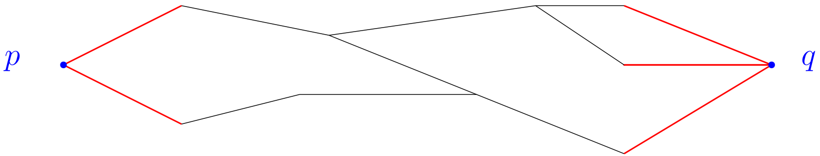

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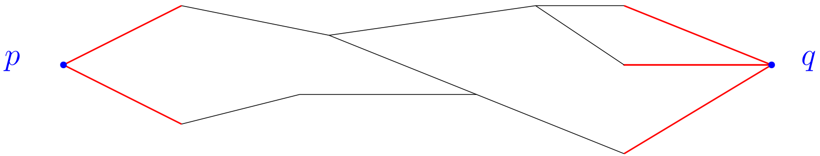

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$$\dim_{1:2:3}(\mathrm{Star}_2) + \dim_{1:2:3}(\mathrm{Star}_3)$$

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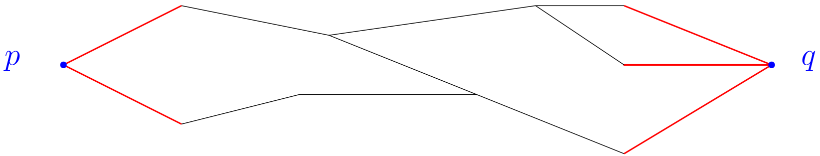

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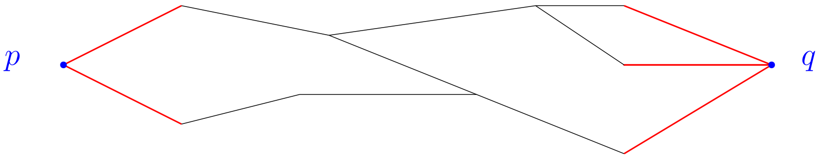

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- The Hausdorff dimension of the set of geodesic networks of type G whose source and sink vertices have degree k and ℓ should be:

$$\dim_{1:2:3}(\mathrm{Star}_k) + \dim_{1:2:3}(\mathrm{Star}_\ell) - \#(\text{Interior faces})$$

Geodesic stars and a final formula

Theorem (D.)

Let $\mathrm{Star}_k \subset \mathbb{R}^2$ *denote the set of geodesic k-stars for* $\mathcal{L}$*. Then:*

$$\dim(\mathrm{Star}_1) = 5, \quad \dim(\mathrm{Star}_2) = 4, \quad \dim(\mathrm{Star}_3) = 2, \quad \mathrm{Star}_4 = \varnothing.$$

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Therefore: the Hausdorff dimension of the set of geodesic networks of type G whose source and sink vertices have degree k and ℓ should be:

$$\dim_{1:2:3}(\mathrm{Star}_k) + \dim_{1:2:3}(\mathrm{Star}_\ell) - \#(\text{Interior faces}) = 12 - \frac{|V| + k^2 + \ell^2}{2}.$$

Other models

- Other random continnum planar metrics have a similar coalescence structure, and so we might expect similar results

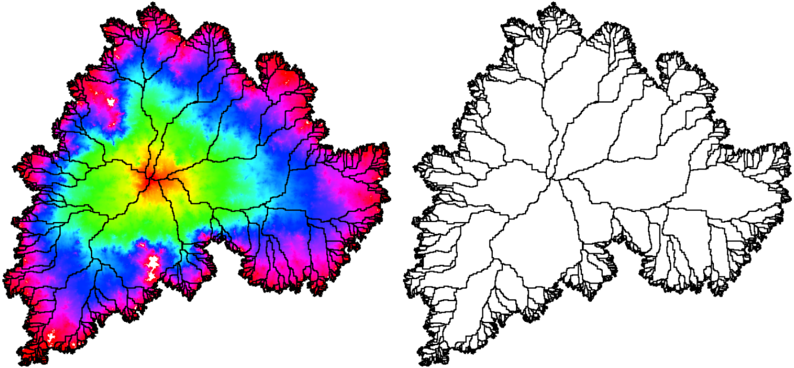

Figure: Geodesics in the Brownian map

The Brownian map and Liouville quantum gravity

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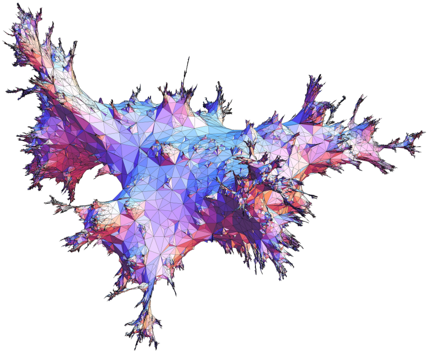
- Liouville quantum gravity is a family of metric spaces parametrized by $\gamma \in (0, 2)$. Conjectured scaling limits of random planar maps sampled from a biased measure



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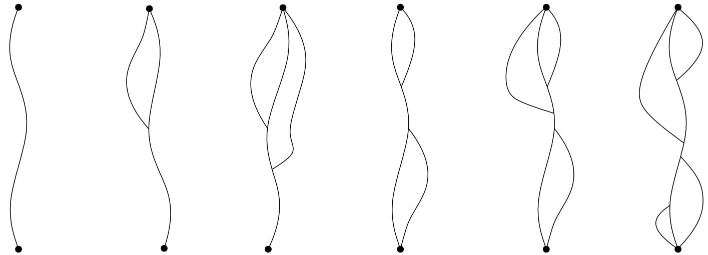
Theorem (Angel-Miermont-Kolesnik, Gwynne, D.)

There are exactly 6 *dense geodesic networks in* X.



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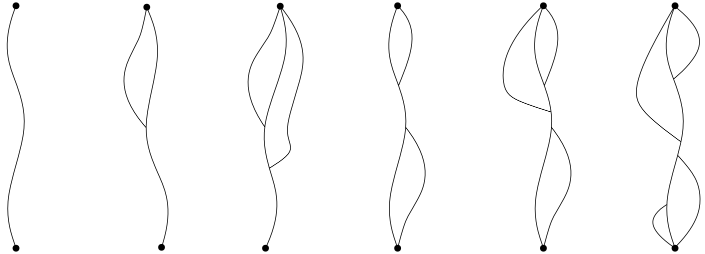

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Conjecture

There are either 27, 28, or 29 geodesic networks in X.