Symmetries of K3 surfaces

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Outline

- Moduli space of **K3 surfaces** and their **symmetry groups**:
  - Kummer surface $\widetilde{T^4/\mathbb{Z}_2} \quad \Leftarrow \quad$ Kummer construction
  - Kummer-like surface $\widetilde{T^4/\mathbb{Z}_3}$

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  - Niemeier lattices

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- **Symmetry surfing** idea
- **Our results** for $\mathbb{Z}_3$-orbifold

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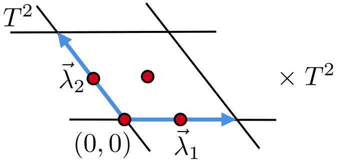

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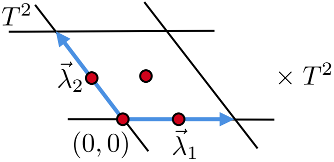

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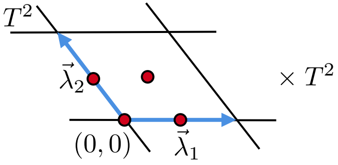

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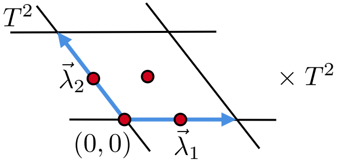

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- The complex surface obtained by minimally resolving the singularities is a K3 surface called a **Kummer surface**
- The Kummer surface carries over the complex and Kähler structures from the underlying torus $T^4(\Lambda)$

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$$H_2(K3, \mathbb{Z})$$

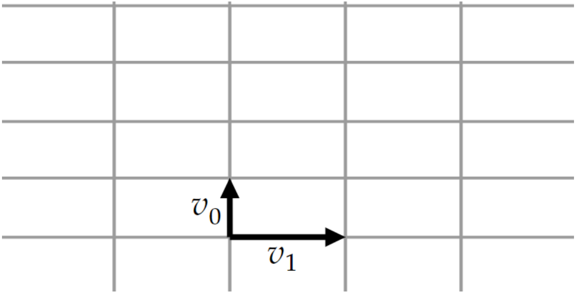
- The integral homology lattice can be described using **gluing**:

$$H_2(K3, \mathbb{Z}) \supset K \oplus P$$

  - K - contributions from the underlying torus $T^4(\Lambda)$
  - P - contributions from the blow-up of singularities

Lattice gluing

Example: **Hyperbolic lattice** U generated by v_0, v_1
Sublattices $A_\pm$ generated by $v_0 \pm v_1$
(w_+, w_-) is a **glue vector**



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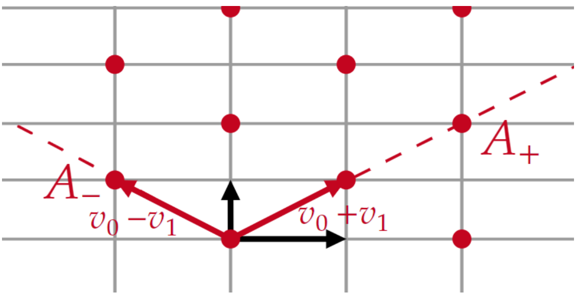

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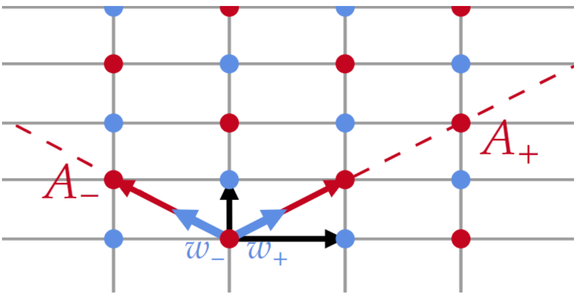

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- 24 even unimodular positive-definite lattices in $\mathbb{R}^{24}$

Roots	**Glue vectors**	**Permutations**
A_1^{24}	4096	M_{24}
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- Action of symmetry group on P can be viewed as a subgroup of **automorphisms of the Niemeier lattice!**

Symmetry surfing

$\mathbb{Z}_2$-orbifold example [Taormina, Wendland '11]:

square Kummer surface

$$\Lambda_S: \quad \vec{\lambda}_1 = (1,0),\ \vec{\lambda}_2 = (i,0)$$
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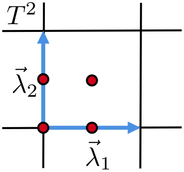

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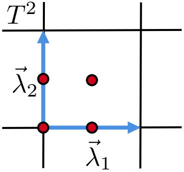

Symmetry group:

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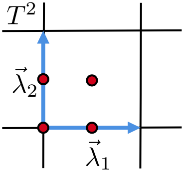

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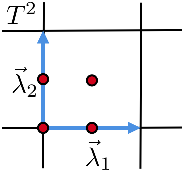

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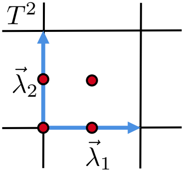

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Can be combined into **overarching symmetry group**:

$$\underbrace{(\mathbb{Z}_2)^4 \rtimes A_7}_{40320} \subset M_{24}$$

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- $T^4(\Lambda)/\mathbb{Z}_3$ orbifold:

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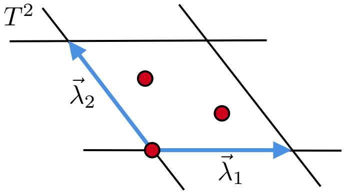

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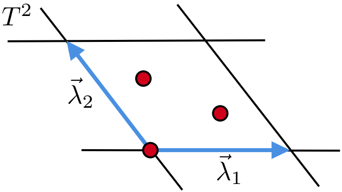

- 9 fixed points under the action of $\mathbb{Z}_3$:

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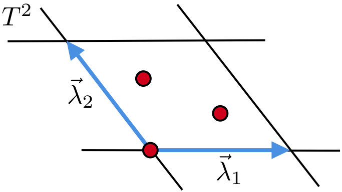

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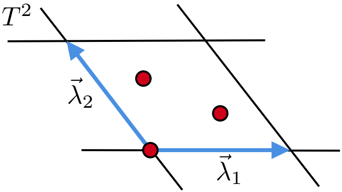

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- The **Kummer-like** K3 surface $\widetilde{T^4(\Lambda)/\mathbb{Z}_3}$ is obtained by minimally resolving the 9 singularities of type A_2

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- The integral homology of the **Kummer-like surface** can be described using gluing techniques:

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- The **Kummer-like lattice** P can be embedded inside the Niemeier lattice $N(A_2^{12})$
- The **symmetry group** of the Kummer-like surface can be realized as a subgroup of **automorphisms of** $N(A_2^{12})$:

$$(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_4 \subset M_{12}$$

Future directions

1. Symmetry surf between $\widetilde{T^4/\mathbb{Z}_2}$ and $\widetilde{T^4/\mathbb{Z}_3}$ cases
    - $(\mathbb{Z}_2)^4 \rtimes A_8 \subset M_{24}$
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Thank you!