On the measures satisfying a monotonicity of surface area with respect to Minkowski sum

Dylan Langharst
Institute of Mathematics of Jussieu [1]

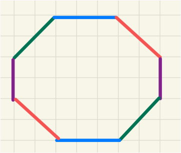
Harmonic Analysis and Convexity BIRS Workshop
20 November 2023

[1] joint with Fradelizi, Madiman, and Zvavitch

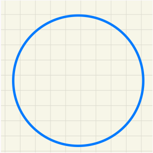
- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)

- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.

- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.



- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.

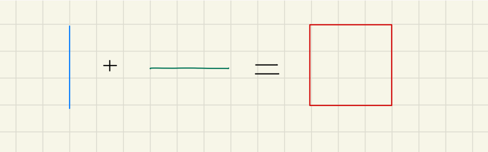


- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.

- We will denote by $\mathrm{vol}_n(K)$ - volume of $K \subset \mathbb{R}^n$
- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.

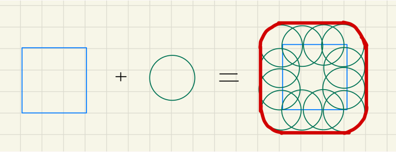
- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0, 1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.

- We will denote by $\mathrm{vol}_n(K)$ - volume of $K \subset \mathbb{R}^n$
- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.



- All of the sets we will consider will be convex (i.e. K is convex if $x, y \in K$ implies $(1-\lambda)x + \lambda y \in K$ for every $\lambda \in [0,1]$.)
- We will usually deal with convex bodies: i.e. convex, compact sets with non-empty interior.

- We will denote by $\mathrm{vol}_n(K)$ - volume of $K \subset \mathbb{R}^n$
- We will often use notion of Minkowski sum: $K + L = \{x + y : x \in K \text{ and } y \in L\}$.



Supermodularity of Volume

Theorem of Fradelizi, Madiman and Zvavitch

Let A be a convex body and B and C compact, convex sets, all in $\mathbb{R}^n$. Then, volume is supermodular, i.e.

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C).$$

Supermodularity of Volume

Theorem of Fradelizi, Madiman and Zvavitch

Let A be a convex body and B and C compact, convex sets, all in $\mathbb{R}^n$. Then, volume is supermodular, i.e.

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C).$$

How would one go about proving this result?

Supermodularity of Volume

Theorem of Fradelizi, Madiman and Zvavitch

Let A be a convex body and B and C compact, convex sets, all in $\mathbb{R}^n$. Then, volume is supermodular, i.e.

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C).$$

How would one go about proving this result?
We all know that $\mathrm{vol}_n(tK) = t^n \mathrm{vol}_n(K)$ for $t \geq 0$, i.e. volume is a homogeneous measure of degree of homogeneity n. But there is much more!!!

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$

Then, volume of Minkowski summation is a polynomial:

$$\mathrm{vol}_n(t_1 K_1 + t_2 K_2 + t_3 K_3) = \sum_{i_1, i_2, \ldots, i_n = 0}^{3} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} t_{i_2} \ldots t_{i_n}.$$

where $V(K_{i_1}, \ldots, K_{i_n})$ is the mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$
Then, volume of Minkowski summation is a polynomial:

$$\text{vol}_n(t_1 K_1 + t_2 K_2 + t_3 K_3) = \sum_{i_1, i_2, \ldots, i_n = 0}^{3} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} t_{i_2} \ldots t_{i_n}.$$

where $V(K_{i_1}, \ldots, K_{i_n})$ is the mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

- $V(K, \ldots, K) = \text{vol}_n(K)$; Mixed volume is symmetric and translation invariant in its arguments.

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$

Then, volume of Minkowski summation is a polynomial:

$$\mathrm{vol}_n(t_1 K_1 + t_2 K_2 + t_3 K_3) = \sum_{i_1, i_2, \ldots, i_n = 0}^{3} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} t_{i_2} \ldots t_{i_n}.$$

where $V(K_{i_1}, \ldots, K_{i_n})$ is the mixed volume of $K_{i_1}, \ldots, K_{i_n}$.

- $V(K, \ldots, K) = \mathrm{vol}_n(K)$; Mixed volume is symmetric and translation invariant in its arguments.
- If $K \subset L$, then $V(K, K_2, K_3, \ldots, K_n) \leq V(L, K_2, K_3, \ldots, K_n)$.

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$

Then, volume of Minkowski summation is a polynomial:

$$\mathrm{vol}_n(t_1 K_1 + t_2 K_2 + t_3 K_3) = \sum_{i_1, i_2, \dots, i_n = 0}^{3} V(K_{i_1}, \dots, K_{i_n}) t_{i_1} t_{i_2} \dots t_{i_n}.$$

where $V(K_{i_1}, \dots, K_{i_n})$ is the mixed volume of $K_{i_1}, \dots, K_{i_n}$.

- $V(K, \dots, K) = \mathrm{vol}_n(K)$; Mixed volume is symmetric and translation invariant in its arguments.
- If $K \subset L$, then $V(K, K_2, K_3, \dots, K_n) \leq V(L, K_2, K_3, \dots, K_n)$.

Notation We denote
$V(K_1, \dots, K_m, K \dots, K) = V(K_1, \dots, K_m, K[n-m])$.

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$
Then, volume of Minkowski summation is a polynomial:

$$\mathrm{vol}_n(t_1 K_1 + t_2 K_2 + t_3 K_3) =$$
$$\sum_{0 \leq k \leq j \leq n} \binom{n}{n-j} \binom{n-j}{n-j-k} V(K_1[n-j-k], K_2[j], K_3[k]) t_1^{n-j-k} t_2^j t_3^k.$$

where $V(K_1[j], K_2[k], K_3[n-j-k])$ is the mixed volume of K_1 j-times, K_2 k-times and K_3 $(n-j-k)$ times.

- $V(K, \dots, K) = \mathrm{vol}_n(K)$; Mixed volume is symmetric and translation invariant in its arguments.
- If $K \subset L$, then $V(K, K_2, K_3, \dots, K_n) \leq V(L, K_2, K_3, \dots, K_n)$.

Notation We denote
$V(K_1, \dots, K_m, K \dots, K) = V(K_1, \dots, K_m, K[n-m])$.

Main Definitions: Mixed Volume

Let K_1, K_2, K_3 be convex bodies in $\mathbb{R}^n$ and $t_1, t_2, t_3 \geq 0$

Then, volume of Minkowski summation is a polynomial:

$$\mathrm{vol}_n(t_1K_1 + t_2K_2 + t_3K_3) =$$
$$\sum_{0 \leq k \leq j \leq n} \binom{n}{n-j}\binom{n-j}{n-j-k} V(K_1[n-j-k], K_2[j], K_3[k]) t_1^{n-j-k} t_2^j t_3^k.$$

where $V(K_1[j], K_2[k], K_3[n-j-k])$ is the mixed volume of K_1 j-times, K_2 k-times and K_3 $(n-j-k)$ times.

- $V(K, \ldots, K) = \mathrm{vol}_n(K)$; Mixed volume is symmetric and translation invariant in its arguments.
- If $K \subset L$, then $V(K, K_2, K_3, \ldots, K_n) \leq V(L, K_2, K_3, \ldots, K_n)$.

Notation We also use $V(K, L) = V(K[n-1], L[1])$ and $V(A, B, C) = V(A[n-2], B[1], C[1])$.

Main Question

To establish

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A+B+C) \geq \mathrm{vol}_n(A+B) + \mathrm{vol}_n(A+C),$$

expand each term with the Minkowski polynomial and obtain known inequalities about the mixed volumes.

Main Question

To establish

$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C),$$

expand each term with the Minkowski polynomial and obtain known inequalities about the mixed volumes.

Can we establish which Radon (locally finite and inner regular Borel) measures on $\mathbb{R}^n$ are supermodular?

Main Question

To establish

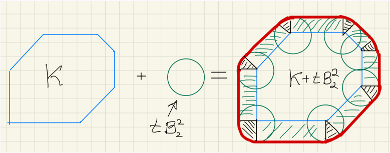
$$\mathrm{vol}_n(A) + \mathrm{vol}_n(A + B + C) \geq \mathrm{vol}_n(A + B) + \mathrm{vol}_n(A + C),$$

expand each term with the Minkowski polynomial and obtain known inequalities about the mixed volumes.

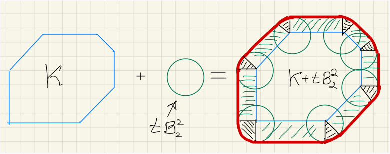
Can we establish which Radon (locally finite and inner regular Borel) measures on $\mathbb{R}^n$ are supermodular?

Turns out, it is connected to another story!

Surface Area

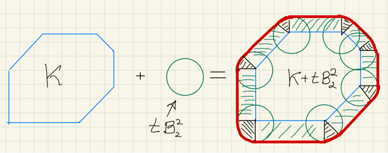


Surface Area



Thus $\mathrm{vol}_2(K + tB_2^2) = \mathrm{vol}_2(K) + \mathrm{vol}_1(\partial K)t + t^2 \mathit{Error}$, or

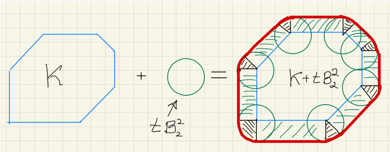
Surface Area



Thus $\mathrm{vol}_2(K + tB_2^2) = \mathrm{vol}_2(K) + \mathrm{vol}_1(\partial K)t + t^2 \mathit{Error}$, or

$$\mathrm{vol}_1(\partial K) = \lim_{t \to 0} \frac{\mathrm{vol}_2(K + tB_2^2) - \mathrm{vol}_2(K)}{t}.$$

Surface Area

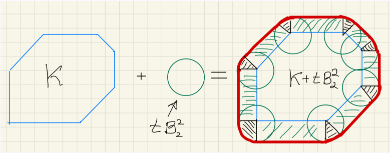


Thus $\mathrm{vol}_2(K + tB_2^2) = \mathrm{vol}_2(K) + \mathrm{vol}_1(\partial K)t + t^2 Error$, or

$$\mathrm{vol}_{n-1}(\partial K) = \lim_{t \to 0} \frac{\mathrm{vol}_n(K + tB_2^n) - \mathrm{vol}_n(K)}{t},$$

where K convex body in $\mathbb{R}^n$ and B_2^n is a unit Euclidean ball.

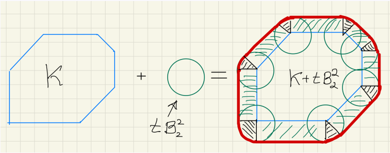
Surface Area



On the other-hand, using Minkowski's polynomial

$$\mathrm{vol}_n(K + tB_2^n) = \sum_{k=0}^{n} \binom{n}{k} V(B_2^n[k], K[n-k]) t^k.$$

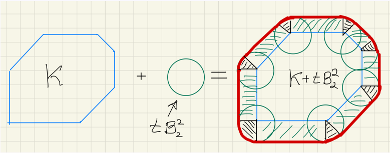
Surface Area



Combining the two:

$$\mathrm{vol}_{n-1}(\partial K) = \lim_{t \to 0} \frac{\mathrm{vol}_n(K + tB_2^n) - \mathrm{vol}_n(K)}{t}$$

Surface Area

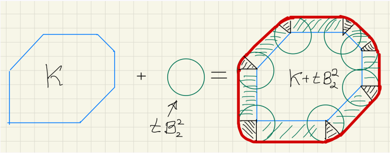


Combining the two:

$$\operatorname{vol}_{n-1}(\partial K) = \lim_{t\to 0} \frac{\operatorname{vol}_n(K + tB_2^n) - \operatorname{vol}_n(K)}{t}$$

$$= \lim_{t\to 0} \frac{\operatorname{vol}_n(K) + tnV(K, B_2^n) + \mathcal{O}(t^2) - \operatorname{vol}_n(K)}{t}$$

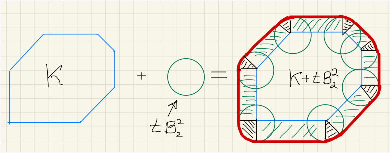
Surface Area



Combining the two:

$$\begin{aligned}\mathrm{vol}_{n-1}(\partial K) &= \lim_{t\to 0}\frac{\mathrm{vol}_n(K+tB_2^n)-\mathrm{vol}_n(K)}{t}\\ &= \lim_{t\to 0}\frac{\mathrm{vol}_n(K)+tnV(K,B_2^n)+\mathcal{O}(t^2)-\mathrm{vol}_n(K)}{t}\\ &= nV(K,B_2^n).\end{aligned}$$

Surface Area



Combining the two:

$$
\begin{aligned}
\mathrm{vol}_{n-1}(\partial K) &= \lim_{t\to 0} \frac{\mathrm{vol}_n(K+tB_2^n) - \mathrm{vol}_n(K)}{t} \\
&= \lim_{t\to 0} \frac{\mathrm{vol}_n(K) + tnV(K,B_2^n) + \mathcal{O}(t^2) - \mathrm{vol}_n(K)}{t} \\
&= nV(K,B_2^n).
\end{aligned}
$$

Thus, from monotonicity of mixed volumes, $K \subseteq L$ implies $\mathrm{vol}_{n-1}(\partial K) \leq \mathrm{vol}_{n-1}(\partial L)$.

Monotonicity of Weighted Surface Area

Definition

Let μ be a Borel measure on $\mathbb{R}^n$ and K a Borel set (convex body). Then, the Minkowski content of K with respect to μ, or its *weighted surface area* is given by

$$\mu^+(\partial K) = \liminf_{\varepsilon \to 0} \frac{\mu(K + \varepsilon B_2^n) - \mu(K)}{\varepsilon}.$$

Monotonicity of Weighted Surface Area

Kryvonos and Langharst ('22): If K is convex body and μ has density ϕ containing ∂K in its Borel set, then the $\liminf$ is a limit and

$$\mu^{+}(\partial K) = \int_{\partial K} \phi(x) d\mathcal{H}^{n-1}(x).$$

Monotonicity of Weighted Surface Area

Kryvonos and Langharst ('22): If K is convex body and μ has density ϕ containing ∂K in its Borel set, then the $\liminf$ is a limit and

$$\mu^+(\partial K) = \int_{\partial K} \phi(x) d\mathcal{H}^{n-1}(x).$$

Theorem of G. Saracco and G. Stefani ('23)

Let μ be a Borel measure on $\mathbb{R}^n$ with continuous density that has the following property: if K and L are convex bodies such that $K \subseteq L$, then $\mu^+(\partial K) \leq \mu^+(\partial L)$. Then, μ is a multiple of the Lebesgue measure.

Mixed Measures

We say a collection of Borel sets is a *class* if it is closed under Minkowski summation and dilation.

Mixed Measures

We say a collection of Borel sets is a *class* if it is closed under Minkowski summation and dilation.

Definition (Mixed Measures; Milman-Rotem and Livshyts)

Let μ be a Borel measure supported on a class of Borel sets $\mathcal{C}$. Then, for $K, L \in \mathcal{C}$,

$$\mu(K,L) := \liminf_{\epsilon\to 0} \frac{\mu(K+\epsilon L)-\mu(K)}{\epsilon}.$$

Mixed Measures

We say a collection of Borel sets is a *class* if it is closed under Minkowski summation and dilation.

Definition (Mixed Measures; Milman-Rotem and Livshyts)

Let μ be a Borel measure supported on a class of Borel sets $\mathcal{C}$. Then, for $K, L \in \mathcal{C}$,

$$\mu(K, L) := \liminf_{\epsilon \to 0} \frac{\mu(K + \epsilon L) - \mu(K)}{\epsilon}.$$

$\mathcal{C}$ will always be some convex sets (all convex bodies, symmetric convex bodies, etc.) in which case, the $\liminf$ is a limit if μ has density that

Mixed Measures

We say a collection of Borel sets is a *class* if it is closed under Minkowski summation and dilation.

Definition (Mixed Measures; Milman-Rotem and Livshyts)

Let μ be a Borel measure supported on a class of Borel sets $\mathcal{C}$. Then, for $K, L \in \mathcal{C}$,

$$\mu(K, L) := \liminf_{\epsilon \to 0} \frac{\mu(K + \epsilon L) - \mu(K)}{\epsilon}.$$

$\mathcal{C}$ will always be some convex sets (all convex bodies, symmetric convex bodies, etc.) in which case, the $\liminf$ is a limit if μ has density that

- is continuous Livshysts ('19)
- contains ∂K in its Lebesgue set (K-L '22)

Mixed Measures

We say a collection of Borel sets is a *class* if it is closed under Minkowski summation and dilation.

Definition (Mixed Measures; Milman-Rotem and Livshyts)

Let μ be a Borel measure supported on a class of Borel sets $\mathcal{C}$. Then, for $K, L \in \mathcal{C}$,

$$\mu(K, L) := \liminf_{\epsilon \to 0} \frac{\mu(K + \epsilon L) - \mu(K)}{\epsilon}.$$

$\mathcal{C}$ will always be some convex sets (all convex bodies, symmetric convex bodies, etc.) in which case, the $\liminf$ is a limit if μ has density that

- is continuous Livshysts ('19)
- contains ∂K in its Lebesgue set (K-L '22)

Notice: $\mu^+(\partial K) = \mu(K; B_2^n)$.

The Connection

Local forms of supermodularity

Let μ be a Radon measure on $\mathbb{R}^n$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be classes of convex sets such that $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$. Then the following are equivalent: for every $A \in \mathcal{A}, B \in \mathcal{B}$ and $C \in \mathcal{C}$,

1. $\mu(A+B+C)+\mu(A) \geq \mu(A+B)+\mu(A+C)$,
2. $\mu(A+C;B) \geq \mu(A;B)$,

The Connection

Local forms of supermodularity

Let μ be a Radon measure on $\mathbb{R}^n$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be classes of convex sets such that $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$. Then the following are equivalent: for every $A \in \mathcal{A}, B \in \mathcal{B}$ and $C \in \mathcal{C}$,

1. $\mu(A + B + C) + \mu(A) \geq \mu(A + B) + \mu(A + C)$,
2. $\mu(A + C; B) \geq \mu(A; B)$,

Local forms of supermodularity with a ball

Let μ be a Radon measure on $\mathbb{R}^n$ and set $\mathcal{B} = \{rB_2^n\}_{r \geq 0}$. Let $\mathcal{A}$ and $\mathcal{C}$ be classes of convex sets such that $\mathcal{B}, \mathcal{C} \subset \mathcal{A}$. Then the following are equivalent: for every $r \geq 0$, $A \in \mathcal{A}$ and $C \in \mathcal{C}$

1. $\mu(A + rB_2^n + C) + \mu(A) \geq \mu(A + rB_2^n) + \mu(A + C)$.
2. $\mu^+(\partial(A + C)) \geq \mu^+(\partial A)$.

Classification Results

Main Theorem

Let μ be a Radon measure on $\mathbb{R}^n$ such that, for every convex body K and compact, convex set L, one has

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K).$$

Then, μ is a multiple of the Lebesgue measure.

Classification Results

Main Theorem

Let μ be a Radon measure on $\mathbb{R}^n$ such that, for every convex body K and compact, convex set L, one has

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K).$$

Then, μ is a multiple of the Lebesgue measure.

We use that the class of dilates of B_2^n is a subset of all convex bodies, and the localization theorem, to obtain the following.

Classification Results

Main Theorem

Let μ be a Radon measure on $\mathbb{R}^n$ such that, for every convex body K and compact, convex set L, one has

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K).$$

Then, μ is a multiple of the Lebesgue measure.

Main Corollary

Let μ be a Radon measure that is supermodular over the class of all convex bodies. Then, μ is a multiple of the Lebesgue measure.

Can we bridge the gap?

An open question

Let μ be a Radon measure on $\mathbb{R}^n$ with the following property: for every convex body K and compact, convex set L such that L contains the origin, one has

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K),$$

it is true that then, μ is a constant multiple of the Lebesgue measure?

A different type of result

Theorem in the plane

Let K be a convex body in $\mathbb{R}^2$ and let μ be the Borel measure with density $|x|^2$. Then, for every symmetric convex, compact set L containing the origin in $\mathbb{R}^2$

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K).$$

A different type of result

Theorem in the plane

Let K be a convex body in $\mathbb{R}^2$ and let μ be the Borel measure with density $|x|^2$. Then, for every symmetric convex, compact set L containing the origin in $\mathbb{R}^2$

$$\mu^+(\partial(K+L)) \geq \mu^+(\partial K).$$

Hint to the proof

Let K be a convex body in $\mathbb{R}^2$. Let μ be the Borel measure with density $\phi(x) = |x|^2$. Then, for every $u \in \mathbb{R}^2$

$$\mu^+(\partial(K+[0,u])) \geq \mu^+(\partial K).$$

Restricting the classes of Convex Bodies

Theorem for Zonoids

Let μ be a Radon measure on $\mathbb{R}^n$ with the following property: for every symmetric convex body A, centered zonoid B and zonoid containing the origin C, one has

$$\mu(A + C; B) \geq \mu(A; B).$$

Then, μ is a constant multiple of the Lebesgue measure.

Restricting the classes of Convex Bodies

Theorem for Zonoids

Let μ be a Radon measure on $\mathbb{R}^n$ with the following property: for every symmetric convex body A, centered zonoid B and zonoid containing the origin C, one has

$$\mu(A+C+B)+\mu(A)\geq\mu(A+B)+\mu(A+C).$$

Then, μ is a constant multiple of the Lebesgue measure.