Travel Time Inverse Problems on Compact Riemannian Manifolds

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Seeing inside the earth with earthquakes

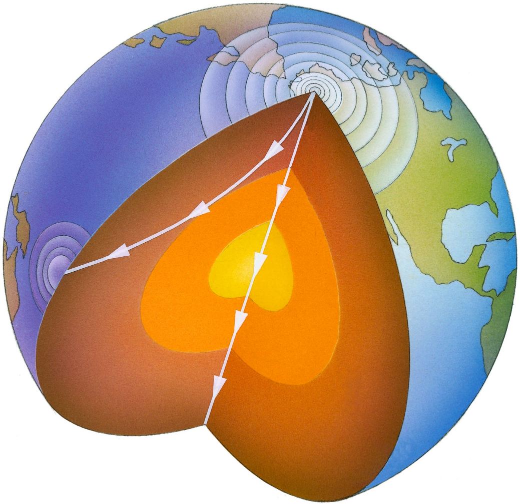


figure from: https://www.dkfindout.com/us/earth/earthquakes/shock-waves/

Model: A smooth, compact and connected Riemannian manifold (M, g) with a smooth boundary ∂M

Data: Travel times of seismic waves

Inverse Problem: Recover the Riemannian manifold (M, g).

This task is known as the **boundary rigidity problem**.

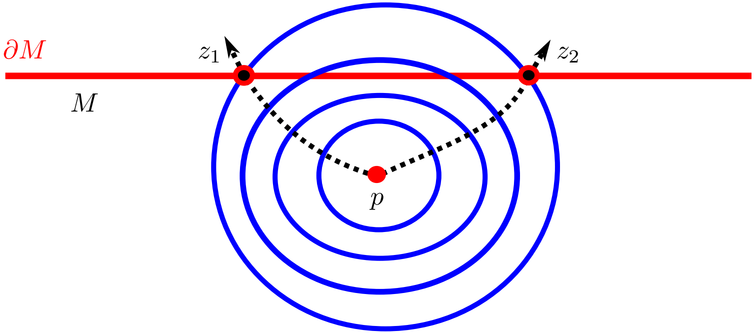
Poor data (sources only at the boundary) makes the boundary rigidity problem extremely difficult!

$$\Rightarrow$$

More data (interior interactions)

Inverse Problems: Recover a compact Riemannian manifold (M, g) (up to change of coordinates) from

1. **Travel Time Data** $\{d(p, \cdot) \colon \partial M \to \mathbb{R} | \, p \in M\}$
   - Uniqueness: Katchalov-Kurylev-Lassas (2001), Hölder stability: Katsuda-Kurylev-Lassas (2007)
   - Optimal uniqueness in Finsler geometries: de Hoop-Ilmavirta-Lassas-S (2021)
2. **Broken Scattering Relations** "Exiting directions, and lengths of broken geodesics"
   - Uniqueness for dimension 3 and up: Kurylev-Lassas-Uhlmann (2010)
   - Uniqueness in Finsler geometries with a foliation condition for dimension 3 and up: de Hoop-Ilmavirta-Lassas-S (2021)



Main results:

1. Uniqueness of the Travel Time Problem with partial data on compact Riemannian manifolds with strictly convex boundary
2. Finite Source Approximation of Simple Riemannian manifolds
3. Lipschitz Stability of the Travel Time Data on Simple Riemannian manifolds
4. Uniqueness of the Broken Scattering Relation on Simple Riemannian manifolds

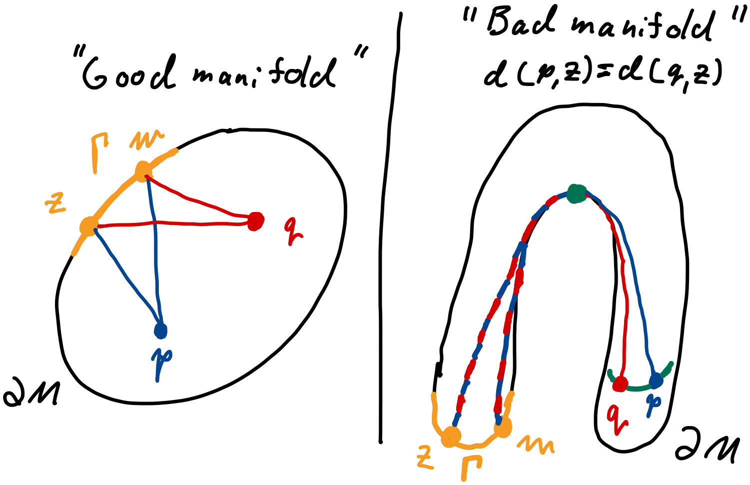
Contents

Partial Travel Time Data:

- $\Gamma \subset \partial M$ is a known open subset of the boundary
- The set of travel time functions $\{d(p,\cdot)\colon \Gamma \to \mathbb{R} \mid p \in M\}$ is given

∂M is strictly convex:

- Geodesics that are tangental to ∂M exit immediately
- Any $p, q \in M$ can be connected by a distance minimizing geodesic (not necessarily unique!)



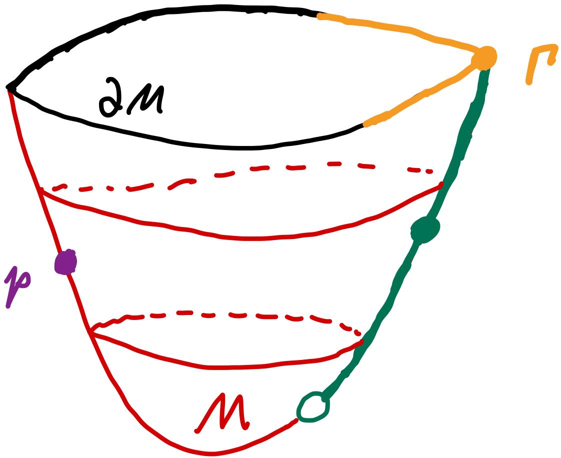
Theorem (Pavlechko-S (2022))

A smooth, compact, connected, and oriented Riemannian manifold of dimension ≥ 2 with smooth and strictly convex boundary is determined upto an isometry by its partial travel time data.

Key of the proof:
The gradient of a distance function is the velocity of a distance minimizing unit speed geodesic.

If we can differentiate the distance function, we can track the traces of some geodesics!

Main obstacle: For $p \in M$, can the set $\Gamma \cap \{x \in M : d(p, \cdot) \text{ is not } C^1\text{-smooth at } x\}$ be very large?



Cut locus: For each $p \in M$

$$\mathrm{cut}(p) = \overline{\{q \in M :\ \text{There are two distance minimizing curves from } p \text{ to } q\}}$$

Proposition: Pavlechko-S (2022)

If ∂M is strictly convex then:

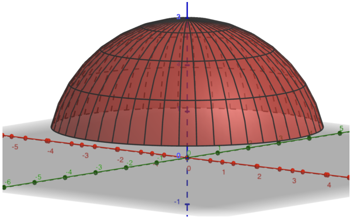
- $\mathrm{cut}(p)$ is closed
- $d(p, \cdot)$ is C^∞ in $M \setminus (\mathrm{cut}(p) \cup \{p\})$
- Hausdorff dimension of $\mathrm{cut}(p) \leq n - 1$
- Hausdorff dimension of $\mathrm{cut}(p) \cap \partial M \leq n - 2$

Implication: We can embed M into $L^\infty(\Gamma)$ with the **partial travel time map**:
$R\colon M \to L^\infty(\Gamma), \quad R(p) = d(p, \cdot)|_\Gamma.$

Contents

Simple Riemannian manifolds

- M is smooth, connected and compact manifold with smooth boundary
- ∂M is strictly convex (all tangential geodesics to the boundary exit immediately)
- Each pair of points is connected by a smoothly varying unique distance minimizing geodesic



1. No trapped geodesics
2. No conjugate points
3. Distance function $d(p, \cdot)$ is smooth on $M \setminus \{p\}$
4. M is diffeomorphic with the Euclidean disc $\mathbb{D}^n$.

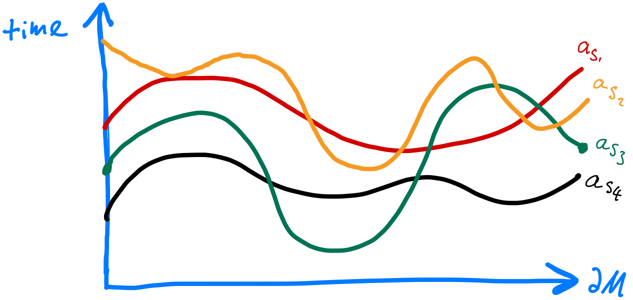
Arrival time data: The set of **unknown** interior point sources $S \subset M^{int} \times (0, \infty)$.
Arrival time function:

$$a_s(z) = d(p, z) + \tau, \quad \text{for } z \in \partial M \text{ and } s = (p, \tau) \in S.$$

Known:

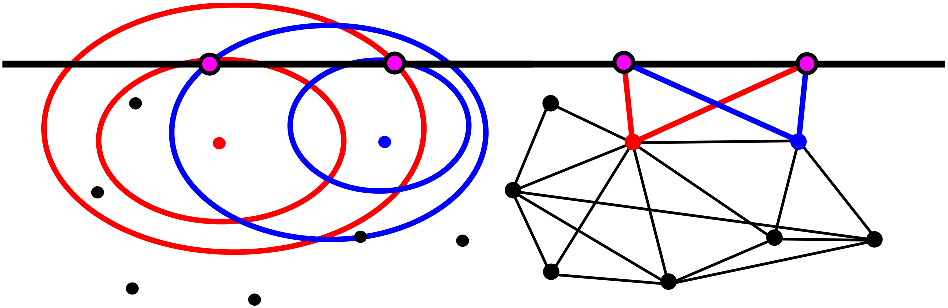
$$Q(S) := \bigcup_{s \in S} \text{Graph}(a_s) \subset \partial M \times (0, \infty), \quad \text{and} \quad (\partial M, g|_{\partial M})$$

Observe: $Q(S)$ does not have any labels and we know it as a point set.



Theorem (de Hoop-Ilmavirta-Lassas-S (2023))

- *We can disentangle the signals and build a metric graph between the source points*
- *We can provide data driven density estimates for the source points*
- *We can show that the metric graph approximate the Riemannian manifold*



Geometric assumptions: (M, g) is a simple Riemannian n-manifold whose sectional curvature is bounded from above by $C_{sec+} > 0$ so that

$$\text{Diam}(M)\sqrt{C_{sec+}} < \pi. \tag{1}$$

Remark: This holds if (M, g) has a negative sectional curvature.

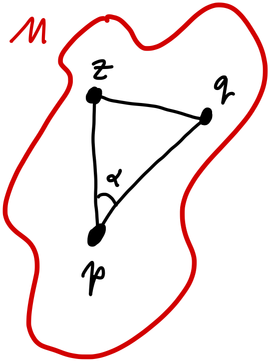
(1) is used to estimate the density of the sources, via Rauch's comparison theorem for n-sphere $S^n(r)$ and spherical law of cosines.

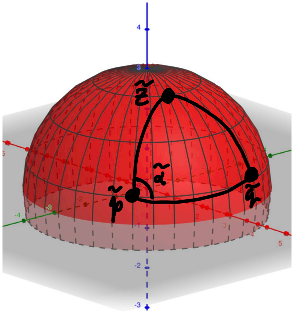
Rauch's Comparison theorem

Let $p, q, z \in M$. There are $\widetilde{p}, \widetilde{q}, \widetilde{z} \in S^n(r)$, for $r = (C_{\text{sec}+})^{-1/2}$ so that

$$d(p,q) = d(\widetilde{p}, \widetilde{q}), \quad d(p,z) = d(\widetilde{p}, \widetilde{z}).$$

- If $\alpha = \widetilde{\alpha}$ then $d(q,z) \leq d(\widetilde{q}, \widetilde{z})$.
- If $d(q,z) = d(\widetilde{q}, \widetilde{z})$ then $\alpha \leq \widetilde{\alpha}$.





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Travel Time Data

Without loss of generality we assume for a simple Riemannian manifold $(M, \partial M, g) = (\mathbb{D}^n, \mathbb{S}^{n-1}, g)$.

For a point $p \in \mathbb{D}^n$ its **travel time function** $r_p : \mathbb{S}^{n-1} \to \mathbb{R}$ is defined by the formula

$$r_p(z) = d(p, z).$$

The point source p is **unknown**.

The travel time map of the simple Riemannian manifold $(\mathbb{D}^n, g)$ is then given by the formula

$$\mathcal{R} : (\mathbb{D}^n, g) \to (C(\mathbb{S}^{n-1}), \|\cdot\|_\infty), \quad \mathcal{R}(p) = r_p.$$

The image set $\mathcal{R}(\mathbb{D}^n) \subset C(\mathbb{S}^{n-1})$ of the travel time map is called the **travel time data** of the Riemannian manifold $(\mathbb{D}^n, g)$.

How to measure the closeness of the travel time data and the metrics?

Distance of travel time data, of two simple Riemannian metrics g_1 and g_2 on $\mathbb{D}^n$, is

$$d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)) \geq 0,$$

where d_H is the **Hausdorff distance** of $(C(\mathbb{S}^{n-1}), \|\cdot\|_\infty)$.

The travel time data of simple Riemannian metrics g_1 and g_2 on $\mathbb{D}^n$ **coincide** if

$$\mathcal{R}_2(\mathbb{D}^n) = \mathcal{R}_1(\mathbb{D}^n) \quad \Leftrightarrow \quad d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)) = 0.$$

To measure the closeness of compact metric spaces X and Y we use the **Gromov–Hausdorff distance**

$$\begin{aligned} d_{GH}(X,Y) := \inf\{d_H^Z(f(X), g(Y)); \quad & Z \text{ is a metric space}, \\ & f\colon X \to Z \text{ and } g\colon Y \to Z \text{ are isometric embeddings}\}. \end{aligned}$$

$d_{GH}(X,Y) = 0$ if and only if the metric spaces X and Y are isometric.

Theorem (Ilmavirta-Liu-S, 2023)

Let $n \geq 2$, and let g_1 and g_2 be two simple Riemannian metrics of $\mathbb{D}^n$. Then

$$d_{GH}((\mathbb{D}^n, g_1), (\mathbb{D}^n, g_2)) \leq d_H^{C(\mathbb{S}^{n-1})}(\mathcal{R}_1(\mathbb{D}^n), \mathcal{R}_2(\mathbb{D}^n)).$$

If the travel time data for two metrics coincide, then they agree up to a boundary fixing diffeomorphism.

Proof:

- $\mathcal{R}_i : (\mathbb{D}^n, d_i) \to (C(\mathbb{S}^{n-1}), \|\cdot\|_\infty)$ is a metric isometry.
- If $\mathcal{R}_2(\mathbb{D}^n) = \mathcal{R}_1(\mathbb{D}^n)$, then $\mathcal{R}_2^{-1} \circ \mathcal{R}_1 : (\mathbb{D}^n, d_1) \to (\mathbb{D}^n, d_2)$ is a bijective distance preserving map.

Theorem (Myers-Steenrod, 1939)

A bijective distance preserving map between Riemannian manifolds is a smooth Riemannian isometry.

- $\mathcal{R}_2^{-1} \circ \mathcal{R}_1 :$ is a smooth Riemannian isometry.
- The stability claim follows from the definition of Gromov-Hausdorff distance.

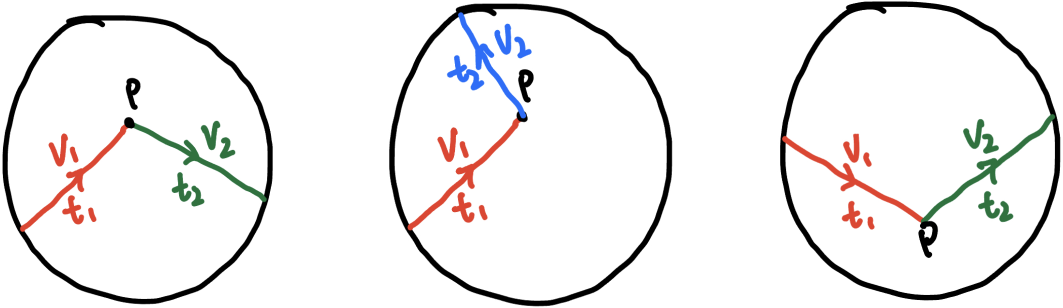
Contents

Broken Scattering Relations

For each $T > 0$ we define a relation R_T on $\partial_{\text{in}} S\mathbb{D}^n$ (the bundle of inward pointing directions) so that $v_1 R_T v_2$ if there are two numbers $t_1, t_2 > 0$ for which

$$t_1 + t_2 = T \quad \text{and} \quad \gamma_{v_1}(t_1) = \gamma_{v_2}(t_2) =: p.$$

We **do not know** the scattering points $p \in M$, or the travel times t_1, t_2.



The family $\{\mathcal{B}_T : T > 0\}$ of relations is called the **Broken Scattering Relations** of Riemannian manifold $(\mathbb{D}^n, g)$.

Uniqueness of the broken scattering relations on simple manifolds

Theorem (Ilmavirta-Liu-S., 2023)

Let $n \geq 2$, and let g_1 and g_2 be two simple Riemannian metrics in $\mathbb{D}^n$ whose first fundamental forms agree on $\mathbb{S}^{n-1}$. If the broken scattering relations of g_1 and g_2 coincide, then there exists a smooth Riemannian isometry $\Psi\colon (\mathbb{D}^n, g_1) \to (\mathbb{D}^n, g_2)$ whose boundary restriction $\Psi\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ is the identity map.

Key of the proof: Reduce the problem to the travel time data.

1. Recover the exit time function and the scattering relation
2. Recover the travel time functions

Recovery of the exit time function and the scattering relation

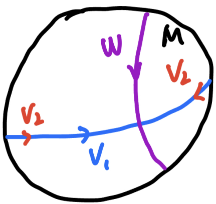
- The broken scattering relations determine the **exit time function**:

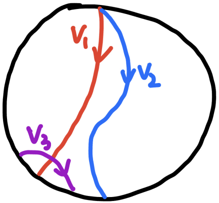
$$\tau_{\text{exit}}(v) := \sup\{t > 0 : \ \gamma_v(t) \in \mathbb{D}^n\} = \sup\left\{\frac{T}{2} : \ v\mathcal{B}_T v\right\}, \quad v \in \partial_{\text{in}} S\mathbb{D}^n.$$

- Let $v_1, v_2 \in \partial_{\text{in}} S\mathbb{D}^n$. In simple geometries the following two statements are equivalent:
  (1) We have $V(v_1) = V(v_2)$, where

$$V(v_i) := \{\text{set of all geodesics intersecting } \gamma_{v_i}\}.$$

  (2) Either $v_1 = v_2$ or $v_2 = -\phi_{\tau_{\text{exit}}(v_1)}(v_1)$.





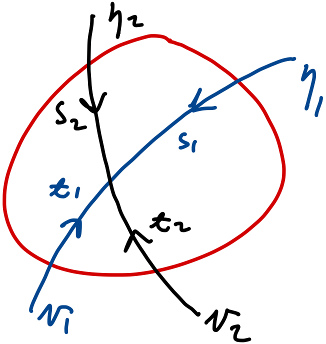
Remark: In a hemisphere any two geodesics intersect! Simplicity is needed!

- The broken scattering relations determine the **scattering relation** $v_1 \mapsto \phi_{\tau_{\text{exit}}(v_1)}(v_1)$.

Recovery of the travel times

- Let $v_1, v_2 \in \partial_{\text{in}} S\mathbb{D}^n$ and let $\eta_i := -\phi_{\tau_{\text{exit}}(v_i)}(v_i)$.
- Suppose that $v_1 \mathcal{B}_T v_2$ for some $T = T(v_1, v_2) > 0$.
- Since g is simple, the geodesics γ_{v_1} and γ_{v_2} intersect exactly once. Thus, there are some numbers $t_1, t_2, s_1, s_2 \geq 0$ satisfying the four equations with the known RHS:

$$t_1 + t_2 = T(v_1, v_2), \quad t_1 + s_1 = T(v_1, \eta_1), \quad t_2 + s_2 = T(v_2, \eta_2), \quad \text{and} \quad t_1 + s_2 = T(v_1, \eta_2).$$



Therefore:

$$t_1 = \frac{1}{2}\left(T(v_1, v_2) - T(v_2, \eta_2) + T(v_1, \eta_2)\right)$$

and

$$t_2 = T(v_1, v_2) - t_1.$$

This talk was based on the following papers

1. **Uniqueness of the partial travel time representation of a compact Riemannian manifold with strictly convex boundary**, with: Ella Pavlechko, *Inverse Problems and Imaging*, 16(5), (October 2022), pp 1325-1357
2. **Stable reconstruction of simple Riemannian manifolds from unknown interior sources**, with: Maarten V. de Hoop, Joonas Ilmavirta and Matti Lassas, Inverse Problems, to appear
3. **Three travel time inverse problems on simple Riemannian manifolds**, with: Joonas Ilmavirta and Boya Liu, *Proceedings of the American Mathematical Society*, to appear

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Thank you for your attention!

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