Joint Spectra and related Topics in Complex Dynamics and Representation Theory

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1 Overview of the Field

Mathematics has a wide range of different disciplines, and new discoveries are made daily in every corner. Some discoveries, however, are able to link several disciplines together and open new fields of interplay. Two of such recent discoveries are self-similar group representations and projective spectrum in Banach algebras. This workshop, which took place at Banff International Research Station from May 22 to 26, 2023, brought together scholars in the fields of spectral theory, representation theory, geometric group theory, complex dynamics and Lie algebras to examine this development. In the overview, we briefly recall several notions of joint spectrum, first for commuting operators and then for noncommuting ones. The theory is vastly different for the two settings.

1.1 The Commuting Case

Consider an abelian unital Banach algebra $\mathcal{B}$. The idea of joint spectrum for several elements $A_1, ..., A_n \in \mathcal{B}$ goes back to the 1960s. The following notion was studied in [33].

Definition 1.1. *For a tuple $A = (A_1, ..., A_n)$ of elements in $\mathcal{B}$, the joint spectrum* $\mathrm{Sp}(A)$ *is the collection of $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ such that the ideal generated by $A_1 - \lambda_1 I, ..., A_n - \lambda_n I$ is proper in $\mathcal{B}$.*

In other words, λ is not in $\mathrm{Sp}(A)$ if and only if there are elements $B_1, ..., B_n$ in $\mathcal{B}$ such that $(A_1 - \lambda_1 I)B_1 + \cdots + (A_n - \lambda_n I)B_n = I$.

A more popular notion of joint spectrum was defined by Taylor in the early 70's using Koszul complex [44, 45]. Let $\oplus_{0 \leq p \leq n} \Lambda^p(V)$ be the exterior algebra on a vector space V of dimension n. Given a Hilbert space $\mathcal{H}$, we let $B(\mathcal{H})$ be the C^*-algebra of bounded linear operators on $\mathcal{H}$. A tuple A of n commuting operators in $B(\mathcal{H})$ gives rise to the following Koszul complex $E(\mathcal{H}, A)$ of cochains:

$$0 \xrightarrow{d_{-1}} \mathcal{H} \otimes \Lambda^0 \xrightarrow{d_0} \mathcal{H} \otimes \Lambda^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \mathcal{H} \otimes \Lambda^n \xrightarrow{d_n} 0, \tag{1}$$

where $d_{-1} = d_n = 0$, and $d_p : \mathcal{H} \otimes \Lambda^p(V) \to \mathcal{H} \otimes \Lambda^{p+1}(V), 0 \leq p \leq n-1$, are the linear maps defined by

$$d_p(x \otimes \omega) = \sum_{i=1}^{n} A_i x \otimes (e_i \wedge \omega), \quad x \in \mathcal{H},\ \omega \in \Lambda^p(V).$$

The reader shall gain some insight by looking at the special cases $n = 1$ and 2. Direct computation can verify that $d_{p+1} d_p = 0$ for each p. The complex (1) is said to be exact if $\ker d_{p+1} = \operatorname{ran} d_p$ for each $0 \leq p \leq n$. For a vector $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$, we let $A - \lambda$ stand for $(A_1 - \lambda_1 I, ..., A_n - \lambda_n I)$.

Definition 1.2. *The Taylor spectrum of a commuting tuple A is defined as*

$$\sigma_T(A) = \{\lambda \in \mathbb{C}^n \mid E(\mathcal{H}, A - \lambda) \textit{ is not exact}\}.$$

The following theorem has many applications.

Theorem 1.3. *For a commuting tuple of operators $A = (A_1, ..., A_n)$, we have*

a) $\sigma_T(A)$ is a nontrivial compact subset of $\mathbb{C}^n$;

b) if a complex domain Ω contains $\sigma_T(A)$ and $f : \Omega \to \mathbb{C}^m$ is holomorphic, then $\sigma_T(f(A)) = f(\sigma_T(A))$.

Part b) is known as the spectral mapping theorem. One observes that both $\mathrm{Sp}(A)$ and $\sigma_T(A)$ are natural generalizations of the classical spectrum. Moreover, if $f(z) = z_j$ is the projection to the jth coordinate, then b) implies the projection property of Taylor spectrum, namely, the projection of $\sigma_T(A)$ onto any axis z_j equals $\sigma(A_j)$. It is known that $\sigma_T(A) \subset \mathrm{Sp}(A)$, and the inclusion can be proper. Some other notions of joint spectrum have also been investigated in the past half century. Curto's mini-course gave an in-depth survey on this subject.

1.2 Non-Commuting Operators

Unfortunately, the two notions of joint spectrum above do not have a good generalization to non-commuting operators. A drastically different approach was taken. The notion of projective spectrum for general (possibly non-commuting) operators came independently from two directions: the spectral theory of self-similar groups and the theory of nonabelian Banach algebras.

Given several matrices $A_1, ..., A_n$ of equal size, the determinant of the linear pencil $A(z) := z_1 A_1 + \cdots + z_n A_n$ was studied as early as the late 19th century and early 20th century. Let λ stand for the regular representation of a finite group $G = \{g_1, ..., g_n\}$. Starting in 1896 [12], Frobenius studied the factorization of $\det(z_1\lambda(g_1) + \cdots + z_n\lambda(g_n))$ in a series of papers. Indeed, this work is the birth place of group representation theory [7]. In 1900 [9], Dixon considered the problem whether a homogeneous polynomial is of the determinantal form $\det A(z)$. The linear pencil related to infinite groups was only seriously considered more than half century later. Kesten [37, 38], published in 1957, studied random walks on groups and discovered a probabilistic criterion of amenability. In the sequel, we let group G be generated by a finite set $\{g_1, ..., g_n\}$. The Markov operator M_λ is the average of $\lambda(g_1), ..., \lambda(g_n)$, namely, $M_\lambda = \frac{1}{n}(\lambda(g_1) + \cdots + \lambda(g_n))$. A weaker version of Kesten's theorem is as follows.

Theorem 1.4. *The following are equivalent for a finitely generated group G:*

a) G is amenable;

b) $1 \in \sigma(M_\lambda)$;

c) the spectral radius of M_λ is 1.

The theorem also holds for the weighted average $M = x_1\lambda(g_1) + \cdots + x_n\lambda(g_n)$, where x_i are nonnegative real numbers such that $x_1 + \cdots + x_n = 1$.

Self-Similar Groups. The discovery of the Grigorchuk group $\mathcal{G}$ in 1980 [14] inspired greater interest in the spectral properties of groups. The group $\mathcal{G}$ is a finitely generated infinite torsion group that possesses obvious self-similarity features. Its construction is given in Example 2.2. Such torsion groups are also known as the Burnside groups, and they are related to one of the most famous problems in algebra posted by Burnside in 1902. In 1983, Grigorchuk discovered that $\mathcal{G}$ has other remarkable properties [15].

Theorem 1.5. *The following hold for the group $\mathcal{G}$.*

a) It has intermediate (between polynomial and exponential) growth.

b) It is amenable but not elementary amenable.

Part a) settles a question of Milnor from 1967 [40], and part b) answers a question of Day from 1957 [8]. Moreover, the construction of $\mathcal{G}$ motivated the definition of self-similar group representation.

Definition 1.6. *Given an integer $d \geq 2$, a unitary representationr $(\pi, \mathcal{H})$ of a group G is said to be d-similar if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}^d$ such that for every $g \in G$ the $d \times d$ block matrix $\hat{\pi}(g) = U\pi(g)U^*$ has all of its entries either equal to 0 or of the form $\pi(x), x \in G$.*

It is clear that $\hat{\pi}$ is a unitary representation of G on $\mathcal{H}^d$. Since each of its nonzero block entries are unitary operators on $\mathcal{H}$, every row or column of $\hat{\pi}(g)$ has precisely one nonzero entry. This is manifested clearly in Examples 2.1 and 2.2 in the next section.

As more examples of self-similar groups are discovered, for instance see Grigorchuk and Gupta-Sidki (1983), efforts have been made to generalize them to the Grigorchuk-Gupta-Sidki (GGS) groups [2], spine groups, and other classes of groups. The idea of self-similarity was thus introduced prominently into group theory, opening a new direction in mathematics that has numerous applications in algebra, dynamical systems, random walks, operator algebras, geometry, computer science, cryptography, and mathematical physics, etc.

In the pioneer paper [4] published in 2000, Bartholdi and Grigorchuk studied the spectra of Schreier graphs associated with self-similar groups. In particular, they considered the invertibility of the pencil

$$A_\pi(z) := z_1\pi(g_1) + \cdots + z_n\pi(g_n), \tag{2}$$

where π is any unitary representation of G and z_i are complex coefficients. This idea has motivated a great amount of subsequent work in geometric group theory. In a broader context, a big part of mathematics is devoted to the study of spectral properties of graphs (finite and infinite), which has applications not only in many areas of mathematics but also in technology, for instance in communication systems via the notion of expanding graphs. Among infinite graphs, a special subclass constitutes graphs with uniformly bounded degree and regular graphs, in which the degrees of all vertices are equal. Such graphs usually have a realization as the Schreier graphs of groups, such as free groups. Schreier graph generalizes the notion of Cayley graph, and it enables much wider applications of group theory methods. The study of Schreier graph is a fast developing frontier in mathematics. Part of this development was presented in the mini-course by Grigorchuk at the workshop.

Projective Spectrum in Banach Algebras. The lack of a proper notion of joint spectrum for non-commuting operators to a large extent confined multivariable operator theory to commuting settings. Attempting to address this issue, and unaware of the early works on group determinant and self-similar groups, Yang defined the notion of projective joint spectrum in 2009 [49].

Definition 1.7. *Given elements $A_1, ..., A_n$ in a Banach algebra $\mathcal{B}$, their projective spectrum is defined as $p(A) = \{z \in \mathbb{P}^{n-1} \mid A(z)$ is not invertible$\}$.*

Here, $\mathbb{P}^{n-1}$ stands for the complex projective space of dimension $n-1$, hence the terminology. This definition has the following obvious features:

1) It is valid for all elements in $\mathcal{B}$, commuting or not.

2) It lets go of the formulation $A - \lambda I$ and treats all elements in a symmetric way.

3) Last but not least, it is easy to compute in many examples.

Projective spectrum has the following general properties.

Theorem 1.8. *For $n \geq 2$, the following holds for any elements $A_1, ..., A_n \in \mathcal{B}$.*

a) $p(A)$ is a nonempty compact subset of $\mathbb{P}^{n-1}$.

b) The complement $\mathbb{P}^{n-1} \setminus p(A)$ is a Stein domain.

c) If the operators are commuting, then $p(A)$ is a union of projective hyperplanes.

This result put the study of projective spectrum on a solid base. Many examples were computed following the definition, for example tuples of compact operators or projections, Cuntz algebra, irrational rotation algebra, free group von Neumann algebra, Coxeter groups, etc. A comprehensive treatment of this subject is presented in the upcoming book [50].

In the case $\dim \mathcal{H} < \infty$, the projective spectrum led naturally to the definition of multivariable characteristic polynomial.

Definition 1.9. *Given square matrices $A_1, ..., A_n$ of equal size, their characteristic polynomial is defined as*

$$Q_A(z) := \det(z_0 I + z_1 A_1 + \cdots + z_n A_n), \quad z = (z_0, ..., z_n) \in \mathbb{C}^{n+1}.$$

The idea is quickly applied to the study of Lie algebras, leading to a new classification of finite dimensional simple Lie algebras [13] as well as new invariants for the classification of solvable Lie algebras (an unsolved problem) [1].

The Origin of the Workshop. The afore mentioned two fronts of study met in fall 2015 when Yang spent a semester of sabbatical leave at Texas A&M University. In collaboration, Grigorchuk and Yang [30] obtained the following result regarding the infinite dihedral group $D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$.

Theorem 1.10. *For the pencil $A_\lambda(z) = z_0 I + z_1 \lambda(a) + z_2 \lambda(t)$, where λ is the regular representation of D_∞, we have*

$$p(A_\lambda) = \bigcup_{-1 \le \xi \le 1} \{z \in \mathbb{P}^2 \mid z_0^2 - z_1^2 - z_2^2 - 2 z_1 z_2 \xi = 0\}.$$

This theorem found applications to C^*-algebras, Maurer-Cartan differential forms, and Fuglede-Kadison determinant. In particular, it led to a rational map linking self-similarity to complex dynamics and Julia set, see Example 2.1 and Goldberg and Yang [32]. Moreover, due to the connection between D_∞ and $\mathcal{G}$, this theorem sheds new light on the spectral picture of $\mathcal{G}$. This collaboration is the original motivation of this Banff workshop.

2 Recent Developments and Open Problems

The focus of this workshop is on the connection between self-similarity, projective spectrum, and complex dynamics. Two important groups, D_∞ and $\mathcal{G}$, underpined the development that led to this workshop. Both groups have self-similar representations on $L^2[0, 1]$. We provide some details here for clarity. Express the numbers in $[0, 1]$ by a binary sequence, omitting the dot, i.e., a sequence (denoted by w in the sequel) of 0s and 1s. To make this expression unique, we assume that a rational number ends with a sequence of 0s instead of 1s. For example, 0.5 is expressed as $100\cdots$ rather than $0111\cdots$. Any measure-preserving action g on $[0, 1]$ lifts canonically to a unitary map $\pi(g) : L^2[0, 1] \to L^2[0, 1]$ such that $\pi(g)f(x) = f(g^{-1}x)$.

Two Examples. The following two examples serve to illuminate the subsequent report.

Example 2.1. *The self-similar representation π of $D_\infty = \langle a, t \mid a^2 = t^2 = 1 \rangle$ on $L^2[0, 1]$ is realized by the following measure preserving action on the interval $[0, 1]$:*

$$a(0w) = 1w, \;\; a(1w) = 0w; \quad t(0w) = 0a(w), \;\; t(1w) = 1t(w).$$

In fact, this action maps dyatic subintervals to dyatic subintervals. Clearly, $L^2[0,1] = L^2[0,1/2] \oplus L^2[1/2,1]$*, and we can identify both summands with* $L^2[0,1]$ *by giving a weight* 2 *to the Lebesgue measure on the two subintervals. This produces a unitary operator* $U : L^2[0,1] \to L^2[0,1] \oplus L^2[0,1]$*. Then in light of Definition 1.6, we have*

$$\hat{\pi}(a) \cong \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \hat{\pi}(t) \cong \begin{bmatrix} \pi(a) & 0 \\ 0 & \pi(t) \end{bmatrix}. \tag{3}$$

Example 2.2. *The Grigorchuk group* $\mathcal{G}$ *is generated by the following four actions on* $[0,1]$*:*

$$\begin{array}{llll} a(0w) = 1w, & a(1w) = 0w; & b(0w) = 0a(w), & b(1w) = 1c(w); \\ c(0w) = 0a(w), & c(1w) = 1d(w); & d(0w) = 0w, & d(1w) = 1b(w). \end{array}$$

Using the same identification U *as that in the previous example, we have*

$$\begin{array}{ll} \hat{\pi}(a) \cong \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \hat{\pi}(b) \cong \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(c) \end{pmatrix}, \\ \hat{\pi}(c) \cong \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(d) \end{pmatrix}, & \hat{\pi}(d) \cong \begin{pmatrix} I & 0 \\ 0 & \pi(b) \end{pmatrix}. \end{array} \tag{4}$$

Two of $\mathcal{G}$'s outstanding properties are described in Theorem 2.2. But two related problems are unsolved.

Problem 1. Is there a finitely generated group whose growth is strictly less than that of the Grigorchuk group $\mathcal{G}$ but faster than polynomials?

Given a representation π of a group G, we denote by $C^*_\pi(G)$ the C^*- algebra generated by the set $\pi(G)$.

Problem 2. Is $C^*_\pi(\mathcal{G})$ just-infinite, meaning that every proper quotient is finite dimensional?

Renormalization Maps. When the invertibility of the pencils $A_\pi(z) = z_0 I + z_1\pi(a) + z_2\pi(t)$ for D_∞ and $B_\pi(z) = z_0 I + z_1\pi(a) + z_2\pi(b) + z_3\pi(c) + z_d\pi(d)$ for $\mathcal{G}$ is considered, the Schur complement leads to the following two rational maps (called the renormalization maps):

$$F_A(z) = \left[z_0(z_0^2 - z_1^2 - z_2^2) : z_1^2 z_2, z_2(z_0^2 - z_2^2)\right], \quad z \in \mathbb{P}^2,$$
$$F_B(z) = \left[z_0\alpha - z_1^2(z_0 + z_4) : z_1^2(z_2 + z_3) : z_4\alpha, z_2\alpha, z_3\alpha\right], \quad z \in \mathbb{P}^4,$$

where $\alpha(z) = (z_0 + z_4)^2 - (z_2 + z_3)^2$.

The connection shown in the two examples above merged three areas of mathematics: spectral theory, holomorphic dynamics, and group theory. Such rational mappings exist for other well-known examples of self-similar groups, such as the Lamplighter group $\mathcal{L}$, the Hanoi towers group $\mathcal{H}^3$, the Basilica group, the iterated monodromy group IMG$(z^2 + i)$. These maps allow an in-depth analysis of the invariant sets and their dynamical behavior. On the other hand, the Basilica group produces a simple 2-dimensional map that is quite hard for investigation. The paper of Dang, Grigorchuk and Lyubich [10] contains a comprehensive study of maps associated with the groups $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{H}^3$. It provides a base for further work. The work in progress on the Basilica group by the same co-authors and Bedford is in a final stage. This part of investigation was presented in the mini-course given by Dang. A plethora of examples of multidimensional rational maps and associated dynamical figures are presented in Grigorchuk and Samarakoon [20]. An important question thus arises.

Problem 3. Can the rational maps, such as F_A and F_B in the examples above, help us understand the spectral property of the groups?

Problem 3 for D_∞ was solved with satisfaction (Theorem 1.10). For $\mathcal{G}$, although only partial results are known, the map F_B, in its various forms, indeed helps, see Grigorchuk and Nekrashevych [19]. For many other groups, such as the Basilica group, the Lamplighter group, the Gupta-Sidki 3-group, the overgroup $\tilde{\mathcal{G}}$, just to name a few, computation of their corresponding joint spectrum sheds light on the spectrum of their associated Markov operator [4]. For example, the following facts have been discovered.

a) For $\mathcal{G}$, the spectrum of the Schreier graph coming from the representation π in Example 2.2 is a union of two intervals.

b) For the Gupta-Sidki 3-group, the corresponding spectrum is a Cantor set.

c) For the Gupta-Fanrikovsky group, the spectrum is a Cantor set union with a countable set of isolated points accumulating to it.

These are new phenomena in the spectral theory of regular graphs. Indeed, the connection between spectral theory, self-similarity and complex dynamics has led to a number of remarkable results concerning the spectral theory of groups and graphs.

d) The self-similar realization of the Lamplighter $\mathcal{L}$ by automaton of Mealy type over binary alphabet found in [21] led to a proof that $\mathcal{L}$ has a system of generators that produces a Cayley graph with a pure point spectrum. This was the first example of this sort, and it was immediately used in [22] to answer Atiyah's question on the existence of closed Riemannian manifolds with non-integer L^2-Betti number.

e) Further, in [23], the result from [21] was generalized to show that the spectrum of a Cayley graph could have infinitely many gaps.

Schreier Graphs. An open problem related to the e) above is as follows.

Problem 4. Is there a finitely generated group whose associated Cayley graph has a Cantor set spectrum?

There is a plenty of examples of Schreier graphs with such property, see b) above and Grigorchuk, Nagnibeda and Perez [24]. Moreover, in some important cases, for instance $\mathcal{G}$, the spectrum of the Schreier graphs coincides with the spectrum of the Cayley graph [11]. Another important discovery is the link between the spectral problem for the Schreier graphs of linear type and that for the random Schroedinger operators, Grigorchuk, Lenz and Nagnibeda [25]. These results were presented in Nagnibeda's talk. The spectral theory of Schreier graphs of self-similar groups is closely related to the spectral theory of Koopman representation π in the Hilbert space $L^2(\partial T, \mu)$, where ∂T is a boundary of the rooted tree T on which the group acts and μ is a uniform Bernoulli measure on ∂T. It is also related to the quasi-regular representations of the type

λ_{G/G_x}, where G_x is a stabilizer of a point x of the boundary. It is a miracle that despite the representation π being a direct sum of finite dimensional subrepresentations, while $\lambda_{G/G_x}, x \in \partial T$, are infinite dimensional irreducible representations in the space $l^2(G/G_x)$, the spectrum of all of them is the same and does not depend on x. Even more interesting are the C^*-algebras generated by these representations. In particular, they are related to the just-infinite trichotomy for C^*-algebras [27]. The technique of joint spectrum allows one to compute the spectra of elements in these algebras, providing important information about their properties. The Hulanicki type theorem for graphs mentioned earlier is a combinatorial analogue of the Hulanicki criterion of amenability: a group is amenable if and only if the regular representation weakly contains every unitary representation of the group.

Problem 5. Find the spectrum of the dendrite type Schreier graphs associated with IMG$(z^2 + i)$.

Observe that for the latter, a 3-dimensional renormalization map was found. Further work is needed to describe the joint spectrum and make conclusion about the spectral properties of of Laplacian on Schreier graphs.

While the interest in group theory focuses on the spectrum, which can be obtained as the limiting set of the zeros of characteristic polynomials associated with the sequences of adjacency matrices, there are other natural polynomials and functions associated with graphs, such as the chromatic polynomials and, more generally, the partition functions. Given a sequence of diamond shaped graphs (called hierarchical), Roeder, in joint work with Bleher and Lyubich, has presented how techniques from holomorphic dynamics can be used to understand the zeros of the partition function associated to a given Ising model. In the same spirit, Peters explained in his talk (based on a joint work with de Boer, Buys and Regts) how one can study zeros of the independence polynomials for sequences of lattices. The main idea is to relate the independence polynomial to the free energy associated with certain quantum field theory. Although these are different polynomials, the nature of the questions are very similar: what is the distribution of the zeros of these polynomials? The method to tackle them involves very recent techniques from holomorphic dynamics in several complex variables.

The characteristic polynomial of Lie algebras. Given a Lie algebra $\mathcal{L} = \text{span}\{x_1, ..., x_n\}$, its characteristic polynomial is defined as

$$Q_{\mathcal{L}}(z) = \det(z_0 I + z_1 \,\text{ad}\, x_1 + \cdots + z_n \,\text{ad}\, x_n),$$

where ad stands for the adjoint representation of $\mathcal{L}$. It is shown in Hu and Zhang [34] that $\mathcal{L}$ is solvable if and only $Q_{\mathcal{L}}$ is a product of linear factors. Since simple Lie algebras have been fully characterized, the following interesting question awaits an answer.

Problem 6. For a simple Lie algebra $\mathcal{L}$, what are the irreducible factors of $Q_{\mathcal{L}}(z)$?

3 List of Participants

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4 Presentation Highlights

Mini-courses. An important feature of this workshop is the presentation of mini-courses by the organizers. For a meeting that involves a wide range of fields, it is necessary to lay a common ground and instigate common interest for the participants. The mini-courses by the four organizers served this purpose very well.

Curto's mini-course discussed the foundational aspects of multivariable spectral theory and provided some applications. It started with a description of the algebraic and spatial spectral theory for several commuting operators, with an emphasis on the axiomatic approach to spatial spectra. The he proved the spectral mapping theorem for spatial spectra, assuming that the projection property holds. Next, the analytic functional calculus was defined for the Taylor spectrum, and the connections with the Bochner-Martinelli kernel was mentioned for the case when the operators belong to a C^*-algebra. The Fredholm theory was also mentioned and applied to subnormal $n$tuples, Bergman $n$tuples, and the $n$tuple M_z of multiplications by the coordinate functions acting on the Bergman space of a Reinhardt domain in $\mathbb{C}^n$.

Yang's mini-course first surveyed some recent work on the projective joint spectrum for linear operators. In finite dimension, the notion in part motivated the definition of joint characteristic polynomial for several matrices. His first lecture presented some examples and described an application of the idea to the classification of finite dimensional Lie algebras. The second lecture examined the connection between self-similarity and holomorphic dynamics, focusing particularly on the case of the dihedral group D_∞. He showed that the Julia set of the "normalized" map F_A turns out to coincide with the projective spectrum $p(A_\lambda)$ in Theorem 1.10.

Certain problems related to the computation of spectrum and joint spectrum of specific self-similar groups can be translated into a study of the dynamics of very particular transformations. In Dang's mini-course, he introduced the main techniques from complex geometry for studying the iterates of rational transformations. He explained how and when one can define some invariant currents by a given rational transformation and then explained in which situation iterative preimages of a hypersurface/subvarieties can converge to these currents.

In his mini-course, Grigorchuk introduced a general framework of the self-similar groups, explained how self-similarity and a classical tool known as the Schur complement led in some cases to either a complete solution of the spectral problem or to a reduction to a multidimensional dynamical problem: computing iterations of a rational function in $\mathbb{C}^n$ (or $\mathbb{R}^n, n \geq 2$), finding invariant subsets and attractors, etc. The process was demonstrated by examples, including the group $\mathcal{G}$, the Lamplighter group, and the Hanoi Towers groups. The lectures gave a ground for understanding the lectures by Dang and some other talks of the workshop.

On the Theme of Self-Similarity. Amenable groups (under different name) were defined by von Neumann in 1929 as a result of his study on a mysterious phenomenon in mathematics known as the Banach-Tarski Paradox. Independently they were discovered in 1939 by Bogolyubov as a class of groups whose action on a compact space always has an invariant probability measure – a remarkable property that generalizes a powerful theorem of Krylov-Bogolyubov in dynamical systems. The notion of amenability has numerous interpretations: in terms of unitary representations, random walks, combinatorics, dynamics, and joint spectrum, etc. A spectral approach to the amenability of self-similar groups led to the discovery of the "Munchhausen trick", Bartholdi and Virag [3], Kaimanovich [36], which was later used to prove the amenability of the Basilica and many other groups. The method also has a renormalization feature and allows one to associate with any self-similar group a continuous self map $\mathcal{K}$ (the Kaimanovich map) of the simplex $\Delta(G)$ of probability measures on the group. Finding $\mathcal{K}$-invariant finite dimensional non-degenerate subsimplexes of $\Delta(G)$ is a challenging problem. Study of the limit behaviour of random walks on self-similar groups is related to the study of the (covering) random walks on the covering group $\tilde{G}$ which could be a free group or a free product of finite groups. The wreath recursions, which equip G with a self-similar structure, lead to the study of random walks and their limit behavior on the direct products of copies of $\tilde{G}$. This is another challenging problem presented in the talk by Kaimanovich.

The talk by Nekrashevych was dedicated to the relation of random walks on self-similar groups and conformal dimension. Random walks on self-similar groups induce random walks on the (Schreier) graphs of their action on the levels of the tree. If the group is contracting, then the graphs converge in a certain sense to the limit space of the group. The geometry of the limit space can be used to study the random walks on the groups. The conformal dimension of the limit space and the critical exponent of contraction play important role in this study. Since the original approach to these questions involves a map equivalent to the Schur complement, there must be a non-trivial connection of the geometry of the limit space and its dimension with the spectral properties. This workshop was a good opportunity to explore it. The talk presented by Nekrashevych was based on joint work with Matte Bon and Zheng, and it presented new developments in that direction.

The talk by Sunic, “On the Schreier spectra of iterated monodromy groups of critically-fixed polynomials,” also discussed the spectra of the Schreier graph of iterated monodromy groups associated with a special type of polynomials. Every self-similar group G of d-ary tree automorphisms induces a sequence of finite Schreier graphs X_n of the action of G on the level n of the tree, along with a sequence of d-to-1 coverings $X_{n+1} \to X_n$. There are interesting examples of self-similar groups for which the spectra of the corresponding Schreier graphs are described by backward iterations of polynomials of degree 2 (the first Grigorchuk group, the Hanoi Towers group, the IMG of the first Julia set, ...). In his talk, for every $r > 1$, Sunic provided examples of self-similar groups for which the spectra of the Schreier graphs are described by backward iterations of polynomials of degree r. The examples come from the world of iterated monodromy groups of critically-fixed polynomials. A critically-fixed polynomial is a polynomial that fixes all of its critical points, and such polynomials are clearly post-critically finite. In general, if we start with any post-critically finite rational map f of degree d on the Riemann sphere, the iterated monodromy group of f (due to Nekrashevych) is a self-similar group acting on the d-ary rooted tree by automorphisms in such a way that the corresponding sequence of Schreier graphs approximates the Julia set of f and the coverings approximate the action of f on the Julia set. In the examples considered, the degree r of the polynomial that describes the spectra of the Schreier graphs coincides with the maximal local degree of f at the critical points.

As iterated monodromy groups can be constructed out of post-critically finite rational maps f, their structure is read out by the action on a certain covering of the punctured sphere. This topological viewpoint has a natural counterpart in arithmetic where a particular Galois group acts on the prime on a field extension. This new direction of research is particularly active in the last decades and many cases of Odoni’s conjecture were established. This was the subject of Juul’s talk.

Self-similar groups are related to many areas of mathematics, in particular, with the theory of Totally Disconnected Locally Compact Groups (TDLC) discovered recently by Willis [6, 48]. An important subclass of the class of self-similar groups constitutes the self-replicating groups. Scale TDLC groups are closed subgroups of the group $Aut(\tilde{T}_{d+1})$ of automorphisms of the $d+1$-regular tree $\tilde{T}_{d+1}, d \geq 2$ acting vertex transitive and fixing the end of the tree. The stabilizers of vertices in scale groups projected on d-regular rooted tree are self-replication groups. Grigorchuk and Savchuk introduced the class of lifting groups and showed that the closures of such groups embeds into scale groups. Moreover, the closures of the associated ascending HNN-extensions embed as scale groups. They showed for instance that the group $\mathcal{G}$ produces scale group that acts 2-transitively on the punctured boundary of the tree (boundary with one end deleted. Meanwhile it is also shown that groups of isometries of the ring $\mathbb{Z}_p$ and field $\mathbb{Q}_p$ of p-adics are isomorphic to important groups of tree automorphisms. Also the group of dilations of $\mathbb{Q}_p$ is identified. These results were presented in the talk by Savchuk.

In the last two decades, new directions in group theory emerged: dynamically defined groups and measurable group theory, and they are related. A recent book of Nekrashevych [42] (as well as his previous book

"Self-similar groups" [41]) contains the main features of these two directions. An important new class of groups of dynamical origin is the class of Topological Full Groups (TFGs) introduced by Giordano, Putnam and Skau [29] as complete algebraic invariant of flip conjugacy for the minimal Cantor systems. They where studied by Matui [39] and other mathematicians. The conjecture of Grigorchuk and Medynets claiming that the TFGs of minimal Cantor systems are amenable was confirmed by Juschenko and Monod [35]. More interesting results about the TFGs are obtained in [39], where it is shown in particular that the commutators of these groups are finitely generated and simple. They are infinitely presented but the presentations by generators and realtors could be written using the Kakutani-Rohlin towers. Also they can be factored into a product of two locally finite groups [26]. The talk of Medynets gave an account of the results on these topics.

The theory of self-similar groups is closely related to the theory of groups generated by finite automata of Mealy type which can be synchronous or asynchronous. Asynchronous invertible automata could generate groups such as the Thompson's group F. The isomorphism type of the group $\mathcal{R}$ of invertible asynchronous automata, introduced in Grigorchuk, Nekrashevych and Suschanskii [28] under the name "group of Rational Homeomorphisms of a Cantor Set", does not depends on the cardinality of the involved finite alphabet $X = \{x_1, \ldots, x_d\}, d \geq 2$. It attracted much attention during the last decade, and a number of results about embedding various types of groups into $\mathcal{R}$ were obtained in the works of Bleak, Belk, Matucci and others.

There is also a way to generalize groups generated by finite synchronous automata, making their higher dimensional analogous. This approach was described in the talk by Vdovina " Higher structures in mathematics: buildings, k-graphs and C^*-algebras." She presented buildings as universal covers of certain infinite families of CW-complexes of arbitrary dimension. Then she showed several different constructions of new families of k-rank graphs and C^*-algebras based on these complexes, for arbitrary k. The underlying building structure allows explicit computation of the K-theory as well as the explicit spectra computation for the k-graphs. Also, she suggested a new definition of higher-dimensional automata motivated by cocompact quotients of buildings, which allows one to construct infinite series of such automata and produce very explicit constructions of Ramanujan higher-dimensional graphs. The talk was based on joint results with Bondarenko, Grigorchuk and Stix.

On the Theme of Spectral Theory. The workshop also consists of inspiring talks in the broader context of spectral theory.

Klep reviewed their solution of the two-sided version of the 2003 conjecture by Hadwin and Larson concerning linear pencils of the form $L = T_0 + x_1T_1 + \cdots + x_mT_m$, where $x_1, ..., x_m$ are matrices. He showed that ranks of such pencils constitute a collection of separating invariants for simultaneous similarity of matrix tuples.

A (non-selfadjoint) operator algebra is said to be residually finite dimensional (RFD) if it embeds into a product of matrix algebras. A theorem of Exel and Loring characterizes RFD C^*-algebras in terms of the state space and in terms of a finite-dimensional approximation property for representations. Hartz talked about a non-selfadjoint version of the Exel-Loring theorem.

Describing the joint invariant subsapces of a tuple A is in general an insurmountable challenge. The situation is more tractible for multiplication operators on holomorphic function spaces. Misra's talk considered unitary representations of the Möbius group acting on the Hilbert space $\mathcal{H}^{(\lambda)} \otimes V$, where $H^{(\lambda)}$ is a weighted Bergman space and V is a finite dimensional Hilbert space. He showed a one-to-one correspondence between such representations and tuples of subnormal homogeneous operators.

Kuhlmann and Infusino's talks provided an extension of the classical truncated moment problem. They showed a criterion for the existence of a representing Radon measure for linear functionals defined on a unital commutative real algebra. This allows one to extend these existence results to infinite dimensional instances, for example the case when the generating vector space is endowed with a nuclear topology. This allows them to prove a Riesz-Haviland type theorem, extending some classical results in the moment problems.

Given a tuple $A = (A_1, \dots, A_n)$ of operators or matrices, how much does the projective spectrum $p(A)$ determine the tuple? The tuple is said to be spectrally rigid if any other tuple (probably with additional conditions) that has the same projective spectrum is equivalent to $(A_1, \dots, A_n)$. Results addressing this question are called spectral rigidity theorems. In his talk, Stessin surveyed some surprising rigidity theorems related to representations of

a) Coxeter groups;

b) certain subgroups of permutation groups which are related to Hadamard matrices of Fourier type;

c) Lie algebra $\mathfrak{sl}(2)$;

4) infinitesimal generators of quantum $SU(2)$ groups.

In addition, Kinser gave an expository introduction to quiver representations and their associated rank schemes. Like joint spectra, they are also defined by determinantal equations. Schiffler's talk aims to link cluster algebras with joint spectrum, and it provided some intriguing examples. Reinke's talk provided a model for random automorphisms of spherically homogeneous rooted trees such that the action on the ends has high emergence almost surely.

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