MINICOURSE: INTRODUCTION TO THE ALGEBRAIC AND SPATIAL SPECTRAL THEORY FOR SEVERAL COMMUTING OPERATORS

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Review from Monday's lecture

Let $\mathcal{B}$ be a unital commutative Banach algebra, and let $M_{\mathcal{B}}$ be the maximal ideal space of $\mathcal{B}$, and $\hat{}: \mathcal{B} \longrightarrow C(M_{\mathcal{B}})$ be the Gelfand transform. For $a \subset \mathcal{B}$, the joint spectrum of a in $\mathcal{B}$ is

$$\sigma_{\mathcal{B}}(a) = \hat{a}(M_{\mathcal{B}}).$$

Moreover, there exists a homomorphism $f \longmapsto f(a)$ from the algebra $H(\sigma_{\mathcal{B}}(a))$ of (germs of) functions analytic in a neighborhood of $\sigma_{\mathcal{B}}(a)$ into $\mathcal{B}$ such that

$$\begin{aligned}
1(a) &= 1; \\
\xi(a) &= a, \quad \text{where } \xi(z) := z \text{ for all } z; \\
f(\hat{a}) &= f \circ \hat{a} \qquad (f \in H(\sigma_{\mathcal{B}}(a))) \\
\sigma_{\mathcal{B}}(f(a)) &= f(\sigma_{\mathcal{B}}(a)) \quad (\text{ Spectral Mapping Theorem}).
\end{aligned}$$

When $\mathcal{B}$ is no longer commutative, say $\mathcal{B} = \mathcal{L}(\mathcal{X})$ ($\mathcal{X}$ a Banach space), one looks for a commutative subalgebra $\mathcal{A}$ containing a.

DEFINITION 1.4. Let K be a bounded subset of $\mathbb{C}^n$. The polynomially convex hull of K is

$$\hat{K} := \{\lambda \in \mathbb{C}^n : |p(\lambda)| \le \|p\|_K \quad \text{for all } p \in \mathbb{C}[z]\}.$$

(Here $\mathbb{C}[z]$ stands for the algebra of polynomials in z.)

Clearly $K \subseteq \hat{K}$. When $K = \hat{K}$ we say that K is polynomially convex.

Observe that $\hat{K}$ is always closed.

THEOREM 1.6. (Oka-Weil) Let K be polynomially convex, and let Ω be an open subset of $\mathbb{C}^n$ containing K. If $f \in H(\Omega)$, the algebra of functions analytic in Ω, then f is the uniform limit on K of a sequence of polynomials.

THEOREM 1.8. Let $a \subset \mathcal{B}^n$ and assume that a generates $\mathcal{B}$. Then $\sigma_{\mathcal{B}}(a)$ is polynomially convex, and will be denoted by $\hat{\sigma}(a)$.

To construct an analytic functional calculus for $\sigma_{\mathcal{B}}$, we first consider the case of a polydomain D:

$$f(a) := \frac{1}{(2\pi i)^n} \int_{\partial D} f(w) \prod_{i=1}^{n} (w_i - a_i)^{-1} \, d_1 \cdot \ldots \cdot dw_n.$$

Next, we look at a polynomially convex set K, and use Oka-Weyl to approximate a function analytic in a neighborhood of K by polynomials.

From polynomially convex sets we jump to a special type of polynomially convex sets, the polynomial polyhedra.

THEOREM 1.12. (Oka-Cartan-Waelbroeck)

$$H(D \times \Delta)/I(g) \simeq H(\Omega(p, D, \Delta))$$

THEOREM 1.13. (Waelbroeck) Let a $\in \mathcal{B}^n$ and let Ω be an open subset of $\mathbb{C}^n$ containing $\sigma_R(a)$. Then there exist polydomains D and Δ and a polynomial mapping $p : D \longrightarrow \Delta$ such that $\sigma_R(a) \subseteq \Omega(p, D, \Delta) \subseteq \Omega$.

To get the functional calculus for $\sigma_{\mathcal{B}}$ one needs the following clever observation.

LEMMA 1.14. (The Arens-Calderón trick) Let $a \in \mathcal{B}^n$ and let $\Omega \supseteq \sigma_{\mathcal{B}}(a)$. Then there exists $b = (b_1, \ldots, b_k) \in \mathcal{B}^k$ such that

$$\Omega \supseteq P_a(\hat{\sigma}(a, b)),$$

where $P_a : \mathbb{C}^{n+k} \longrightarrow \mathbb{C}^n$ takes (z, w) to z, and where $\hat{\sigma}(a, b) := (\sigma_{\mathcal{B}}(a, b))^{\wedge}$.

To further appreciate the value of this result, recall that $\hat{\sigma}$ is the largest of the algebraic spectra. Actually,

$$\sigma_R \subseteq \sigma_{\mathcal{B}} \subseteq \hat{\sigma}.$$

Thus, if we start with $\Omega \supset \sigma_{\mathcal{B}}(a)$, a priori we don't know that $\Omega \supset \sigma_{\mathcal{B}}(a)$. However, as we transition from (a) to $((a, b))$, the likelihood that a would be invertible increases.

The Arens-Calderón trick states that for some b, Ω will contain the projection of $\hat{\sigma}(a, b)$.

THEOREM 1.15. (Shilov, Arens-Calderón, Waelbroeck) There exists a continuous homomorphism $f \mapsto f(a)$ from $H(\sigma_{\mathcal{B}}(a))$ into $\mathcal{B}$ such that

(i) $1(a) = 1$;

(ii) $z_i(a) = a_i \quad (i = 1, \ldots, n)$; and

(iii) $f(\hat{a}) = f \circ \hat{a}$, for all $f \in H(\sigma_{\mathcal{B}}(a))$.

Consequently,

$$\sigma_{\mathcal{B}}(f(a)) = f(\sigma_{\mathcal{B}}(a)).$$

THEOREM 1.17. (Uniqueness of the Functional Calculus) ([59, III.4.1]) Suppose that $f \mapsto \tilde{f}(a)$ is another functional calculus satisfying (i), (ii), and

(iv) $\tilde{f}(a) = (f \circ P_a)(a, b)$, for all a, b and $f \in H(\sigma_{\mathcal{B}}(a))$. Then $\tilde{f}(a) = f(a)$ for all $f \in H(\sigma_{\mathcal{B}}(a))$.

EXAMPLE 2.2. (E. Albrecht) There exist a Banach space, $\mathcal{X}$, an n–tuple $a \subset \mathcal{L}(\mathcal{X})$, and maximal abelian subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ containing a such that

$$\sigma_{\mathcal{A}_1}(a) \neq \sigma_{\mathcal{A}_2}(a).$$

Given a set X, we shall let $\mathcal{P}(X)$ denote the power set of X. Also, $\mathbb{C}^\omega$ will denote the Cartesian product of denumerably many copies of $\mathbb{C}$).

DEFINITION 2.3. Let $\mathcal{B}$ be a Banach algebra. A spectral system for $\mathcal{B}$ is a map

$$\tilde{\sigma} : \bigcup_{n=1}^{\infty} \mathcal{B}^n_{\mathrm{com}} \longrightarrow \mathcal{P}\left(\mathbb{C}^\omega\right)$$

such that

(i) a $\subset \mathcal{B} \Longrightarrow \tilde{\sigma}(a) \neq \emptyset$

(ii) a $\in \mathcal{B}^n_{\mathrm{com}} \Longrightarrow \tilde{\sigma}(a) \subseteq \mathbb{C}^n \subseteq \mathbb{C}^\omega$ and

(iii) a $\subset \mathcal{B} \Longrightarrow \tilde{\sigma}(a)$ is compact.

DEFINITIONS 2.4. Let $\mathcal{B}$ be a Banach algebra. A spectral system $\tilde{\sigma}$ for $\mathcal{B}$ possesses the projection property if

$$P_a[\tilde{\sigma}(a,b)] = \tilde{\sigma}(a)$$

for all $a, b \subset \mathcal{B}$, where $P_a : \mathbb{C}^n \times \mathbb{C}^k \longrightarrow \mathbb{C}^n, \ \ (z,w) \mapsto z$.

$\tilde{\sigma}$ possesses the spectral mapping property for polynomials if $\tilde{\sigma}(p(a)) = p(\tilde{\sigma}(a))$ for every polynomial map $p : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ and for every n–tuple $a \in \mathcal{B}^n_{\mathrm{com}}$.

EXAMPLES 2.5. (i) If $\mathcal{B}$ is commutative, $\sigma_{\mathcal{B}}$, σ_R and $\hat{\sigma}$ are all spectral systems. However, only $\sigma_{\mathcal{B}}$ has the projection property.

(ii) For $a \subset \mathcal{B}$ we let (a), $(a)'$ and $(a)''$ denote the unital Banach subalgebras of $\mathcal{B}$ generated by $a_1, \ldots, a_n$, its commutant (relative to $\mathcal{B}$) and its double commutant, respectively. We define

$$\hat{\sigma}(a) := \sigma_{(a)}(a) \quad \text{(using the Arens-Calderón trick)}$$

$$\sigma''(a) := \sigma_{a''}(a),$$

and

$$\sigma'(a) := \left\{ \lambda \in \mathbb{C}^n : (a - \lambda) \circ (a)' \neq (a)' \right\}.$$

More generally, if $\mathcal{A}$ is any (closed) subalgebra of $\mathcal{B}$ containing a in its center, we let

$$\sigma_{\mathcal{A}}(a) := \{\lambda \in \mathbb{C}^n : \ (a - \lambda) \circ \mathcal{A} \neq \mathcal{A}\}.$$

Observe that $\hat{\sigma}, \sigma'$ and σ'' are spectral systems without the projection property (Slodkowski - Żelazko). Also,

$$\sigma' \subseteq \sigma'' \subseteq \hat{\sigma}.$$

In general, when we refer to an algebraic spectrum we mean $\sigma_{\mathcal{A}}$ for some $\mathcal{A}$.

(iii) The left spectrum is defined by

$$\sigma_\ell(a) := \left\{\lambda \in \mathbb{C}^n : \ \mathcal{B} \circ (a - \lambda) \neq \mathcal{B}\right\} \quad (a \subset \mathcal{B}).$$

Similarly, the right spectrum is given by

$$\sigma_r(a) := \left\{\lambda \in \mathbb{C}^n : (a - \lambda) \circ \mathcal{B} \neq \mathcal{B}\right\} \quad (a \subset \mathcal{B}).$$

The Harte spectrum is

$$\sigma_H := \sigma_\ell \cup \sigma_{\mathbf{r}}.$$

We easily see that

$$\sigma_\ell, \sigma_{\mathbf{r}} \subseteq \sigma_H \subseteq \sigma';$$

moreover, σ_ℓ, σ_r and σ_H all possess the projection property.

(iv) If $\mathcal{B} = \mathcal{L}(\mathcal{X})$, where $\mathcal{X}$ is a Banach space, the approximate point spectrum and defect spectrum are defined by

$$\sigma_\pi(a) := \{\lambda \in \mathbb{C}^n : a - \lambda \text{ is not jointly bounded below}\}$$

and

$$\sigma_\delta(a) := \{\lambda \in \mathbb{C}^n : a - \lambda \text{ is not jointly onto}\}.$$

$a \subset \mathcal{L}(\mathcal{X})$ is jointly bounded below if there exists a positive constant $\epsilon > 0$ such that

$$\sum_{i=1}^{n} \|a_i x\| \geq \epsilon \|x\| \quad (\text{ all } x \in \mathcal{X});$$

a is jointly onto if

$$\sum_{i=1} a_i \mathcal{X} = \mathcal{X}.$$

σ_π and σ_δ are spectral systems with the projection property.

(v) Let $\{\sigma_{\pi,k}\}_{k=0}^{n-1}$ and $\{\sigma_{\delta,k}\}_{k=1}^{n}$ be the generalized approximate point and defect spectra introduced by Z. Slodkowski. As shown by Slodkowski, $\sigma_{\pi,k}$ and $\sigma_{\delta,k}$ are spectral systems with the projection property.

(vi) For $a \subset \mathcal{L}(\mathcal{X})$ let

$$\sigma_{\prod}(a, \mathcal{X}) := \sigma(a_1, \mathcal{X}) \times \cdots \times \sigma(a_n, \mathcal{X}),$$

where $\sigma(a_i, \mathcal{X})$ denotes the ordinary spectrum of a_i as an element of $\mathcal{L}(\mathcal{X})$.

$\sigma_{\prod}$ is a spectral system with the projection property; however, $\sigma_{\prod}$ does not have the spectral mapping property for polynomial mappings. (**Example**. Let p be a nontrivial idempotent in $\mathcal{L}(\mathcal{X})$. Then $\sigma_{\prod}(p, p) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, so that

$$\{\lambda_1 + \lambda_2 : (\lambda_1, \lambda_2) \in \sigma_{\prod}(p, p)\} = \{0, 1, 2\},$$

while $\sigma(2p) = \{0, 2\}$.

(vii) (Taylor's spectrum) Let $\Lambda = \Lambda[e] = \Lambda_n[e]$ be the exterior algebra on n generators $e_1, \ldots, e_n$, with identity $e_0 = 1$. Λ is the algebra of forms in $e_1, \ldots, e_n$ with complex coefficients, subject to the collapsing property

$$e_i e_j + e_j e_i = 0 \quad (1 \leq i, j \leq n);$$

in particular, $e_i^2 = 0 \quad (i = 1, \ldots, n)$.

Let $E_i : \Lambda \longrightarrow \Lambda$ be given by $E_i \xi = e_i \xi \quad (i = 1, \ldots, n)$.

$E_1, \ldots, E_n$ are the so-called creation operators. Clearly,

$$E_i E_j + E_j E_i = 0 \quad (1 \leq i, j \leq n).$$

Λ can be regarded as a Hilbert space, if we declare

$\{e_{i_1} \cdot \ldots \cdot e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ as an orthonormal basis. Observe that every form $\xi \in \Lambda$ admits a unique decomposition

$$\xi = e_i \xi' + \xi''$$

Actually, each E_i is a partial isometry, and

$$E_i^* E_j + E_j E_i^* = \delta_{ij} \quad (1 \leq i, j \leq n).$$

(In a C^*–algebra $\mathcal{A}$, an element v is a partial isometry if $v^* v v^* = v^*$; equivalently, if

$v^* v$ is an orthogonal projection p, i.e., $p^2 = p$ and $p^* = p$.)

For $\mathcal{X}$ a vector space, we let $\Lambda(\mathcal{X}) := \mathcal{X} \otimes_{\mathbb{C}} \Lambda$. For $A \subset \mathcal{L}(\mathcal{X})$, we define

$D_A : \Lambda(\mathcal{X}) \longrightarrow \Lambda(\mathcal{X})$ by

$$D_A := \sum_{i=1}^{n} A_i \otimes E_i.$$

Then for $x \in \mathcal{X}$ and $\xi \in \Lambda$, we get

$$D_A^2(x \otimes \xi) = \sum_{i,j=1}^{n} A_j A_i x \otimes E_j E_i \xi = \sum_{i<j} A_i A_j x \otimes (E_i E_j + E_j E_i)\, \xi = 0.$$

It follows that $R(D_A) \subseteq N(D_A)$, where R and N denote range and kernel, resp.

We say that A is nonsingular on $\mathcal{X}$ if $R(D_A) = N(D_A)$.

When $n = 1$, for instance, A is nonsingular if and only if A is one-to-one and onto, and therefore invertible.

The Taylor spectrum of A on $\mathcal{X}$ is

$$\sigma_T(A, \mathcal{X}) := \{\lambda \in \mathbb{C}^n : R(D_{A-\lambda}) \neq N(D_{A-\lambda})\}.$$

One can also construct a cochain complex $K(A, \mathcal{X})$, called the Koszul complex associated to A on $\mathcal{X}$, as follows:

$$K(A, \mathcal{X}) : 0 \longrightarrow \Lambda^0(\mathcal{X}) \xrightarrow{D_A^0} \Lambda^1(\mathcal{X}) \xrightarrow{D_A^1} \Lambda^2(\mathcal{X}) \cdots \Lambda^{n-1}(\mathcal{X}) \xrightarrow{D_A^{n-1}} \Lambda^n(\mathcal{X}) \longrightarrow 0,$$

where $\Lambda^k(\mathcal{X})$ is the tensor product of the vector space of k forms and $\mathcal{X}$. Then

$$\sigma_T(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : K(A - \lambda, \mathcal{X}) \text{ is not exact}\}.$$

Illustrative Example: Consider the space $C_0^\infty(\mathbb{R}^3)$, the operators $\partial_x, \partial_y, \partial_z$ of partial differentiation, and the de Rham complex associated with them; that is,

$$0 \longrightarrow C_0^\infty(\mathbb{R}^3) \xrightarrow{\nabla} C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \xrightarrow{\text{curl}} C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3) \xrightarrow{\text{div}} C_0^\infty(\mathbb{R}^3) \longrightarrow 0,$$

with $\text{curl}\,\nabla = 0$ and $\text{div}\,\text{curl} = 0$.

(J.L. Taylor) $\sigma_T(A, \mathcal{X})$ is compact and nonempty, and $\sigma_T(A, \mathcal{X}) \subseteq \sigma'(A)$. Moreover, σ_T carries an analytic functional calculus (so that in particular σ_T has the projection property).

(J.L. Taylor) If $\mathcal{B}$ is a commutative Banach algebra and $a \subset \mathcal{B}$, and if we let L_a denote the n–tuple of left multiplications by the a_i's, then $\sigma_T(L_a, \mathcal{B}) = \sigma_{\mathcal{B}}(a)$.

Thus, every algebraic joint spectrum $\sigma_{\mathcal{A}}(a)$ can be thought of as $\sigma_T(L_a, \mathcal{A})$.

For $n = 2$,

$$D_A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 \\ 0 & -A_2 & A_1 & 0 \end{bmatrix} \begin{matrix} e_0 \\ e_1 \\ e_2 \\ e_1 e_2 \end{matrix}$$

and

$$N(D_A) = [N(A_1) \cap N(A_2) \oplus \{(x_1, x_2) : A_2 x_1 = A_1 x_2\} \oplus \mathcal{X}$$

$$R(D_A) = 0 \oplus \{(A_1 x_0, A_2 x_0) : x_0 \in \mathcal{X}\} \oplus [R(A_1) + R(A_2)] .$$

Observe that

$$\sigma_\pi(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : D^0_{A-\lambda} \text{ is not bounded below}\}$$

and

$$\sigma_\delta(A, \mathcal{X}) = \{\lambda \in \mathbb{C}^n : D^{n-1}_{A-\lambda} \text{ is not onto}\}$$

so that $\sigma_\pi, \sigma_\delta \subseteq \sigma_T$.

(Spectral Permanence; RC 1982) When $\mathcal{B}$ is a C^*–algebra, say represented as $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H})$, and $a \subset \mathcal{B}$, then

$$\sigma_T(L_a, \mathcal{B}) = \sigma_T(L_a, \mathcal{B}(\mathcal{H})) = \sigma_T(a, \mathcal{H}) \quad (RC, [37]).$$

In particular, if $a \subset \mathcal{B} \subseteq \mathcal{C}$, then $\sigma_T(L_a, \mathcal{B}) = \sigma_T(L_a, \mathcal{C})$.

This fact, known as spectral permanence for the Taylor spectrum, is a consequence of the following result:

L_a is nonsingular on $\mathcal{B}$ if and only if $D_a + D_{a^*}^t$ is invertible (as a matrix over $\mathcal{B}$),

where $a^* = (a_1^*, \ldots, a_n^*)$ and ^{t} denotes the transpose of a matrix.

As a consequence, we also see that $D_a + D_{a*}^t$ determines the nonsingularity of a on $\mathcal{H}$.

It is because of this that we shall usually refer to nonsingularity as invertibility when a is an n-tuple of C^*-algebra elements.

In the special case of a normal k–tuple $N \equiv (N_1, \ldots, N_n)$, we know that $C^*(N)$ is abelian, and

$$\sigma_{C^*(N)}(N) = \sigma_T(N);$$

that is, the Taylor spectrum agrees with the algebraic joint spectrum.

It is also the case that for normal k–tuples on Hilbert space,

$$\sigma_\ell(N) = \sigma_r(N) = \sigma_H(N) = \sigma_{\pi,i}(N) = \sigma_{\delta,j}(N) = \sigma_T(N) \quad (0 \le i \le k-1, 1 \le j \le k).$$

More can be said:

$\sigma_{C^*(N)}(N)$ is intimately related to $\sigma_{C^*(N)}(N, N^*)$.

For, $C^*(N)$ is generated, as a Banach algebra, by N and N^*. Therefore, $\sigma_{C^*(N)}(N, N^*)$ is polynomially convex. Moreover, the map

$$\Phi : \sigma_{C^*(N)}(N) \longrightarrow \sigma_{C^*(N)}(N, N^*) \quad (\lambda \mapsto (\lambda, \bar{\lambda}))$$

is a homeomorphism. (Recall that if $\varphi \in M_{C^*(N)}$, then $\varphi(N^*) = \overline{\varphi(N)}$.)

Now, any compact nonempty set can be regarded as the apectrum of a normal operator. Thus. although $\sigma_{C^*(N)}(N)$ is just an arbitrary compact nonempty set, $\sigma_{C^*}(N)(N, N^*)$ is always polynomially convex.

We have, therefore, an operator theoretic proof of the following fact:

(RC) Given a compact nonempty set $K \subset \mathbb{C}^n$, the set $\{(\lambda, \bar{\lambda}) : \lambda \in K\}$ is polynomially convex in $\mathbb{C}^{2n}$.

When $\mathcal{X}$ is a finite dimensional space and $A \subset \mathcal{L}(\mathcal{X})$, then

$$\begin{aligned} \sigma_p(A,\mathcal{X}) = & \quad \sigma_\ell(A,\mathcal{X}) = \sigma_\pi(A,\mathcal{X}) = \sigma_r(A,\mathcal{X}) = \sigma_\delta(A,\mathcal{X}) \\ = & \quad \sigma_T(A,\mathcal{X}) = \sigma'(A) = \sigma''(A) = \hat{\sigma}(A) \end{aligned}$$

The reason for this lies in the fact that $A_1, \ldots, A_n$ can be simultaneously triangularized

as

$$A_i = \begin{bmatrix} \lambda_i^{(1)} & * & \cdots & * \\ 0 & \lambda_i^{(2)} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i^{(k)} \end{bmatrix} \quad (i = 1, \ldots, n),$$

and it follows that all of the above spectra equal $\left\{\lambda^{(1)}, \ldots, \lambda^{(k)}\right\}$.

To conclude this preliminary discussion of σ_T, we list another important property:

(Taylor) Let $\mathcal{X}$ be a Banach space, let $A \subset \mathcal{L}(\mathcal{X})$, and let $\mathcal{Y}$ be a subspace of X invariant under A (i.e., $A_i \mathcal{Y} \subseteq \mathcal{Y}, \quad (i = 1, \ldots, n)$. Then union of any two of the sets $\sigma_T(A, \mathcal{X})$, $\sigma_T(A, \mathcal{Y})$ and $\sigma_T(A, \mathcal{X}/\mathcal{Y})$ contains the third.

(RC) Put differently, if A $\subset \mathcal{L}(\mathcal{X}), \quad B \subset \mathcal{L}(\mathcal{Y})$, and if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \subset \mathcal{L}(\mathcal{X} \oplus \mathcal{Y})$$

then

$$\sigma_T \left[\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \mathcal{X} \oplus \mathcal{Y} \right] \subseteq \sigma_T(A, \mathcal{X}) \cup \sigma_T(B, \mathcal{Y}).$$

To end this section, we shall present a table summarizing the various notions of joint spectra and their properties.

Spectral Systems on $\mathcal{X}$

($\mathcal{B}$ unital, commutative Banach algebra; $\mathcal{X}$ Banach space)

	$\sigma_{\mathcal{B}}$	$\sigma_\pi, \sigma_\delta$	σ_ℓ, σ_r	σ_H	σ_T	σ'	σ''	σ_R	$\hat{\sigma}$	$\sigma_{\prod}$
Proj. Prop.	✓	✓	✓	✓	✓					✓
SMP for Polyn. Maps	✓	✓	✓	✓	✓					
Funct. Calculus	✓				✓					
SMP for Anal. Fctns.	✓				✓					

We will study the functional representation of spectral systems with the projection property. W. Żelazko defined a subspectrum on a Banach algebra $\mathcal{B}$ as a spectral system $\tilde{\sigma}$ on $\mathcal{B}$ possessing the spectral mapping property for polynomials and such that $\tilde{\sigma}(a) \subseteq \prod_{i=1}^{n} \sigma_{\mathcal{B}}(a_i)$ for all a $\subset \mathcal{B}$. Żelazko then showed that subspectra admit a functional representation in terms of the Gelfand transform, as follows:

If $\mathcal{A}$ is a maximal abelian subalgebra of $\mathcal{B}$, then there exists a compact nonempty subset $M_{\tilde{\sigma}}(\mathcal{A})$ of $M_{\mathcal{A}}$ such that

$$\tilde{\sigma}(a) = \hat{a}(M_{\tilde{\sigma}}(\mathcal{A}))$$

for all $a \subset \mathcal{A}$. As a consequence, $\tilde{\sigma}(a) \subseteq \hat{a}(M_a) = \sigma_{\mathcal{A}}(a)$. (Żelazko proved this inclusion first, and then used it in the proof of the functional representation).

Our approach to this circle of ideas will be the following:

(i) We start with a spectral system $\tilde{\sigma}$ on a Banach space $\mathcal{X}$ possessing the projection property and such that $\tilde{\sigma} \subseteq \sigma_{\mathcal{B}}$ for some commutative Banach algebra $\mathcal{B}$ (see Definition 3.1 below).

(ii) We find the functional representation for $\tilde{\sigma}$.

(iii) We prove the uniqueness of $M_{\tilde{\sigma}}(\mathcal{B})$.

Our hypotheses are weaker than those used by Żelazko, but we will still recover the same results.

DEFINITION 3.1. Let $\mathcal{X}$ be a Banach space and let $\tilde{\sigma}$ be a spectral system on $\mathcal{X}$ possessing the projection property. Let $\mathcal{B}$ be a commutative Banach algebra acting on $\mathcal{X}$, i.e., there exists a map $\Phi : \mathcal{B} \longrightarrow \mathcal{L}(\mathcal{X})$ (alternatively, assume that $\mathcal{X}$ is a Banach $\mathcal{B}$–module). Assume that

$$\tilde{\sigma}(\Phi(a)) \subseteq \sigma_{\mathcal{B}}(a) \quad (\text{ for all } a \subset \mathcal{B}).$$

Then

$$M_{\tilde{\sigma}} \equiv M_{\tilde{\sigma}}(\mathcal{B}, \mathcal{X}, \Phi) := \bigcap_{a \subset \mathcal{B}} \hat{\mathrm{a}}^{-1}[\tilde{\sigma}(\Phi(a))] \subseteq \mathrm{M}_{\mathcal{B}}.$$

LEMMA 3.2. $M_{\tilde{\sigma}}$ is a compact and nonempty subset of $M_{\mathcal{B}}$.

The proof uses the finite intersection property.

THEOREM 3.3. Let $\mathcal{X}, \mathcal{B}, \Phi$ and $\tilde{\sigma}$ be as before. If $a \subset \mathcal{B}$ then

$$\tilde{\sigma}(\Phi(a)) = \hat{a}(M_{\tilde{\sigma}})$$

Moreover, $M_{\tilde{\sigma}}$ is the only compact nonempty subset of $M_{\mathcal{B}}$ having this property.

REMARKS 3.4. (i) The importance of the functional representation of $\tilde{\sigma}$ is that one recovers the formula "$\sigma(a) = \text{range}\, \hat{a}$"; in the noncommutative case, however, one must restrict $\hat{a}$ to a certain subset of a maximal ideal space. The uniqueness of $M_{\tilde{\sigma}}$ is of significance in some applications (see Corollary 3.11 below). (ii) For the special case of σ_T, J.L. Taylor found a corresponding version of Theorem 3.3. We basically follow his outline for our proof below.

Recall:

THEOREM 3.3. Let $x, \mathcal{B}, \Phi$ and $\tilde{\sigma}$ be as before. If a $\subset \mathcal{B}$ then

$$\tilde{\sigma}(\Phi(a)) = \hat{a}\,(M_{\tilde{\sigma}})$$

Moreover, $M_{\tilde{\sigma}}$ is the only compact nonempty subset of $M_{\mathcal{B}}$ having this property.

Proof of Theorem 3.3.

$$\hat{a}\,(M_{\tilde{\sigma}}) \subseteq \hat{a}\left(\hat{a}^{-1}[\tilde{\sigma}(\Phi(a))]\right) \subseteq \tilde{\sigma}(\Phi(a)).$$

Conversely, let $\lambda \in \widetilde{\sigma}(\Phi(a))$. We claim that there exists $\varphi \in M\tilde{\sigma}$ such that $\varphi(a) = \lambda$.

Consider the family $\left\{M_b \cap \hat{a}^{-1}(\lambda)\right\}_{b \subset \mathcal{B}}$. One shows that

$$\bigcap_{b \subset \mathcal{B}} \left(M_b \cap \hat{a}^{-1}(\lambda)\right) \neq \emptyset$$

provided $M_b \cap \hat{a}^{-1}(\lambda) \neq \emptyset$ for all $b \subset \mathcal{B}$.

To prove the latter fact, consider $(a, b) \subset \mathcal{B}$. Then

$$P_a[\tilde{\sigma}(\Phi(a), \Phi(b))] = \tilde{\sigma}(\Phi(a)),$$

so that for some $\mu \in \tilde{\sigma}(\Phi(b))$,

$$(\lambda, \mu) \in \tilde{\sigma}(\Phi(a), \Phi(b)) \subseteq \sigma_{\mathcal{B}}(a, b),$$

and therefore there exists $\varphi \in M_{\mathcal{B}}$ such that $\varphi(a) = \lambda$, and $\varphi(b) = \mu$.

In particular, $\varphi \in M_b \cap \hat{a}^{-1}(\lambda)$.

We have thus shown that

$$\tilde{\sigma}(\Phi(a)) \subseteq \hat{a}\,(M_{\tilde{\sigma}})\,.$$

To prove that $M_{\tilde{\sigma}}$ is unique, one uses the fact that $\{\hat{a} : a \subset \mathcal{B}\}$ separates points in $M_{\mathcal{B}}$.

COROLLARY 3.5. (Spectral Mapping Theorem for Polynomial Mappings) Let $\mathcal{X}, \mathcal{B}, \Phi$ and $\tilde{\sigma}$ be as before. Let a $\subset \mathcal{B}$ and let $p : \mathbb{C}^n \longrightarrow \mathbb{C}^k$ be a polynomial mapping. Then

$$\tilde{\sigma}[\Phi(p(a))] = p[\tilde{\sigma}(\Phi(a))]$$

Proof.

$$\tilde{\sigma}[(p(a)] = [p(a)]^{\wedge}(M_{\tilde{\sigma}}) = (p \circ \hat{a})(M_{\tilde{\sigma}}) = p[\hat{a}(M_{\tilde{\sigma}})] = p[\tilde{\sigma}(\Phi(a))].$$

REMARK 3.6. What the corollary really shows is that whenever the formula $f(a) = f \circ \hat{a}$ holds then the spectral mapping theorem also holds. For instance, that is the case for $f \in H(\sigma_{\mathcal{B}}(a))$.

The following special case of Corollary 3.5 encompasses the basic ideas of this section.

COROLLARY 3.7. Let $\tilde{\sigma}$ be a spectral system on a Banach space $\mathcal{X}$. Assume that $\tilde{\sigma}$ possesses the projection property and that $\tilde{\sigma}(a) \subseteq \hat{\sigma}(a)$ for all a $\subset \mathcal{L}(\mathcal{X})$. Then $\tilde{\sigma}$ possesses the spectral mapping property for polynomial mappings.

APPLICATION 3.8. (RC) Let $\tilde{\sigma}$ be a spectral system on $\mathcal{X}$ possessing the projection property. Assume that $\tilde{\sigma}(A) = \sigma(A)$ for every $A \in \mathcal{L}(\mathcal{X})$. Then

$$\tilde{\sigma} \circ \tilde{\sigma} = \sigma_T \circ \sigma_T;$$

in other words,

$$\tilde{\sigma}(A) \circ \tilde{\sigma}(B) = \sigma_T(A) \circ \sigma_T(B)$$

for every $A, B \in \mathcal{L}(\mathcal{X})^{(n)}_{\text{com}}$. Moreover, $\tilde{\sigma}(A)$ and $\sigma_T(A)$ have identical rationally convex hulls, namely $\sigma_R(A)$, for all A $\subset \mathcal{L}(\mathcal{X})$.

APPLICATION 3.9. (RC) For $A \equiv (A_1, \ldots, A_n), B \equiv (B_1, \ldots, B_n) \subset \mathcal{L}(\mathcal{X})$, let

$$\mathcal{E}(T) := \sum_{i=1}^{n} A_i T B_i \quad (T \in \mathcal{L}(\mathcal{X}))$$

be the elementary operator associated with the n–tuples A and B. Then

$$\sigma(E) = \sigma_T(A) \circ \sigma_T(B).$$

To conclude this section we shall establish a functorial property of $M_{\tilde{\sigma}}$.

Section 4. The Projection Property for σ_T

We shall devote this section to a proof of the projection property for the Taylor spectrum. We shall present a simplification of F.-H. Vasilescu's proof, making more evident the use of nilpotent operators of index 2 and pointing out the relevance of Voiculescu's Theorem in this context.

Vasilescu's proof, which basically follows the outline of J. Bunce's proof of the projection property for the left spectrum, makes implicit use of Voiculescu's Theorem. Our proof shows explicitly the role played by that theorem.

We shall begin with a series of lemmas.

LEMMA 4.1. Let T be an operator on a Hilbert space $\mathcal{H}$, and assume that $T^2 = 0$.

The following statements are equivalent:

(i) $R(T) = N(T)$.

(ii) $R_T := T + T^*$ is invertible.

(iii) $\Box_T := T^*T + TT^*$ is invertible.

$\Box$ is called the Laplacian of T. When (i) holds, we say that T is exact.

Proof. (i) $\Rightarrow$ (ii) Let $y \in \mathcal{H}$. Since $R(T) = N(T)$, we can write $y = Tx + T^*z$, where $x \perp N(T)$ and $z \perp N(T^*)$. Then $y = (T + T^*)(x + z)$, since $x \in R(T^*)$ and $z \in R(T)$' Thus $T + T^*$ is onto, i.e., $T + T^*$ is invertible.

(ii) $\Rightarrow$ (iii): $R_T^2 = (T + T^*)^2 = T^*T + TT^* = \Box_T$.

(iii) $\Rightarrow$ (i): Let $x \in N(T)$. Then $u_T x = T^*\ Tx + TT^*\mathbf{x} = TT^*x$, so that $x = a_T^{-1}T^*x$.

Since $\Box_T$ commutes with T^*, we have $x = T^*a_T^{-1}x \in R(T)$. Therefore $N(T) \subseteq R(T)$.

The other inclusion follows from the condition "$T^2 = 0$."

COROLLARY 4.2. Let A $\subset \mathcal{L}(\mathcal{H})$. Then $\sigma_T(A, \mathcal{H})$ is a closed subset of $\mathbf{C}^n$.

Proof. The map A $\Leftrightarrow R_{D_A}$ is real analytic in the norm topology. Since the set of invertible operators in $\mathcal{L}(\Lambda(\mathcal{H}))$ is open, it follows that

$$\{\lambda \in \mathbb{C}^n : R(D_{A-\lambda}) = N(D_{A-\lambda}\}$$

is an open subset of $\mathbb{C}^n$.

The following result contains a new formula for the orthogonal projection onto the range of T, in the case when T is exact.

LEMMA 4.3. Let $T \in \mathcal{L}(\mathcal{H})$ with $T^2 = 0$. Then

(i) $\overline{R(T)} = N(T) \Leftrightarrow N(\Box_T) = 0$.

(ii) R_T invertible $\Rightarrow P_{R(T)} = TR_T^{-1}$. (Here $P_{R(T)}$ is the orthogonal projection of $\mathcal{H}$ onto $R(T)$.)

LEMMA 4.4. Let $T \in \mathcal{L}(\mathcal{X})$ with $T^2 = 0$. If there exists $S \in \mathcal{L}(\mathcal{X})$ such that $TS + ST = I$, then $N(T) = R(T)$.

LEMIA 4.5. (J.L. Taylor) Let A $\subset \mathcal{L}(\mathcal{X})$. Then $\sigma_T(A, \mathcal{X}) \subseteq \sigma'(A)$.

Proof. Assume A is invertible relative to (A)', i.e., there exists $B \subset (A)'$ such that $A \circ B = I$. Let

$$S := \sum_{i=1}^{n} B_i E_i^*.$$

Then S satisfies the equation

$$\begin{aligned} D_A S + S D_A &= \sum_{i,j} \left(A_i B_j E_i E_j^* + B_j A_i E_j^* E_i \right) \\ &= \sum_{i,j} A_i B_j \left(E_i E_j^* + E_j^* E_i \right) \\ &= \sum_{i=1}^{n} A_i B_i = I. \end{aligned}$$

By the previous lemma, $N(D_A) = R(D_A)$, so that A is Taylor invertible.

In the next lemma, we establish that for $(A, B) \subset \mathcal{L}(\mathcal{H})$, the invertibility of A implies the invertibility of (A, B); as a result, $\sigma_T((A, B), \mathcal{H}) \subseteq \sigma_T(A, \mathcal{H}) \times \sigma_T(B, \mathcal{H})$.

LEMMA 4.6. Let $A, B \subset \mathcal{L}(\mathcal{H})$ and assume that $N(D_A) = R(D_A)$ and that $(A, B) \subset \mathcal{L}(\mathcal{H})$. Then $N\left(D_{(A,B)}\right) = R(D(A, B)$.

Proof. Observe that $D_{(A,B)} = D_A + \widetilde{D}_B$, where $\widetilde{D}_B := \sum_{j=1}^m B_j E_{n+j}$.

We shall show that if $\eta \in \Lambda_{n+m}(\mathcal{H})$ and $(D_A + \widetilde{D}_B)\eta = 0$ then $\eta = (D_A + \tilde{D}_B)\xi$, for some $\xi \in \Lambda_{n+m}(\mathcal{H})$. (We shall actually give a formula for ξ in terms of η). It suffices to assume that η is homogeneous of degree $p (0 \le p \le n+m)$, i.e., η is a linear combination of forms of degree p in $e_1, \ldots, e_{n+m}$; this is because $D_A + \widetilde{D}_B$ maps homogeneous p–forms to homogeneous $(p+1)$–forms. Now write $\eta = \eta_0 + \cdots + \eta_p$, where η_i has degree i in $e_{n+1}, \ldots, e_{n+m}$ We get:

$$\begin{cases} D_A \eta_O &= 0 \\ D_A \eta_1 + \widetilde{D}_B \eta_O &= 0 \\ \cdots & . \quad \cdots \\ D_A \eta_p + \widetilde{D}_B \eta_{p-1} &= 0 \\ \widetilde{D}_B \eta_p &= 0. \end{cases}$$

Standard manipulations with these equations and the well-known properties of kernel and range for operators on Hilbert space lead to the following formula for the components of ξ:

$$(*)\ \xi_{p-1} = \sum_{k=0}^{p-1} (-1)^k R_A^{-1} (\tilde{D}_B R_A^{-1})^k \eta_{p-1-k}.$$

Then

$$\left(D_A + \tilde{D}_B\right)(\xi_0 + \cdots + \xi_{p-1}) = \eta,$$

as desired.

To prove the remaining half of the projection property (i.e., that $\sigma_T(A, \mathcal{H}) \subseteq P_A \sigma_T((A, B), \mathcal{H})$, one must show that whenever $R\,(D_A) \neq N\,(D_A)$ then $R\left(D_{(A,B-\mu)}\right) \neq N(D(A, B-\mu)$ for at least one $\mu \in \mathbb{C}^m$.

Suppose for simplicity that $\mathcal{H}$ is finite dimensional, $A = (A_1)$ and $B = (B_1)$. If A is not invertible then $N(A) \neq 0$; since B maps $N(A)$ into $N(A)$, we can look at $B|_{N(A)}$ and find an eigenvalue $\mu \in \mathbb{C}$. Then $(0, \mu)$ is a joint eigenvalue for (A,B), which shows that $(0, \mu) \in \sigma_T((A, B), \mathcal{H})$.

The general case is of course much more complicated than this simple example, but the idea of the proof is basically the same. One finds, however, that there are two distinctive cases whenever A is not invertible on $\mathcal{H}$:

either $\overline{R(D_A)} \neq N(D_A)$ or $\overline{R(D_A)} = N(D_A)$.

In the former case, one can imitate the above construction with $N(A)$ replaced by $N(D_A) \ominus R(D_A)$. In the latter case, however, there is no enough room between $N(D_A)$ and $R(D_A)$!

It is to handle this case that one needs Voiculescu's Theorem.

First, we need a definition.

DEFINITION 4.7. Let $A, \tilde{A} \subset \mathcal{L}(\mathcal{H})$. We say that A is approximately unitarily equivalent to $\tilde{A}$ (in symbols, $A \simeq_a \tilde{A}$) if there exists a sequence $\{U_k\}_{k=1}^{\infty}$ of unitary operators on $\mathcal{H}$ such that $U_k A U_k^* := (U_k A_1 U_k^*, \cdots, U_k A_n U_k^*) \longrightarrow \tilde{A}$, as $k \longrightarrow \infty$ (naturally, this implies that $\widetilde{A}\mathcal{L}(\mathcal{H})$.

$\simeq a$ is an equivalence relation on commuting n–tuples of operators on $\mathcal{H}$, which preserves all spectral properties, as the reader can easily verify. In particular, $\sigma_T(A) = \sigma_T(\tilde{A})$ whenever $A \simeq_a \tilde{A}$.

The next lemma, which slightly generalizes the Gap Theorem proved by RC and L. Fialkow, states that if A is not invertible on $\mathcal{H}$, one can always assume $\overline{R(D_A)} \neq N(D_A)$, up to approximate unitary equivalence. (For $n = 1$, this fact was first proved by C. Apostol).

LEMMA 4.8. Let $(A, B) \subset \mathcal{L}(\mathcal{H})$. If $R(D_A) \neq \overline{R(D_A)} = N(D_A)$ then there exists $(\tilde{A}, \widetilde{B}) \simeq a(A, B)$ such that $\overline{R(D_{\tilde{A}})} \neq N(D_{\tilde{A}})$.

Proof. Recall that

$$N(\Box_A) = N(D_A) \cap N(D_A^*) = \{0\},$$

so that $\Box_A$ cannot have closed range (being a self-adjoint operator). Therefore, there exists a projection $Q \in \mathcal{L}(\Lambda_n(\mathcal{H}))$ such that Q is not compact but $\Box_A Q$ is a compact operator.

Let $\mathcal{A}$ be the C*-algebra generated by $A_1, \ldots, A_n, B_1, \ldots, B_m, I$ and $\{Q_{ij}\}_{i,j=1}^{2^n} \cdot$ and let $p : \pi(\mathcal{A}) \longrightarrow \mathcal{L}(\mathcal{H})$ be a faithful *-representation of $\pi(\mathcal{A})$ on a separable Hilbert space $\mathcal{K}$, where π is the Calkin map. By Voiculescu's Theorem,

$$\mathrm{id}_{\mathcal{A}} \simeq_a \mathrm{id}_{\mathcal{A}} \oplus (p \circ \pi)$$

so that $(A,B) \simeq a\,(A',B') \subset \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. Let

$$Q' = 0 \oplus (p \circ \pi)(Q)$$

Then $\Box_{A'} Q' = 0$ and $Q' \neq 0$, so that $\overline{R\,(D_{A'})} \neq N\,(D_{A'})$. Since $\mathcal{H} \oplus \mathcal{H}$ is isometrically isomorphic to $\mathcal{H}$, we obtain $(\widetilde{A}, \widetilde{B}) \subset \mathcal{L}(\mathcal{H}), (\widetilde{A}, \widetilde{B}) \simeq_a (A,B)$ and $N\left(D_{\tilde{A}}\right) \neq \overline{R\,(D_{\tilde{A}})}$.

THEOREM 4.9 (J.L. Taylor). Let $(A, B) \subset \mathcal{L}(\mathcal{H})$. Then

$$P_A(\sigma_T((A, B), \mathcal{H}) = \sigma_T(A, \mathcal{H}).$$

PROOF OF THEOREM 4.9 (Conclusion). Since $R(Q(\widetilde{B} - \mu)Q) \neq R(Q)$, Lemma 4.12 (iii) says that $R(C) \neq R(P)$, where $P = P_{N(D_{A-\lambda}}$ and $C = PD_{A-\lambda} + P(\tilde{B} - \mu)P$' By Lemma 4.11, this then implies that $N\,(D_{A-\lambda}) \neq R\,(D_{A-\lambda}) + (\widetilde{B} - \mu)N\,(D_{A-\lambda})$, and by Lemma 4.10, we must therefore have $N\,(D_{A-\lambda,B-\mu}) \neq R_{A-\lambda,B-\mu})$, so that $(\lambda, \mu) \in \sigma_T((A, B), \mathcal{H})$. This completes the proof.

Section 5. The Analytic Functional Calculus for σ_T

J.L. Taylor developed the analytic functional calculus for σ_T. He based the construction on a Cauchy-Weil integral formula in homology. Taylor later simplified the construction, but it was F.-H. Vasilescu who found a much simpler version (for the case of n–tuples of elements in a C*-algebra). We shall present here an outline of Vasilescu's construction of the functional calculus.

The main problem is to make sense out of $f(a)$, where a is a commuting n–tuple and f is analytic in a neighborhood of $\sigma_T(a)$.

Of course, if f is analytic in a neighborhood of $\sigma_{\mathcal{B}}(a)$, where $\mathcal{B}$ is some commutative Banach algebra containing a, $f(a)$ should agree with Waelbroeck's and Arens and Calderón's "$f(a)$." In particular, if f is analytic in a polydisc D containing $\sigma_T(a)$, then $f(a)$ must be given by the formula

$$f(a) = \frac{1}{(2\pi i)^n} \int_{\partial D} f(w) \prod_{i=1}^{n} \frac{1}{w_i - a_i} dw_i,$$

where ∂D denotes the distinguished boundary of D.

We will discuss Vasilescu's construction in the case of Hilbert spaces. By spectral permanence, there is no loss of generality in doing so. We first need to recall some facts about the Martinelli kernel.

DEFINITION 5.1. Let Ω be a bounded and open subset of $\mathbb{C}^n$ and assume that the boundary of Ω, $\partial\Omega$, is C^1. Let $f \in C^1(\Omega)$. We define

$$\partial f := \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f}{\partial z_n} dz_n$$

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}_1}\ d\bar{z}_1 + \cdots + \frac{\partial f}{\partial \bar{z}_n} d\bar{z}_n$$

and

$$df := \partial f + \bar{\partial} f$$

More generally, if ξ is a $C^\infty(\Omega)$-valued (k,l)-form in $dz_1 \dots, dz_n \cdot d\bar{z}_1 \dots . d\bar{z}_n$, say

$$\xi = f\, dz_{i_1} \wedge \cdots \wedge dz_{i_k} \wedge d\bar{z}_j \wedge \cdots \wedge d\bar{z}_j{}^{\ell},$$

we let

$$\partial\xi := \partial f \wedge dz_{i_1} \wedge \cdots \wedge dz_{i_\mathbf{k}} \wedge d\bar{z}_{j_1 \wedge \cdots \wedge} d\bar{z}_{j_\ell}$$

and similarly for $\bar{\partial}$ and d. It follows that each of $\partial, \bar{\partial}$ and d gives rise to operators on $C^\infty(\Omega) \otimes \Lambda[dz, d\bar{z}]$ such that $\partial^2 = \bar{\partial}^2 = d^2 = 0$.

The Koszul complex associated with d is called the de Rham complex over Ω; the one associated with $\bar{\partial}$ is the $\bar{\partial}$-complex. For us, the $\bar{\partial}$-complex will be the restriction of the $\bar{\partial}$-complex to $C^\infty(\Omega)\Lambda[d\bar{z}]$, that is $\bar{\partial}$ acting on $(0,\ell)$ forms. (This complex is sometimes called the Dolbeault complex.) Also, $H(\Omega) = \left\{ f \in \Lambda_0\left(C^\infty(\Omega)\right) : \bar{\partial} f = 0 \right\}$.

DEFINITION 5.2. For $z \neq 0, z \in \mathbb{C}^n$, let

$$M(z) := (n-1)! \sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{z}_j}{\|z\|^{2n}} \bigwedge_{\substack{1 \leq k \leq n \\ k \neq j}} d\bar{z}_k,$$

where $\|z\|^2 := \sum_{i=1}^{n} |z_i|^2$. M is the Martinelli kernel.

THEOREM 5.3. (Bochner-Martinelli formula) ([74, Theorem 1.14]). Let $f \in C^1(\bar{\Omega})$.

Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial\Omega} f(w) M(w-z) \wedge dw - \frac{1}{(2\pi i)^n} \int_{\Omega} \bar{\partial} f(w) \wedge M(w-z) \wedge dw \quad (z \in \Omega)$$

(generalization of Cauchy's formula).

REMARKS 5.5. (i) M is an (n-1)-differential form in $d\bar{z}_1, \ldots, d\bar{z}_n$ such that $\bar{\partial} M = 0$.

(ii) If $n = 1, M(z) = \frac{1}{z}$, the Cauchy kernel.

(iii) If $n \geq 2$, M is not analytic (in the sense that its C^∞-coefficients are not analytic functions) and, therefore, M does not "create" analytic functions (as the Cauchy kernel does), but it nevertheless reproduces analytic functions.

We now give the definition of the Martinelli kernel, which generalizes the resolvent $(z - A)^{-1}$, used when $n = 1$.

DEFINITION 5.9. Let $\beta \subset A(\Omega, \mathcal{L}(\mathcal{H}))$ and let $z \notin \sigma_T^\Omega(\beta, \mathcal{H})$. We let

$$M(\beta)(z) := R_{\beta(z)}^{-1} (\bar{\partial}_z R_{\beta(z)}^{-1})^{n-1} E|_{\mathcal{H} \otimes \Lambda^0}.$$

$M(\beta)$ is the Martinelli kernel associated with β.

THEOREM 5.18 (Taylor). Let $A \subset \mathcal{L}(\mathcal{H})$ and let $\Omega \supseteq \sigma_T(A, \mathcal{H})$. Let Ω' be such that $\sigma_T(A, \mathcal{H}) \subseteq \Omega' \subseteq \bar{\Omega} \subseteq \Omega$ and Ω' has C^1-boundary. Then the map

$$H(\Omega) \to \mathcal{L}(\mathcal{H})$$

$$f \mapsto f(A) := v_\beta(f)$$

(where $\beta(z) := z - A$) is a unital continuous homomorphism such that:

(i) $z_i(A) = A_i \quad (i = 1, \ldots, n)$

(ii) $f(A) \in (A)''$

(iii) If $f = 0$ on a neighborhood of $\sigma_T(A, \mathcal{H})$ then $f(A) = 0$.

(iv) For every relatively compact open subset Φ of Ω containing $\sigma_T(A, \mathcal{H})$ there exists a constant $C_\Phi > 0$ such that

$$\|f(A)\| \leq C_\Phi \sup\{|f(z)| : z \in \Phi\},$$

for all $f \in H(\Omega)$.

We shall now state the uniqueness of the analytic functional calculus for σ_T.

THEOREM 5.20. (M. Putinar) Let $A \subset \mathcal{L}(\mathcal{H})$ and let $\Omega \supseteq \sigma_T(A, \mathcal{H})$. Then there exists a unique continuous unital $H(\Omega)$–functional calculus extending the polynomial calculus and satisfying the spectral mapping theorem.

COROLLARY 5.21. (W. Zame) For $\mathcal{B}$ a commutative unital Banach algebra with identity, the Shilov-Arens-Calderón-Waelbroeck functional calculus is unique (subject to the conditions (i), (ii) and (iii) of Theorem 1.15).

Proof. $\sigma_{\mathcal{B}}(a) = \sigma_T\,(L_a, \mathcal{B})$, for all $a \subset \mathcal{B}$.

REMARK 5.22. The importance of Putinar's uniqueness result lies in the fact that no matter which construction we use for $f(A)$ we are bound to obtain the same element of $\mathcal{L}(\mathcal{H})$. For instance, if $f \in H(\sigma_R(A))$ (recall that $\sigma_R(A)$ is the rational spectrum of A), then $f(A)$ has a priori three meanings: one in the sense of Arens-Calderón-Waelbroeck, one in the sense of Taylor, and one in the sense of Vasilescu. As it turns out, they all agree and the value $f(A)$ can the be computed in whichever way.

We shall now give some applications of the results in this section.

APPLICATION 5.24. (The Shilov Idempotent Theorem for σ_T) ([105, Theorem 4.9]))

Let $A \subset \mathcal{L}(\mathcal{H})$ and assume that

$$\sigma_T(A, \mathcal{H}) = K_1 \dot{U} K_2,$$

where K_1 and K_2 are two nonempty disjoint compact subsets of $\sigma_T(A, \mathcal{H})$. Then there exist subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{H}$ such that

$$\mathcal{M}_1 \cap \mathcal{M}_2 = (0),$$

$$\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{H},$$

$$A\mathcal{M}_i \subseteq \mathcal{M}_i \quad (i = 1, 2)$$

and

$$\sigma_T\left(A|_{\mathcal{M}_i}, \mathcal{M}_i\right) = K_i \quad (i = 1, 2).$$

APPLICATION 5.27. (Fialkow) The Harte spectrum σ_H does not carry an analytic functional calculus. Consider the following pair of operators on $\mathcal{H} = \ell^2(\mathbb{N}_0) \otimes \ell^2(\mathbb{N}_0)$, where $\ell^2(\mathbb{N}_0)$denotes the Hilbert space of square summable sequences:

$$A_1 = \begin{bmatrix} U_+^* & & & \\ & U_+^* & & \\ & & U_+^* & \\ & & & \ddots \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & & & \\ U_+^* & 0 & & \\ & I & 0 & \\ & & & \ddots \end{bmatrix}$$

where $U_+$is the unilateral shift on $\ell^2(\mathbb{N}_0)$. Suppose $P \in \mathcal{L}(\mathcal{H})$, $P^2 = P$ and $PA_i = A_iP \quad (i = 1, 2)$. A matricial computation shows that P must be 0 or I. Therefore $\sigma_T(A, \mathcal{H})$ is connected (by Application 5.24); indeed, $\quad \sigma_T(A, H) = \overline{\mathbb{D}}^2$. However,

$$\sigma_H(A, \mathcal{H}) = \{(0, 0)\} \cup (\mathbb{T} \times \mathbb{D}) \cup (\mathbb{D} \times \mathbb{T}),$$

so that $(0, 0)$ is isolated in $\sigma_H(A, \mathcal{H})$. Therefore, $(0, 0)$ cannot be excised by means of an analytic functional calculus.

Let $\mathcal{A}$ be a unital C*-algebra, let L_a, R_b denote the *left* and *right* multiplication operators induced by $a, b \in \mathcal{A}$ (i.e., $L_a(x) := ax,\ R_b(x) := xb,\ x \in \mathcal{A}$), and set $M_{a,b} := L_a R_b$.

An ideal $\mathcal{I}$ in $\mathcal{A}$ is said to be *prime* if, whenever $\mathcal{I}_1, \mathcal{I}_2$ are ideals in $\mathcal{A}$ such that $\mathcal{I}_1 \mathcal{I}_2 \subseteq \mathcal{I}$, it follows that $\mathcal{I}_1 \subseteq \mathcal{I}$ or $\mathcal{I}_2 \subseteq \mathcal{I}$; $\mathcal{A}$ is said to be prime if (0) is a prime ideal.

THEOREM

(M. Mathieu) Let $\mathcal{A}$ be a unital C-algebra. The following statements are equivalent.*

(i) $\mathcal{A}$ is prime.

(ii) $\|M_{a,b}\| = \|a\| \, \|b\|$ for all $a, b \in \mathcal{A}$.

(iii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$ for all $a, b \in \mathcal{A}$.

It turns out that there is a more foundational result at the heart of Mathieu's theorem.

Theorem

(RC, C. Hernández G.) Let $\mathcal{A}$ be a unital C-algebra. The following statements are equivalent.*

(i) $\mathcal{A}$ is prime.

(ii) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \quad (a, b \in \mathcal{A})$.

Corollary

Let $\mathcal{A}$ be a unital C-algebra, and let $a, b \in \mathcal{A}$. The following statements are equivalent.*

(i) $\sigma_T((L_a, R_b), \mathcal{A}) = \sigma(a) \times \sigma(b) \ (a, b \in \mathcal{A})$.

(ii) $\sigma(M_{a,b}) = \sigma(a)\sigma(b)$.

SECTION 8. M_z ON REINHARDT DOMAINS

DEFINITION 8.1. Let Ω be a domain in $\mathbb{C}^n$ Ω is said to be Reinhardt if it is invariant under the action of the n–torus $\mathbb{T}^n$, i.e., if whenever $z \in \Omega$ and $e^{i\theta} \in \mathbb{T}^n$, it follows that $e^{i\theta_z} := \left(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n\right) \in \Omega$. Reinhardt domains are important in several complex variables because of their intrinsic relation to power series. For instance, pseudoconvex Reinhardt domains containing the origin are the natural domains of convergence of power series. Moreover, if Ω is a complete Reinhardt domain containing the origin, then $\mathtt{int.}\,\hat{\Omega} = M_\Omega = \tilde{\Omega}$, where $\tilde{\Omega}$ is the pseudoconvex hull of Ω. In what follows, we shall assume that $0 \in \Omega$.

OBSERVATION 8.2. For $\alpha \in \mathbb{Z}_+^n$ let $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$; let us also denote by z^α the function whose value at z is z^α. If $c_\alpha = \|z^\alpha\|_A^2 (\Omega)$ then $\left\{ \frac{z^\alpha}{c_\alpha} \right\}_{\alpha \in \mathbb{Z}_+^n}$ is an orthonormal basis for $A^2(\Omega)$. Moreover,

$$M_{z_i} \left[\frac{z^\alpha}{c_\alpha} \right] = \frac{c_{\alpha+\epsilon_i}}{c_\alpha} \left[\frac{z^{\alpha+\epsilon_i}}{c_{\alpha+\epsilon_i}} \right] \quad (\text{all} \quad \alpha \in \mathbb{Z}_+^n)$$

where $\epsilon_i := (0, \dots, 1 \dots, 0)$, so that M_{z_i} is a weighted shift with weights

$$w_i(\alpha) = \frac{c_{\alpha+\epsilon_i}}{c_\alpha}, \alpha \in \mathbb{Z}_+^n, \quad i = 1, \dots, n.$$

EXAMPLE 8.3. Let $0 < p, q < \infty$ and let

$$\Omega_{p,q} := \left\{ z \in \mathbb{C}^2 : |z_1|^p + |z_2|^q < 1 \right\}.$$

$\Omega_{p,q}$ is pseudoconvex, because

$$\log |\Omega_{p,q}|| := \{(\log |z_1| , \log |z_2|) : (z_1, z_2) \in \Omega_{p,q} \text{ and } z_1 \neq 0, z_2 \neq 0\}$$

is a convex set; when $p, q \geq 2$, $\Omega_{p,q}$ is Levi pseudoconvex, and $\Omega_{p,q}$ is strongly pseudoconvex if and only if $p = q = 2$. For $\Omega_{p,q}$, the coefficients c_α can be calculated explicitly:

$$c_\alpha = \frac{2\pi^2}{p(\alpha_2 + 1)} \cdot B(\frac{2\alpha_1 + 2}{p}, \frac{2\alpha_2 + 2}{q} + 1),$$

where B is the Beta function.

For subnormal 2-variable weighted shifts, RC-K. Yan gave in 1995 a complete description of the spectral picture, by exploiting the groupoid machinery in Muhly-Renault and RC-Muhly, and the presence of the Berger measure, which was used to analyze the asymptotic behavior of sequences of weights.

Notation

μ : compactly supported finite positive Borel measure on $\mathbb{C}^n$ ($n \geqslant 1$)

$P^2(\mu)$: norm closure in $L^2(\mu)$ of $\mathbb{C}[z_1, \cdots, z_n]$

$M_{\mathbf{z}} \equiv M_{\mathbf{z}}^{(\mu)} := (M_{z_1}^{(\mu)}, \cdots, M_{z_n}^{(\mu)})$: multiplication operators acting on $P^2(\mu)$

$M_{\mathbf{z}}$ on $P^2(\mu)$ is the universal model for cyclic subnormal n-tuples

Definition

(i) $E \subseteq \mathbb{C}^n$ is Reinhardt if for every $\mathbf{z} \in E$ and every $\theta \in \mathbb{R}^n$,

$e^{i\theta}\mathbf{z} := (e^{i\theta_1} z_1, \cdots, e^{i\theta_n} z_n) \in E$.

(ii) μ is Reinhardt if $\mu(e^{i\theta}E) = \mu(E)$ for every Borel subset $E \subseteq \mathbb{C}^n$ and every $\theta \in \mathbb{R}^n$

(supp μ is always a Reinhardt set).

Theorem

(RC-K. Yan, 1995) *Let μ be a Reinhardt measure on $\mathbb{C}^2$, and let $C := \log \left|\widehat{K}\right|$. Assume that $\partial\widehat{K} \bigcap (z_1 z_2 = 0)$ contains no 1-dimensional open disks. Then*

(i) $\partial\widehat{K} \supseteq \sigma_\ell(M_{\mathbf{z}}, P^2(\mu)) = \sigma_{\ell_e}(M_{\mathbf{z}}, P^2(\mu)) \supseteq (\exp(\partial^0 C \times \mathbb{T}^2))^-$

(i') If, in addition, μ is well-behaved, then $\sigma_{\ell_e}(M_{\mathbf{z}}, P^2(\mu)) = (\exp(\partial^0 C \times \mathbb{T}^2))^-$.

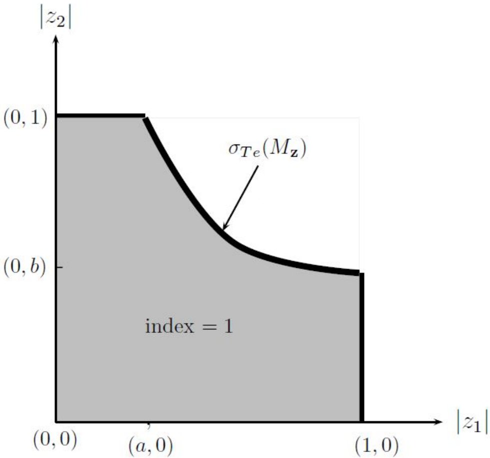
(ii) $\sigma_T(M_{\mathbf{z}}, P^2(\mu)) = \sigma_r(M_{\mathbf{z}}, P^2(\mu)) = \widehat{K}$

(iii) $\sigma_{Te}(M_{\mathbf{z}}, P^2(\mu)) = \sigma_{r_e}(M_{\mathbf{z}}, P^2(\mu)) = \partial\widehat{K}$

(iv) $int.\widehat{K} \subseteq \sigma_p(M_{\mathbf{z}}, P^2(\mu)) = b.p.e.(\mu) \subseteq \widehat{K}$

(v) $index(M_{\mathbf{z}} - \lambda) = \begin{cases} 1 & \text{if } \lambda \in int.(\widehat{K}) \\ 0 & \text{if } \lambda \notin int.(\widehat{K}) \end{cases}$

(vi) $kerD^1_{M_{\mathbf{z}}-\lambda} = ranD^0_{M_{\mathbf{z}}-\lambda}$ for all $\lambda \in int.(\widehat{K})$.



Spectral picture of a typical subnormal 2-variable weighted shift

We shall conclude this section with an example of a domain that has turned out to be quite an interesting one.

EXAMPLE 8.5. Let $0 < \delta, \epsilon < 1$ and let

$$\Omega_{\delta,\epsilon} := \{z \in \mathbb{C}^2 : (|z_1| < \delta \text{ and } |z_2| < 1) \text{ or } (|z_1| < 1 \text{ and } |z_2| < \epsilon)\}$$

$\Omega_{\delta,\epsilon}$ is an L-shaped domain, and

$$\hat{\Omega}_{\delta,\epsilon} = \{z \in \mathbb{C}^2 : |z_1| \leq |z_2| \leq 1 \text{ and } |z_1|^{\log \epsilon} |z_2|^{\log \delta} \leq \delta^{\log \epsilon}\}$$

M_z acting on $A^2(\Omega_\delta, \epsilon)$ is a bivariate weighted shift whose weights are associated with the coefficients

$$c_\alpha = \pi \left[\frac{\delta_1^{2\alpha_1+2} + \epsilon^{2\alpha_2+2} - \delta_1^{2\alpha_1+2} 2\alpha_2 + 2}{(\alpha_1 + 1)(\alpha_2 + 1)} \right]^{1/2} .$$

Using the theory of groupoids, RC and P. Muhly studied the ideal structure of the C^*-algebra generated by multivariable weighted shifts. As an application, they obtained the following result.

THEOREM 8.6. (RC - P. Muhly) Let $\Omega_{\delta,\epsilon}$ be as above. Then $C^*(M_z)$ is type I if and only if $\frac{\log \delta}{\log \epsilon}$ is a rational number.

REMARK 8.7. $\Omega_{\delta,\epsilon}$ is the first known example of a domain for which the Bergman C^*-algebra is not type I. Previous works indicated that $C^*(M_z)$ might always be type I.