Can quasi-static evolutions of perfect plasticity be derived from brittle damage evolutions?

Élise Bonhomme

Laboratoire de mathématiques d'Orsay, Université Paris-Saclay

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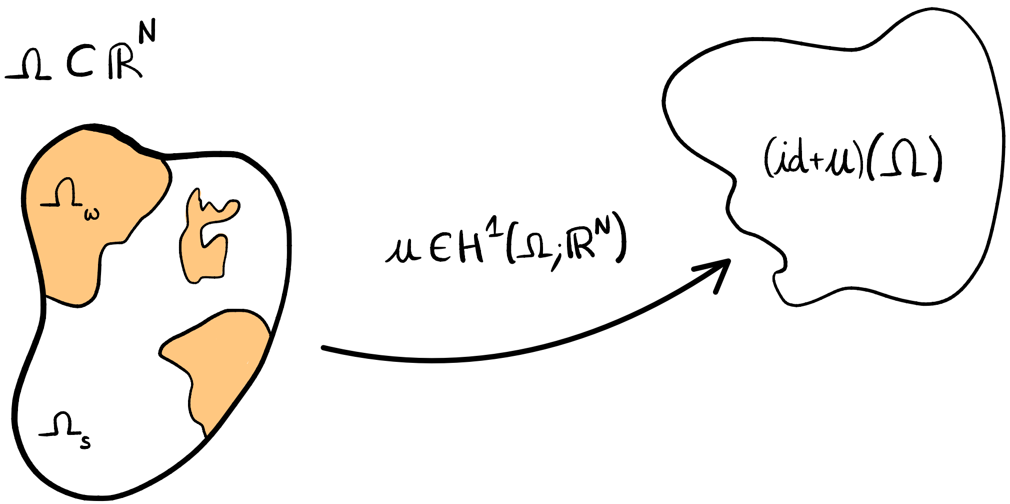
General context of Brittle Damage



$$A_\omega \leq A_s$$

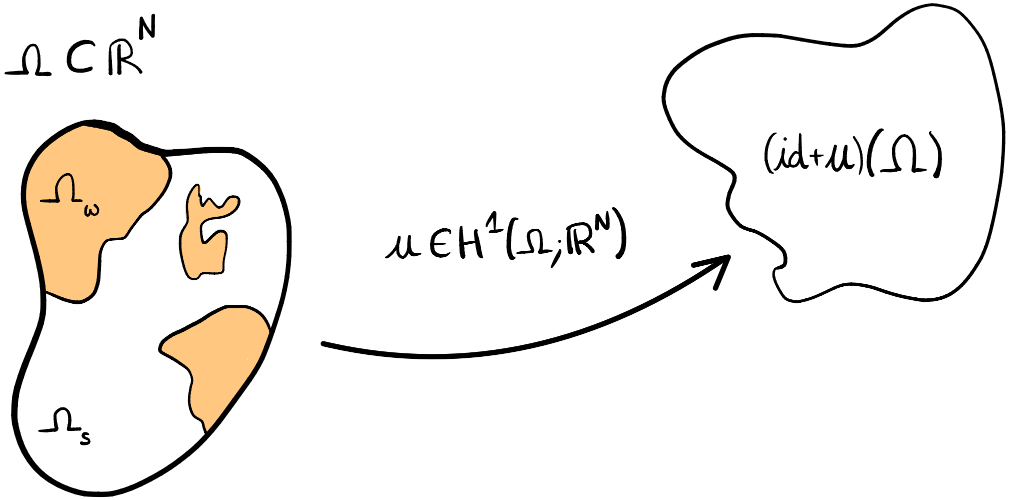
$$\chi = \mathbb{1}_{\Omega_\omega} \in L^\infty(\Omega;\{0,1\})$$

$$A_\chi = \chi A_\omega + (1-\chi) A_s$$

[Francfort-Marigo, 1993]

$$\mathcal{E}(u,\chi) = \frac{1}{2}\int_\Omega A_\chi eu : eu \, dx + K\int_\Omega \chi \, dx$$

Concentration and elastic degeneracy of weak material



$$\varepsilon A_\omega \leq A_s$$

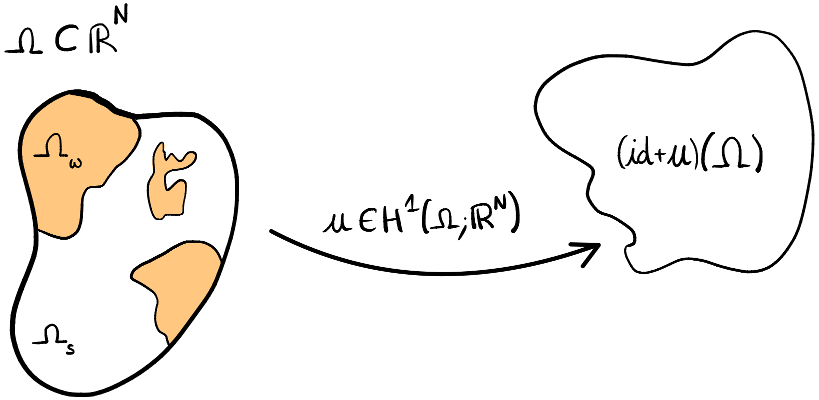
$$\chi = \mathbb{1}_{\Omega_\omega} \in L^\infty(\Omega;\{0,1\})$$

$$A^\varepsilon_\chi = \chi \varepsilon A_\omega + (1-\chi) A_s$$

Scaling Law
[Babadjian-Iurlano-Rindler]

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega A^\varepsilon_\chi eu : eu \, dx + \frac{\kappa}{\varepsilon}\int_\Omega \chi \, dx$$

Concentration and elastic degeneracy of weak material



$\varepsilon A_\omega \leq A_s$

$\chi = \mathbb{1}_{\Omega_\omega} \in L^\infty(\Omega;\{0,1\})$

$A^\varepsilon_\chi = \chi \varepsilon A_\omega + (1-\chi) A_s$

Scaling Law
[Babadjian-Iurlano-Rindler]

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega A^\varepsilon_\chi eu : eu \, dx + \frac{K}{\varepsilon}\int_\Omega \chi \, dx$$

$\Gamma\text{-cv}$ $L^1 \times L^1$

$\varepsilon \searrow 0$

(Hencky)
perfect plasticity

$$u \in BD(\Omega) \longmapsto \int_\Omega \frac{1}{2} A_s e : e \, dx + \int_\Omega I^*_K \left(\frac{dp}{d|p|}\right) d|p| \qquad \text{if } \chi = 0 \text{ a.e.}$$

closed convex set of plasticity

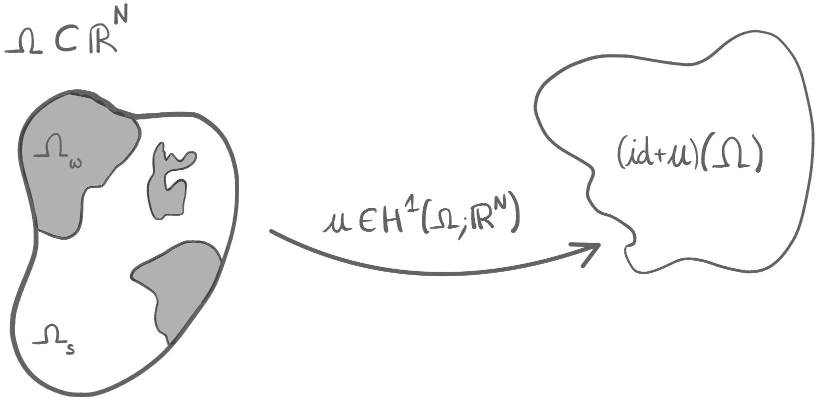
where $\quad Eu = e + p$

elastic strain

plastic (permanent) strain

Concentration and elastic degeneracy of weak material

Scaling Law
[Babadjian-Iurlano-Rindler]



$\varepsilon A_\omega \leq A_s$

$\chi = \mathbb{1}_{\Omega_\omega} \in L^\infty(\Omega;\{0,1\})$

$A^\varepsilon_\chi = \chi\,\varepsilon A_\omega + (1-\chi) A_s$

$$\mathcal{E}_\varepsilon(u,\chi) = \frac{1}{2}\int_\Omega A^\varepsilon_\chi\, eu : eu \, dx + \frac{K}{\varepsilon}\int_\Omega \chi \, dx$$

Γ- cv $\quad \varepsilon \searrow 0$

(Hencky)
perfect plasticity

$$u \in BD(\Omega) \longmapsto \int_\Omega \frac{1}{2} A_s\, e:e \, dx + \int_\Omega I_K^*\left(\frac{dp}{d|p|}\right) d|p| \quad \text{if } \chi = 0 \text{ a.e.}$$

What about Evolution models?

Quasi-static **Brittle Damage** evolution
[Francfort-Garroni]

Scaling Law
[BIR]

coupled with

$\big\downarrow\ \varepsilon \searrow 0$

QS perfect plasticity evolution [Dal Maso-DeSimone-Mora]

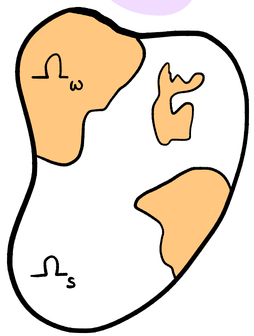
$$(u, e, p, \sigma)$$

?

QS Brittle Damage Evolution [F-G]

$0 = t_0 \quad t_1 \quad \cdots \quad t_i \quad \cdots \quad t_M = T$

t_{-1}



$A_\omega \leq A_s$

$\xrightarrow{u_0}$

t_0

$A_\omega \leq A_0 \leq A_s$

$$\mathcal{E}_0 = \inf_{u,\chi} \frac{1}{2}\int_\Omega A_\chi eu:eu \, dx + K\int_\Omega \chi \, dx \overset{H\text{-}S}{=} \min_{\substack{u,\Theta \\ A \in \mathcal{G}_{1-\Theta}(A_\omega; A_s)}} \int_\Omega \left(\frac{1}{2} A eu:eu + K(1-\Theta)\right) dx$$

$\boxed{u_0 \quad \Theta_0 \quad A_0}$

$\cdots$

$\xrightarrow{u_{i+1}}$

$$\mathcal{E}_{i+1} = K\int_\Omega (1-\Theta_i)\,dx + \underbrace{\inf_{u,\chi} \int_\Omega \left(\frac{1}{2}(\chi A_\omega + (1-\chi)A_i)\, eu:eu + K\chi\Theta_i\right) dx}_{\overset{H\text{-}S}{=} \min\limits_{\substack{u,\Theta \\ A \in \mathcal{G}_{1-\Theta}(A_\omega; A_i)}} \int_\Omega \left(\frac{1}{2} A eu:eu + K(1-\Theta)\Theta_i\right) dx}$$

t_{i+1}

$\boxed{u_{i+1} \quad \Theta_{i+1} \quad A_{i+1}}$

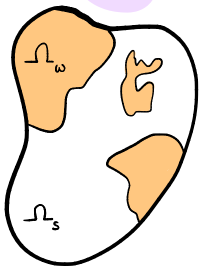
$A_\omega \leq A_{i+1} \leq A_i \leq \cdots A_0 \leq A_s$

QS Brittle Damage Evolution [F-G]

$0=t_0 \quad t_1 \ \cdots \ t_i \ \cdots \ t_M=T$

$$\mathcal{E}_0 = \min_{\substack{u,\Theta \\ A\in\mathcal{G}_{1-\Theta}(A_\omega;A_s)}} \int_\Omega \left(\tfrac{1}{2} A e u : e u + K(1-\Theta)\right) dx \qquad \boxed{u_0 \quad \Theta_0 \quad A_0}$$

t_{-1}



$A_\omega \le A_s$

$\xrightarrow{u_0}$ t_0

$A_\omega \le A_0 \le A_s$

$\cdots$ $\xrightarrow{u_{i+1}}$ t_{i+1}

$$\mathcal{E}_{i+1} = K\int_\Omega (1-\Theta_i)\,dx + \min_{\substack{u,\Theta \\ A\in\mathcal{G}_{1-\Theta}(A_\omega;A_i)}} \int_\Omega \left(\tfrac{1}{2} A\, eu : eu + K(1-\Theta)\Theta_i\right) dx \qquad \boxed{u_{i+1} \quad \Theta_{i+1} \quad A_{i+1}}$$

$A_\omega \le A_{i+1} \le A_i \le \ldots A_0 \le A_s$

Piecewise constant in time interpolation
+
"Helly "

$\longrightarrow$

$\sup_i |t_{i+1}-t_i| \xrightarrow[M]{} 0$

$$u : [0,T] \to H^1(\Omega) \qquad \left.\begin{array}{l} \Theta : [0,T] \to L^\infty(\Omega;[0,1]) \\ A : [0,T] \to L^\infty(\Omega) \end{array}\right\} \text{ in time}$$

$$\mathcal{E}(t) = \frac{1}{2}\int_\Omega A(t)\, eu(t) : eu(t)\, dx + K\int_\Omega (1-\Theta(t))\, dx$$

QS Brittle Damage Evolution [F-G] + Scaling Law [BIR]

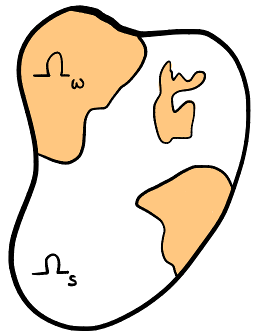
$0 = t_0 \quad t_1 \ \cdots \ t_i \ \cdots \ t_M = T$

$\varepsilon > 0$

$$\mathcal{E}_0^\varepsilon = \min_{\substack{u,\Theta \\ A \in \mathcal{G}_{1-\Theta}(\varepsilon A_\omega ; A_s)}} \int_\Omega \left(\frac{1}{2} A e u : e u + \frac{K}{\varepsilon} (1 - \Theta) \right) dx$$

$$\boxed{u_0^\varepsilon \quad \Theta_0^\varepsilon \quad A_0^\varepsilon}$$

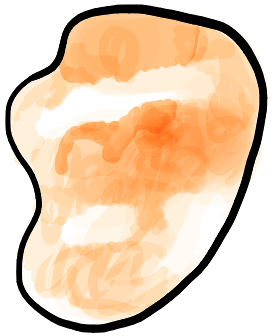
t_{-1}



$\varepsilon A_\omega \leq A_s$

u_0^ε

t_0



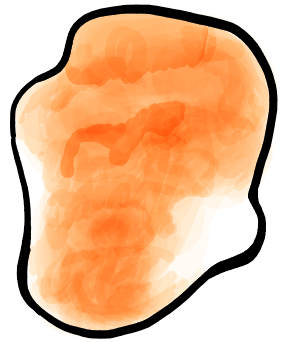
$\varepsilon A_\omega \leq A_0 \leq A_s$

$\cdots$

u_{i+1}^ε

$\varepsilon A_\omega \leq A_{i+1}^\varepsilon \leq A_i^\varepsilon \leq \dots A_0^\varepsilon \leq A_s$

t_{i+1}



$$\mathcal{E}_{i+1}^\varepsilon = \frac{K}{\varepsilon} \int_\Omega (1 - \Theta_i^\varepsilon) dx + \min_{\substack{u,\Theta \\ A \in \mathcal{G}_{1-\Theta}(\varepsilon A_\omega ; A_i^\varepsilon)}} \int_\Omega \left(\frac{1}{2} A e u : e u + \frac{K}{\varepsilon} (1 - \Theta) \Theta_i^\varepsilon \right) dx$$

$$\boxed{u_{i+1}^\varepsilon \quad \Theta_{i+1}^\varepsilon \quad A_{i+1}^\varepsilon}$$

QS Brittle Damage Evolution [F-G] + Scaling Law [BIR] in 1D

$T > 0$

Ω $[0, L]$

$0 < \varepsilon a_w \leq a_s \qquad \frac{K}{\varepsilon}$

Dirichlet boundary condition:

$\omega \in AC([0,T]; H^1(\mathbb{R}))$

$\xrightarrow{\text{[F-G]}}$

$$u_\varepsilon : [0,T] \longrightarrow H^1((0,L))$$

$$\Theta_\varepsilon : [0,T] \longrightarrow L^\infty((0,L);[0,1])$$

$$a_\varepsilon = \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_w} + \frac{\Theta_\varepsilon}{a_s}\right)^{-1} : [0,T] \longrightarrow L^\infty((0,L))$$

$\searrow$ in time

Dirichlet boundary condition: $\quad u_\varepsilon(t)_{|\{0,L\}} = \omega(t)_{|\{0,L\}}$

One-sided minimality: $\quad \forall v \in \omega(t) + H^1_0((0,L)), \ \forall \theta \in L^\infty((0,L);[0,1]),$

$$\frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx \leq \frac{1}{2}\int_0^L \left(\frac{1-\theta}{a_\varepsilon(t)} + \frac{\theta}{\varepsilon a_0}\right)^{-1} |v'|^2 dx + \frac{K}{\varepsilon}\int_0^L \theta\,\Theta_\varepsilon(t)\, dx$$

Energy balance: $\quad \mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx + \frac{K}{\varepsilon}\int_0^L (1-\Theta_\varepsilon(t))\,dx = \mathcal{E}_\varepsilon(0) + \int_0^t\int_0^L a_\varepsilon(s)\, u_\varepsilon'(s)\,(\dot{\omega})'(s)\, dx\, ds$

? Quid when $\varepsilon \searrow 0$?

QS Brittle Damage Evolution [F-G] + Scaling Law [BIR] in 1D

$T>0$

Ω: $[0, L]$

$0 < \varepsilon a_\omega \leq a_s \qquad \frac{K}{\varepsilon}$

Dirichlet boundary condition:

$\omega \in AC([0,T]; H^1(\mathbb{R}))$

[F-G] $\longrightarrow$

$$u_\varepsilon : [0,T] \longrightarrow H^1((0,L))$$

$$\Theta_\varepsilon : [0,T] \longrightarrow L^\infty((0,L);[0,1])$$

$$a_\varepsilon = \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_\omega} + \frac{\Theta_\varepsilon}{a_s}\right)^{-1} : [0,T] \longrightarrow L^\infty((0,L))$$

in time

Dirichlet boundary condition: $\quad u_\varepsilon(t)_{|\{0,L\}} = \omega(t)_{|\{0,L\}}$

One-sided minimality: $\quad \forall v \in \omega(t) + H^1_0((0,L)), \quad \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx \leq \frac{1}{2}\int_0^L a_\varepsilon(t)|v'|^2 dx$

$\rightsquigarrow \quad \sigma_\varepsilon := a_\varepsilon u_\varepsilon' : [0,T] \longrightarrow \mathbb{R}$ homogeneous in space $\qquad (\sigma_\varepsilon(t)' = 0$ in $H^{-1}((0,L)), \ \forall t \in [0,T])$

$\rightsquigarrow \quad u_\varepsilon' = \sigma_\varepsilon \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_\omega} + \frac{\Theta_\varepsilon}{a_s}\right)$

$$e_\varepsilon = \frac{\sigma_\varepsilon}{a_s}\Theta_\varepsilon$$

$$p_\varepsilon = \frac{\sigma_\varepsilon}{a_\omega}\frac{1-\Theta_\varepsilon}{\varepsilon}$$

Energy balance: $\quad \mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx + \frac{K}{\varepsilon}\int_0^L (1-\Theta_\varepsilon(t))\, dx = \mathcal{E}_\varepsilon(0) + \int_0^t\int_0^L a_\varepsilon(s)\, u_\varepsilon'(s)(\dot{\omega})'(s)\, dx\, ds$

QS Brittle Damage Evolution [F-G] + Scaling Law [BIR] in 1D

$T>0$

Ω: 0 ——— L

$0 < \varepsilon a_\omega \le a_s \qquad \dfrac{K}{\varepsilon}$

Dirichlet boundary condition:

$\omega \in AC([0,T]; H^1(\mathbb{R}))$

[F-G] $\longrightarrow$

$$\left.\begin{aligned} u_\varepsilon &: [0,T] \longrightarrow H^1((0,L)) \\ \Theta_\varepsilon &: [0,T] \longrightarrow L^\infty((0,L);[0,1]) \\ a_\varepsilon = \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_\omega} + \frac{\Theta_\varepsilon}{a_s}\right)^{-1} &: [0,T] \longrightarrow L^\infty((0,L)) \end{aligned}\right\} \searrow \text{ in time}$$

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2\,dx + \frac{K}{\varepsilon}\int_0^L (1-\Theta_\varepsilon(t))\,dx$$

$$\sigma_\varepsilon := a_\varepsilon u_\varepsilon' : [0,T] \longrightarrow \mathbb{R}$$

$$u_\varepsilon' = \sigma_\varepsilon\left(\frac{1-\Theta_\varepsilon}{\varepsilon a_\omega} + \frac{\Theta_\varepsilon}{a_s}\right)$$

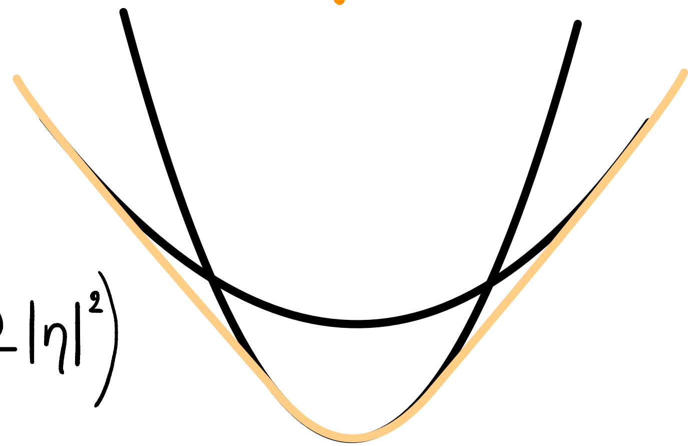
One-sided minimality:

Aumann Selection Criterion

$$\frac{1}{2}a_\varepsilon(t)\,|u_\varepsilon'(t)|^2 \overset{\text{a.e. in } (0,L)}{=} \min_{\theta\in[0,1]}\left\{\frac{K\,\Theta_\varepsilon(t)}{\varepsilon}\theta + \frac{1}{2}\left(\frac{\theta}{\varepsilon a_\omega} + \frac{1-\theta}{a_\varepsilon(t)}\right)^{-1}|u_\varepsilon'(t)|^2\right\}$$

$$= \boxed{\operatorname{Conv} W_\varepsilon^t(x, u_\varepsilon'(t))}$$

$\Longrightarrow$ explicit formula



$$W_\varepsilon^t(x,\eta) := \min\left(\frac{K\,\Theta_\varepsilon(t,x)}{\varepsilon} + \frac{\varepsilon a_\omega}{2}|\eta|^2 \,;\ \frac{a_\varepsilon(t,x)}{2}|\eta|^2\right)$$

Compactness

$T > 0$

Ω: 0 ——— L

$0 < \varepsilon a_w \le a_s$ $\quad \frac{K}{\varepsilon}$

Dirichlet boundary condition:

$\omega \in AC([0,T]; H^1(\mathbb{R}))$

[F-G] $\longrightarrow$

$$\left.\begin{aligned} &u_\varepsilon : [0,T] \longrightarrow H^1((0,L)) \\ &\Theta_\varepsilon : [0,T] \longrightarrow L^\infty((0,L);[0,1]) \\ &a_\varepsilon = \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_w} + \frac{\Theta_\varepsilon}{a_s}\right)^{-1} : [0,T] \longrightarrow L^\infty((0,L)) \end{aligned}\right\} \searrow \text{ in time}$$

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx + \frac{K}{\varepsilon}\int_0^L \left(1-\Theta_\varepsilon(t)\right) dx$$

$$\sigma_\varepsilon := a_\varepsilon u_\varepsilon' : [0,T] \longrightarrow \mathbb{R}$$

$$u_\varepsilon' = \sigma_\varepsilon \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_w} + \frac{\Theta_\varepsilon}{a_s}\right)$$

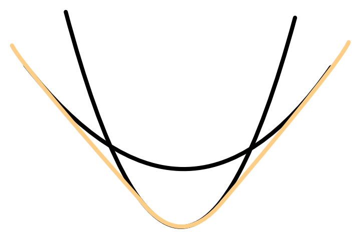
Uniform bounds:

$$\sup_{\substack{t\in[0,T] \\ \varepsilon>0}} \mathcal{E}_\varepsilon(t) < +\infty \implies \sup_{\substack{t\in[0,T] \\ \varepsilon>0}} \Big|\underbrace{\mu_\varepsilon(t) := \frac{1-\Theta_\varepsilon(t)}{\varepsilon}\mathbb{1}_{(0,L)}}_{\nearrow \text{ in time}}\Big|([0,L]) < +\infty \implies \Theta_\varepsilon(t) \xrightarrow[\varepsilon \searrow 0]{L^1((0,L))} 1 \quad \forall t \in [0,T]$$

$\nearrow$ *in time* **& Helly**

$$\sup_{\varepsilon>0} \|\sigma_\varepsilon\|_{L^\infty([0,T])} < +\infty$$

& One-sided minimality: $\frac{1}{2} a_\varepsilon(t)\,|u_\varepsilon'(t)|^2 = \mathrm{Conv}W_\varepsilon^t(x, u_\varepsilon'(t))$



$$\sup_{\substack{t\in[0,T] \\ \varepsilon>0}} \|u_\varepsilon(t)\|_{BV((0,L))} < +\infty$$

Compactness

$$\mathcal{E}_\varepsilon(t) := \frac{1}{2}\int_0^L a_\varepsilon(t)|u_\varepsilon'(t)|^2 dx + \frac{k}{\varepsilon}\int_0^L (1-\Theta_\varepsilon(t))dx$$

$$\sigma_\varepsilon := a_\varepsilon u_\varepsilon' : [0,T] \longrightarrow \mathbb{R}$$

$$u_\varepsilon' = \sigma_\varepsilon\left(\frac{1-\Theta_\varepsilon}{\varepsilon a_w} + \frac{\Theta_\varepsilon}{a_s}\right)$$

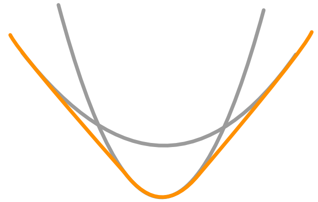
↗ in time & **Helly**

Uniform bounds:

$$\sup_{t,\varepsilon} \mathcal{E}_\varepsilon(t) < +\infty \implies \sup_{t,\varepsilon}\left|\mu_\varepsilon(t) := \frac{1-\Theta_\varepsilon(t)}{\varepsilon}\mathbb{1}_{(0,L)}\right|([0,L]) < +\infty \implies \Theta_\varepsilon(t) \xrightarrow[\varepsilon\searrow 0]{L^1((0,L))} 1 \quad \forall t\in[0,T]$$

$$\sup_{\varepsilon} \|\sigma_\varepsilon\|_{L^\infty([0,T])} < +\infty$$

& **One-sided minimality:** $\frac{1}{2}a_\varepsilon(t)|u_\varepsilon'(t)|^2 = \mathrm{Conv}\, W_\varepsilon^t(x, u_\varepsilon'(t))$



Subsequence independent of time

$$\implies \forall t\in[0,T],$$

$$\sigma_\varepsilon(t) \xrightarrow[\varepsilon]{\mathbb{R}} \sigma(t) \in K := \left[-\sqrt{2ka_0}, \sqrt{2ka_0}\right]$$

$$\mu_\varepsilon(t) = \frac{1-\Theta_\varepsilon(t)}{\varepsilon}\mathbb{1}_{(0,L)} \xrightarrow[\varepsilon]{\mathcal{M}^*([0,L])} \mu(t) \quad \nearrow \text{in time}$$

$$p_\varepsilon(t) = \frac{\sigma_\varepsilon(t)}{a_w}\mu_\varepsilon(t) \xrightarrow[\varepsilon]{\mathcal{M}^*([0,L])} p(t) = \frac{\sigma(t)}{a_w}\mu(t)$$

$$e_\varepsilon(t) = \frac{\sigma_\varepsilon(t)}{a_s}\Theta_\varepsilon(t) \xrightarrow[\varepsilon]{L^2((0,L))} e(t) = \frac{\sigma(t)}{a_s}$$

$$Du_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t)\lfloor(0,L) \xrightarrow[\varepsilon]{\mathcal{M}^*((0,L))} e(t) + p(t)\lfloor(0,L) = \sigma(t)\left(\frac{\mu(t)}{a_w}\lfloor(0,L) + \frac{\mathcal{L}^1}{a_s}\lfloor(0,L)\right)$$

Compactness

$\sigma_\varepsilon := a_\varepsilon u'_\varepsilon : [0,T] \longrightarrow \mathbb{R}$

$u'_\varepsilon = \sigma_\varepsilon \left(\frac{1-\Theta_\varepsilon}{\varepsilon a_\omega} + \frac{\Theta_\varepsilon}{a_s} \right)$

Subsequence independent of time

$\Longrightarrow \forall t \in [0,T],$

$$\sigma_\varepsilon(t) \xrightarrow[\varepsilon]{\mathbb{R}} \sigma(t) \in K := \left[-\sqrt{2Ka_0}, \sqrt{2Ka_0}\right]$$

$$\mu_\varepsilon(t) = \frac{1-\Theta_\varepsilon(t)}{\varepsilon} \mathbb{1}_{(0,L)} \xrightarrow[\varepsilon]{\mathcal{M}^*([0,L])} \mu(t) \quad \nearrow \text{in time}$$

$$p_\varepsilon(t) = \frac{\sigma_\varepsilon(t)}{a_\omega} \mu_\varepsilon(t) \xrightarrow[\varepsilon]{\mathcal{M}^*([0,L])} p(t) = \frac{\sigma(t)}{a_\omega} \mu(t)$$

$$e_\varepsilon(t) = \frac{\sigma_\varepsilon(t)}{a_s} \Theta_\varepsilon(t) \xrightarrow[\varepsilon]{L^2((0,L))} e(t) = \frac{\sigma(t)}{a_s}$$

$$\boxed{Du_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t)\lfloor(0,L) \xrightarrow[\varepsilon]{\mathcal{M}^*((0,L))} e(t) + p(t)\lfloor(0,L)}$$

&

$$\boxed{\sup_{\substack{t \in [0,T] \\ \varepsilon > 0}} \|u_\varepsilon(t)\|_{BV((0,L))} < +\infty}$$

Along the same subsequence

$$\Longrightarrow \forall t \in [0,T], \quad u_\varepsilon(t) \xrightarrow[\varepsilon]{BV((0,L))} u(t)$$

Good candidates

Subsequence independent of time

Uniform bounds & O-S minimality & Helly	$\sigma_\varepsilon(t)$	$\xrightarrow{\mathbb{R}}$	$\sigma(t)$	Good candidates
	$e_\varepsilon(t)$	$\xrightarrow{L^2}$	$e(t) = \dfrac{\sigma(t)}{a_s}$	
	$\mu_\varepsilon(t)$	$\xrightarrow{\mathcal{M}_b \times ([0,L])}$	$\mu(t)$	
	$p_\varepsilon(t)$	$\xrightarrow{\mathcal{M}_b \times ([0,L])}$	$p(t)$	
	$u_\varepsilon(t)$	$\xrightarrow{BV((0,L))}$	$u(t)$	

$$(u, e, \eta, \mu, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}_b([0,L]) \times \mathcal{M}_b([0,L]) \times K \text{ is absolutely continuous on } [0,T]$$

Additive decomposition $\quad Du(t) = e(t)\,\mathcal{L}^1 \llcorner (0,L) + p(t) \llcorner (0,L) \quad \text{in } \mathcal{M}_b((0,L))$

Relaxed Dirichlet boundary condition $\quad \eta(t) \llcorner \{0,L\} = (w(t) - u(t))\, \nu \, \mathcal{H}^0 \llcorner \{0,L\} \quad \text{in } \mathcal{M}_b(\{0,L\})$

Constitutive Equation $\quad \sigma(t) = a_s\, e(t)$

Equilibrium Equation $\quad \sigma'(t) = 0 \quad \text{in } H^{-1}((0,L))$

Stress constraint $\quad \sigma(t) \in K$, ie: $|\sigma(t)| \leq \sqrt{2 K a_w}$

Energy Balance ?

$$\frac{L}{2} a_s e(t)^2 + \sqrt{2Ka_w}\, \mathcal{V}(p;0,t) \overset{?}{=} \frac{L}{2} a_s e(0)^2 + \int_0^t \int_0^L \sigma(s) (\dot{w})'(s) \, dx \, ds$$

$$\mathcal{V}(p;0,t) = \sup\left\{ \sum_{i=1}^{n} |p(t_i) - p(t_{i-1})|([0,L]) \;;\; 0 = t_0 < \cdots < t_n = t, \; n \in \mathbb{N}^* \right\}$$

Good candidates

Uniform bounds

O-S minimality

Helly

$\sigma_\varepsilon(t) \quad e_\varepsilon(t) \quad \mu_\varepsilon(t) \quad p_\varepsilon(t) \quad u_\varepsilon(t)$

Subsequence independent of time

$\varepsilon \searrow 0$

Good candidates

$\sigma(t) \quad e(t) = \dfrac{\sigma(t)}{a_s} \quad \mu(t) \quad p(t) \quad u(t)$

$(u, e, p, \mu, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}([0,L]) \times \mathcal{M}([0,L]) \times K$ is absolutely continuous on $[0,T]$

Additive decomposition $\quad Du(t) = e(t)\,\mathcal{L}^1 \llcorner (0,L) + p(t) \llcorner (0,L) \quad$ in $\mathcal{M}((0,L))$

Relaxed Dirichlet boundary condition $\quad p(t) \llcorner \{0,L\} = (w(t) - u(t))\,\nu\, \mathcal{H}^0 \llcorner \{0,L\} \quad$ in $\mathcal{M}(\{0,L\})$

Constitutive Equation $\quad \sigma(t) = a_s\, e(t)$

Equilibrium Equation $\quad \sigma'(t) = 0 \quad$ in $H^{-1}((0,L))$

Stress constraint $\quad \sigma(t) \in K$, ie: $|\sigma(t)| \leq \sqrt{2Ka_w}$

Energy Balance

?

$$\frac{L}{2} a_s e(t)^2 + \sqrt{2Ka_w}\, \mathcal{V}(p; 0, t) \geq \frac{L}{2} a_s e(0)^2 + \int_0^t \int_0^L \sigma(s) (\dot{w})'(s) \, dx \, ds$$

$$\mathcal{V}(p; 0,t) = \sup \left\{ \sum_{i=1}^{n} |p(t_i) - p(t_{i-1})|([0,L]) \; ; \; 0 = t_0 < \cdots < t_n = t, \quad n \in \mathbb{N}^* \right\}$$

Energy Balance?

Uniform bounds

O-S minimality

Helly

$\xrightarrow[\varepsilon \searrow 0]{\text{Subsequence independent of time}}$

Good candidates

$\sigma(t) \quad e(t) = \frac{\sigma(t)}{a_s} \quad \mu(t) \quad p(t) \quad u(t)$

$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}([0,L]) \times K$

absolutely continuous on $[0,T]$

Additive decomposition

Relaxed Dirichlet boundary condition

Constitutive Equation

Equilibrium Equation

Stress constraint

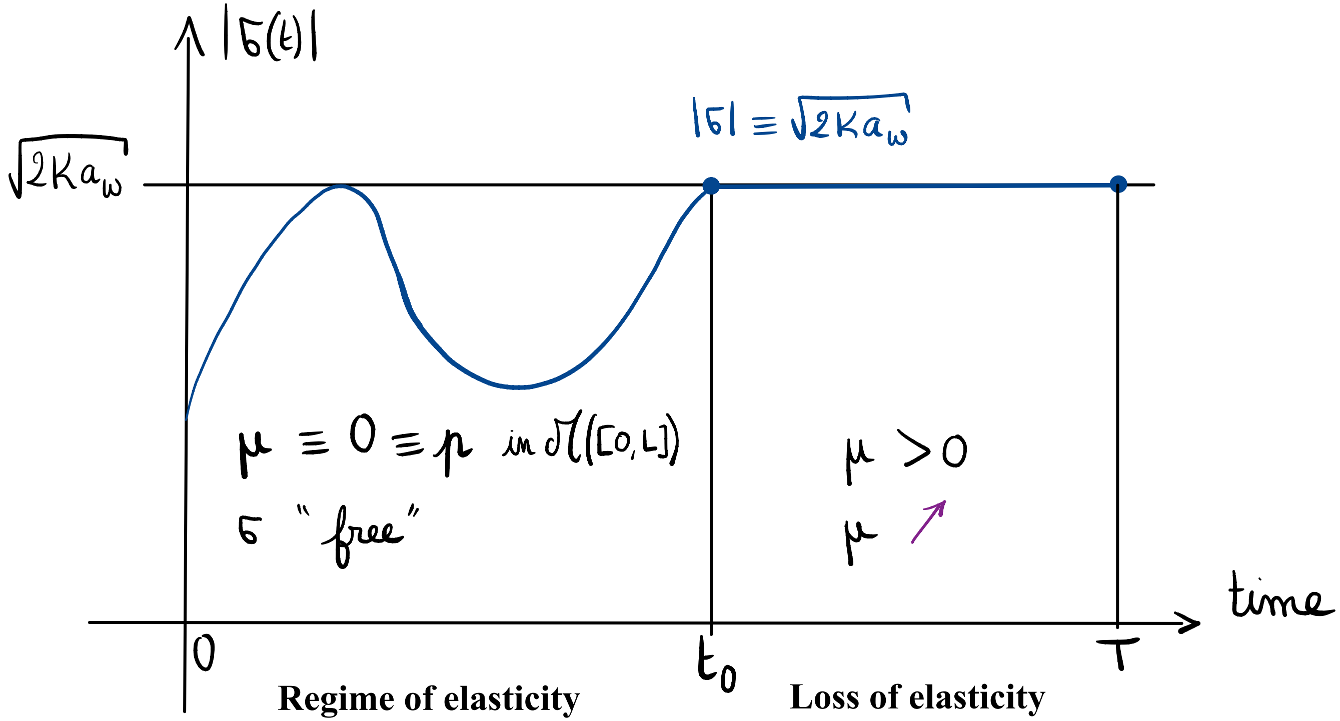
Energy Balance $\Longleftrightarrow$ **Flow Rule:** $\sigma(t)\, \mathring{p}(t)([0,L]) = \sqrt{2Ka_w}\, |\mathring{p}(t)|([0,L])$ for $\mathcal{L}^1$-a.e. $t \in [0,T]$

$\Longleftrightarrow$

$$t_0 := \sup\{t \in [0,T],\ \mu(t)([0,L]) = 0\}$$

$p = \frac{\sigma}{a_w}\mu$

$\nearrow$ in time



Energy Balance?

Uniform bounds
O-S minimality
Helly

Subsequence independent of time $\xrightarrow[\varepsilon \searrow 0]{}$ Good candidates $\sigma(t)\ e(t)\ \mu(t)\ p(t)\ u(t)$

$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}([0,L]) \times K$

absolutely continuous on $[0,T]$

Additive decomposition
Relaxed Dirichlet boundary condition
Constitutive Equation
Equilibrium Equation
Stress constraint

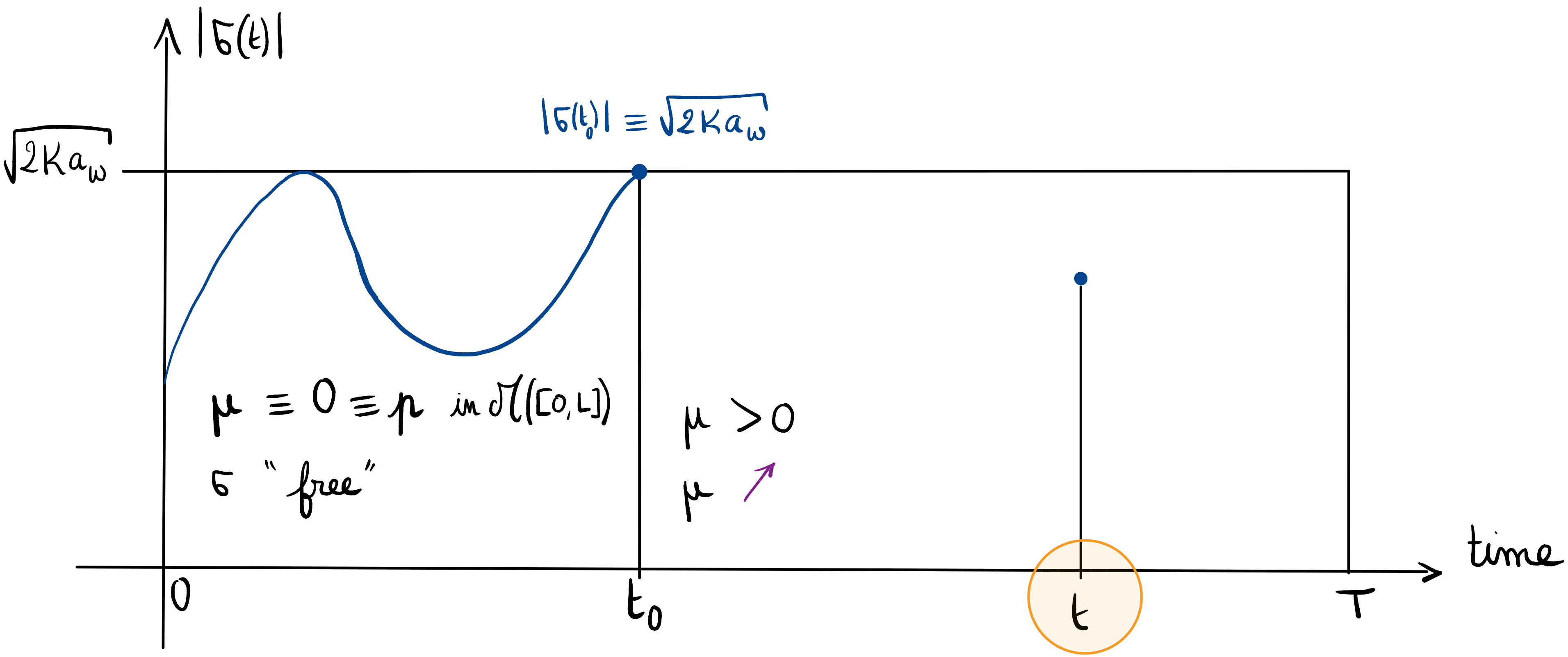
✓

$$p = \frac{\sigma}{a_w}\mu$$

$$t_0 := \sup\left\{ t \in [0,T],\ \mu(t)([0,L]) = 0 \right\}$$

Assume

$$\exists t > t_0,\ |\sigma(t)| < \sqrt{2Ka_w}$$



Uniform bounds

O-S minimality

Helly

Subsequence independent of time $\xrightarrow[\varepsilon \searrow 0]{}$

Good candidates $\sigma(t)\ e(t)\ \mu(t)\ p(t)\ u(t)$

Energy Balance?

$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}_b([0,L]) \times K$

absolutely continuous on $[0,T]$

Additive decomposition
Relaxed Dirichlet boundary condition
Constitutive Equation
Equilibrium Equation
Stress constraint

$$t_0 := \sup\{t \in [0,T],\ \mu(t)([0,L]) = 0\}$$

$$p = \frac{\sigma}{a_w}\mu$$

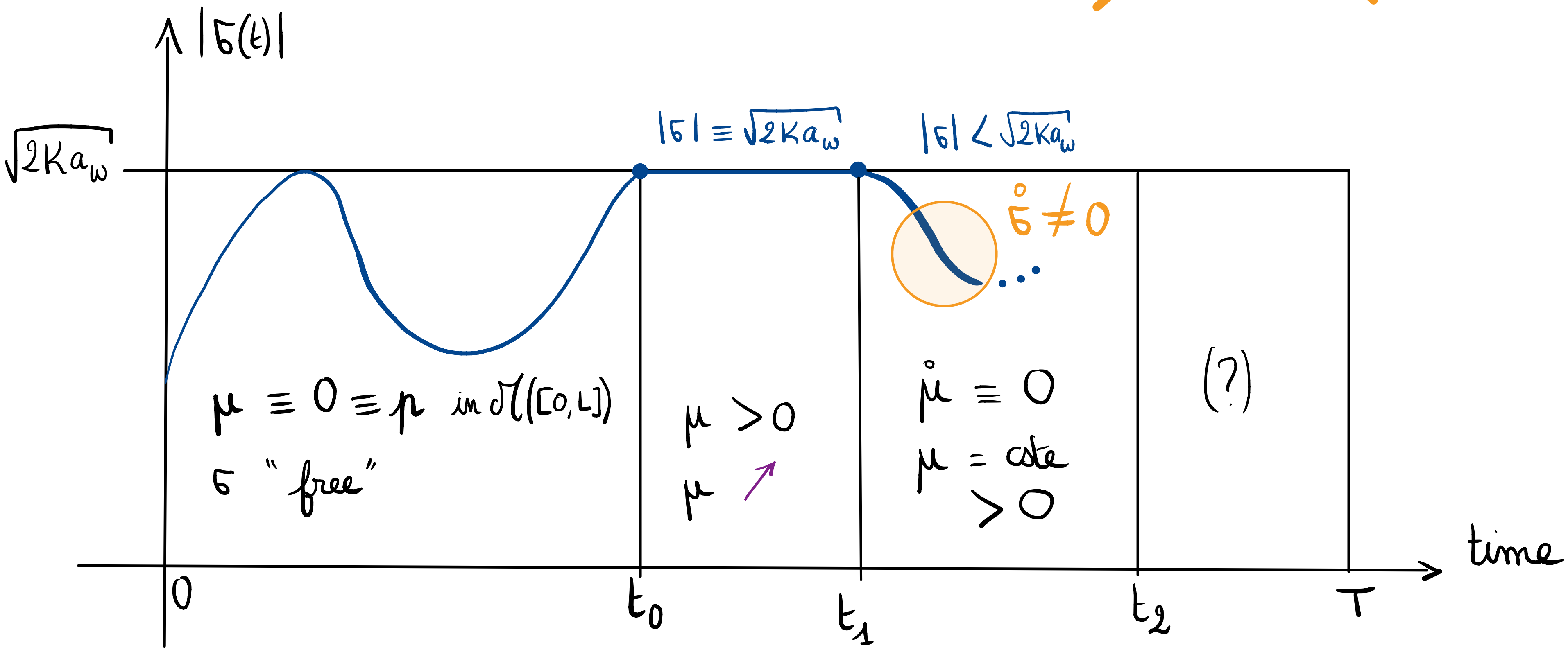
Assume

$$\exists t > t_0,\ |\sigma(t)| < \sqrt{2Ka_w}$$

$\Longrightarrow$

~~**Flow Rule:**~~

$$\cancel{\sigma(t)\ \dot{p}(t)([0,L]) = \sqrt{2Ka_w}\ |\dot{p}(t)|([0,L])}$$



Energy Balance?

Uniform bounds
O-S minimality
Helly

$\xrightarrow[\varepsilon \searrow 0]{\text{Subsequence independent of time}}$

Good candidates
$\sigma(t)\ e(t)\ \mu(t)\ p(t)\ u(t)$

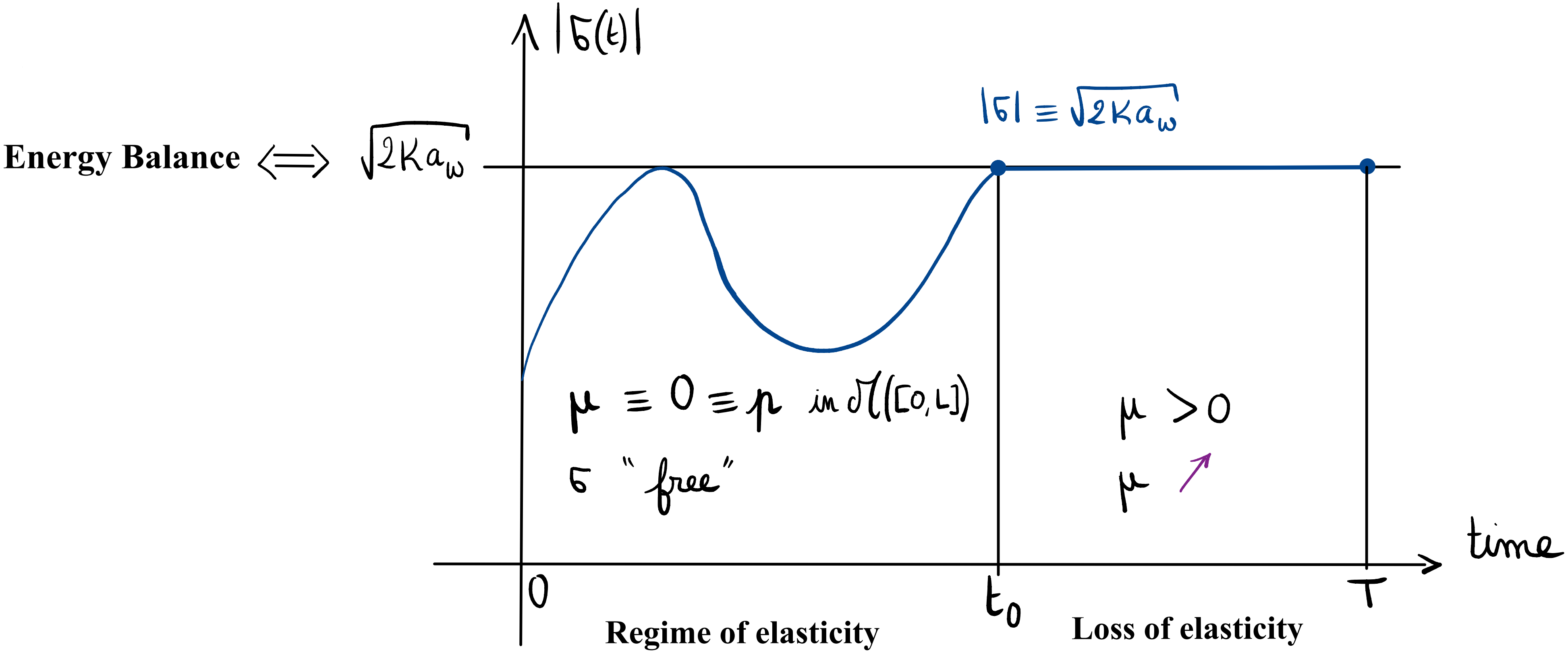
$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}_b([0,L]) \times K$

absolutely continuous on $[0,T]$

✓
Additive decomposition
Relaxed Dirichlet boundary condition
Constitutive Equation
Equilibrium Equation
Stress constraint

$p = \frac{\sigma}{a_w} \mu$

$\nearrow$ in time



$$t_0 := \sup\left\{t \in [0,T],\ \mu(t)([0,L]) = 0\right\}$$

Uniform bounds
O-S minimality
Helly

Subsequence independent of time $\longrightarrow$ $\varepsilon \searrow 0$

Good candidates $\sigma(t)\ e(t)\ \mu(t)\ p(t)\ u(t)$

Energy Balance?

$(u, e, \mu, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}_b([0,L]) \times K$

absolutely continuous on $[0,T]$

Additive decomposition
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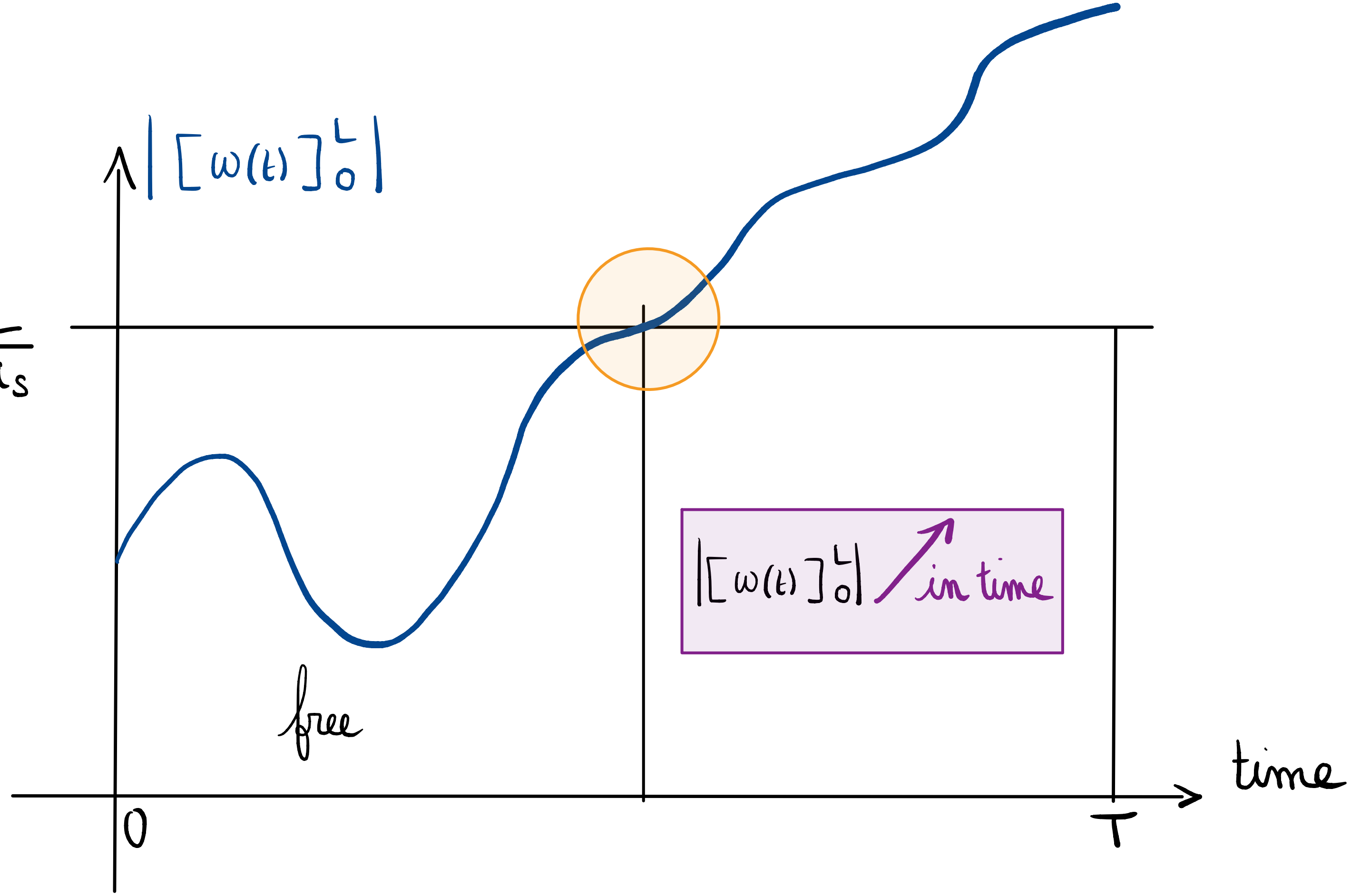
Stress constraint

$\mu = \frac{\sigma}{a_w}\mu$

$\nearrow$ in time

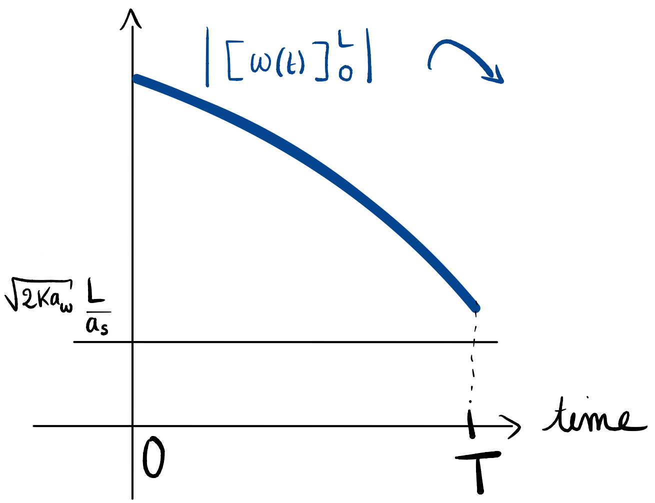
Energy Balance $\Longleftrightarrow$ $\sqrt{2Ka_w}\,\frac{L}{a_s}$



Conclusion

Additive decomposition
Relaxed Dirichlet boundary condition
Constitutive Equation
Equilibrium Equation
Stress constraint

$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L)\times\mathbb{R}\times\mathcal{M}([0,L])\times K$ absolutely continuous on $[0,T]$



$$\Longrightarrow \frac{L}{2}a_s e(t)^2 + \sqrt{2Ka_w}\,\mathcal{V}(p;0,t) > \frac{L}{2}a_s e(0)^2 + \int_0^t\int_0^L \sigma\,(\dot w)'\,dx\,ds$$

$\Longrightarrow$ **QS Brittle Damage Evolution**
[F-G]
+
Scaling Law [BIR]

$\xrightarrow[\varepsilon \searrow 0]{}$ ~~**QS perfect plasticity evolution**~~

$\boxed{\textbf{QS Damage Evolution}}$

$:= a(t)$ "inverse" effective rigidity $\nearrow$ in time

Constitutive Equation $\quad Du(t) = \sigma(t)\left(\overbrace{\frac{\mu(t)\lfloor(0,L)}{a_w} + \frac{\mathcal{L}^1\lfloor(0,L)}{a_s}}^{:=a(t)}\right)$ in $\mathcal{M}((0,L))$

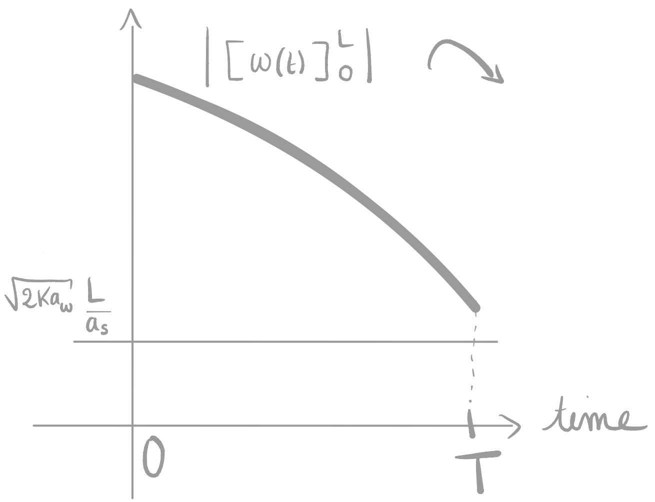
Griffith Evolution Law $\quad \dot a(t)\left(2Ka_w - \sigma(t)^2\right) = 0$ in $\mathcal{M}([0,L])$

Conclusion

Additive decomposition
Relaxed Dirichlet boundary condition
Constitutive Equation
Equilibrium Equation
Stress constraint

✓

$(u, e, p, \sigma) : [0,T] \longrightarrow BV(0,L) \times \mathbb{R} \times \mathcal{M}_b([0,L]) \times K$ absolutely continuous on $[0,T]$



$$\Longrightarrow \frac{L}{2} a_s e(t)^2 + \sqrt{2Ka_w}\,\mathcal{V}(p;0,t) \geq \frac{L}{2} a_s e(0)^2 + \int_0^t \int_0^L \sigma\,(\dot{w})'\,dx\,ds$$

$\Longrightarrow$ **QS Brittle Damage Evolution**
[F-G]
+
Scaling Law [BIR]

$\xrightarrow[\varepsilon \searrow 0]{}$ ~~**QS perfect plasticity evolution**~~

QS Damage Evolution

$$\text{Constitutive Equation} \qquad Du(t) = \sigma(t)\underbrace{\left(\frac{\mu^{(t)} \llcorner (0,L)}{a_w} + \frac{\mathcal{L}^1 \llcorner (0,L)}{a_s}\right)}_{:= a(t)} \quad \text{in } \mathcal{M}_b((0,L))$$

$:= a(t)$ "inverse" effective rigidity ↗ in time

$$\text{Griffith Evolution Law} \qquad \mathring{a}(t)\left(2Ka_w - \sigma(t)^2\right) = 0 \quad \text{in } \mathcal{M}_b([0,L])$$

Thank you for your attention!