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On Liouville theorems :

- $\Delta u = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n$

$$\Longrightarrow \quad u \equiv \text{Constant}.$$

Liouville

- $-\Delta u = n(n-2) u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \mathbb{R}^n, \; n \geq 3$

$$\Longrightarrow \quad u(x) = \left(\frac{a}{1 + a^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

Caffarelli, Gidas, Spruck
1989

$a > 0, \quad \bar{x} \in \mathbb{R}^n$

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The equations are conformally invariant.

Let $u > 0$ be a function in $\mathbb{R}^n$,

$$\varphi : \mathbb{R}^n \cup \{\infty\} \longrightarrow \mathbb{R}^n \cup \{\infty\}$$

a Möbius transformation,

i.e. generated by

$$x \longmapsto x + \bar{x} \qquad (\bar{x} \in \mathbb{R}^n)$$

$$x \longmapsto a x \qquad (a > 0)$$

$$x \longmapsto \frac{x}{|x|^2}.$$

$$u_\varphi := (u \circ \varphi) \, |J_\varphi|^{\frac{n-2}{2n}}.$$

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- $$u_\varphi(x) = u(x+\bar{x}) \qquad (\varphi: x \longmapsto x+\bar{x})$$
- $$u_\varphi(x) = a^{\frac{n-2}{2}} u(ax) \qquad (\varphi: x \longmapsto ax)$$
- $$u_\varphi(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) \qquad \left(\varphi: x \longmapsto \frac{x}{|x|^2}\right)$$

$$\frac{\Delta u_\varphi}{(u_\varphi)^{\frac{n+2}{n-2}}} = \left(\frac{\Delta u}{u^{\frac{n+2}{n-2}}}\right) \circ \varphi$$

$\forall$ Möbius φ

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Consider :

$$A^u := -\frac{2}{n-2}\frac{\nabla^2 u}{u^{\frac{n+2}{n-2}}} + \frac{2n}{(n-2)^2}\frac{\nabla u \otimes \nabla u}{u^{\frac{2n}{n-2}}} - \frac{2}{(n-2)^2}\frac{|\nabla u|^2}{u^{\frac{2n}{n-2}}} I .$$

$$\lambda(A^{u_\varphi}) = \lambda(A^u \circ \varphi), \quad \forall \text{ Möbius } \varphi .$$

$$f(\lambda(A^{u_\varphi})) \equiv f(\lambda(A^u)) \circ \varphi, \quad \forall \text{ Möbius } \varphi$$

f symmetric in $\lambda = (\lambda_1, \cdots, \lambda_n)$

Take $f(\lambda) = \sigma_1(\lambda) \equiv \lambda_1 + \cdots + \lambda_n$,

then $$f(\lambda(A^u)) = -\frac{2}{n-2}\frac{\Delta u}{u^{\frac{n+2}{n-2}}} .$$

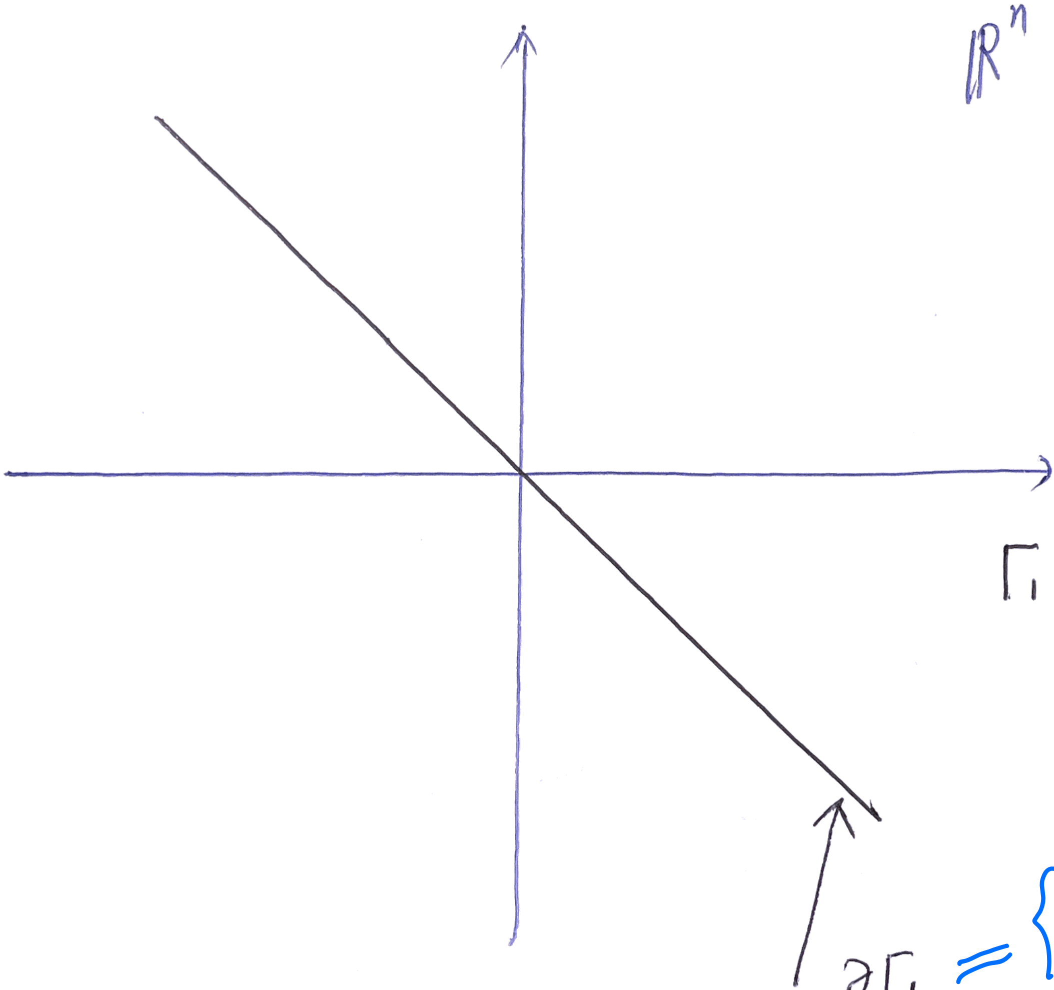
Aobing Li, L. 2002 : If $H(\cdot, u_\varphi, \nabla u_\varphi, \nabla^2 u_\varphi) \equiv H(\cdot, u, \nabla u, \nabla^2 u) \circ \varphi$

$\forall$ Möbius φ,

then

$H(\cdot, u, \nabla u, \nabla^2 u) = f(\lambda(A^u))$ for some symmetric f.

- Rephrasing the Liouville theorem in terms of A^u:



$$\Gamma_1 := \{\lambda \in \mathbb{R}^n \,/\, \underbrace{\sigma_1(\lambda)}_{\lambda_1 + \cdots + \lambda_n} > 0\}$$

$$\partial\Gamma_1 = \{\lambda_1 + \cdots + \lambda_n = 0\}$$

$\lambda(A^u) \in \partial\Gamma_1$, $u > 0$, $\mathbb{R}^n$

$\Longrightarrow$ $u \equiv$ Constant.

Will work with cone Γ instead of Γ_1:

(*) $\phi \neq \Gamma \subsetneqq \mathbb{R}^n$, open symmetric cone with vertex at the origin,

$$\Gamma + \Gamma_n \subset \Gamma.$$

$\Gamma_n = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1, \cdots, \lambda_n > 0\}$

- <u>Theorem 1</u> (Baozhi Chu, L., Zongyuan Li)

For $n \geq 3$, let Γ satisfy (*).

Assume $(1, -1, \cdots, -1) \notin \partial\Gamma$,

$$\lambda(A^u) \underset{\substack{\uparrow \\ \text{viscosity sense}}}{\in} \partial\Gamma, \quad \underset{\substack{\uparrow \\ C(\mathbb{R}^n)}}{u} > 0, \quad \mathbb{R}^n$$

$\Longrightarrow$ $u \equiv$ Constant

- Optimal : If $(1,-1,\cdots,-1) \in \partial\Gamma$, the conclusion fails.

- Examples of Γ :

$k = 1, 2, \cdots, n$

$$\begin{cases} \sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \\ \Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_1(\lambda), \cdots, \sigma_k(\lambda) > 0 \} \end{cases}$$

$$\Gamma_n \subset \Gamma_{n-1} \subset \cdots \subset \Gamma_1$$

- Chang, Gursky, Yang 2002 : Γ_2 in $\mathbb{R}^4$, $u \in C^{1,1}_{loc}(\mathbb{R}^4)$

- L. 2009, $\Gamma \subset \Gamma_1$

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- **Theorem 2** (Baozhi Chu, L., Zongyuan Li)

For $n \geq 3$, let Γ satisfy $(*)$.

Assume $(1, -1, \cdots, -1) \notin \Gamma$.

Assume

$$(**) \begin{cases} f \in C^1(\Gamma), \text{ symmetric}, \\ \dfrac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n. \end{cases}$$

elliptic

$$f(\lambda(A^u)) = 1, \quad u > 0, \quad \mathbb{R}^n$$

$C^{1,1}_{loc}(\mathbb{R}^n)$

$$\Longrightarrow \quad u(x) \equiv \left(\frac{a}{1 + b^2 |x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

$\bar{x} \in \mathbb{R}^n$, $a, b > 0$, $f(2b^2 a^{-2} e) = 1$,

$e = (1, 1, \cdots, 1)$

$A^u \equiv 2b^2 a^{-2} I$

- **Optimal**: If $(1,-1,\cdots,-1) \in \bar{\Gamma}$, $\exists f$ satisfies $(**)$ and a smooth solution u of $f(\lambda(A^u))=1$, $\lambda(A^u)\in\Gamma$, and u is not of the form.

- Caffarelli, Gidas, Spruck 1989: $n\geq 3$, $\Gamma=\Gamma_1$, $f=\sigma_1$.
  (Obata; Gidas, Ni, Nirenberg) under assumption near ∞ u
- Chang, Gursky, Yang 2002: $n=4$, $\Gamma=\Gamma_2$, $f=\sigma_2$.
- Aobing Li, L. 2003, 2005
  2003: $n\geq 3$, all Γ_k, $f=\sigma_k$
  2005: $n\geq 3$, $\Gamma\subset\Gamma_1$.

Thank You!

Proof of Theorem 2:

- Moving sphere ...

$u \in C^2(B_1)$, $v \in C^2(B_1 \setminus \{0\})$

a key new ingredient

- $f(\lambda(A^u)) = 1$, $u > 0$, B_1

$f(\lambda(A^v)) = 1$, $v > 0$, $B_1 \setminus \{0\}$

in Γ

$u < v$ $\quad B_1 \setminus \{0\}$

$\lambda(A^v) \in \overline{\Gamma}$ $\Longrightarrow$ $\liminf_{x \to 0} (v-u)(x) \geq 0$.

if $(1, -1, \cdots, -1) \notin \overline{\Gamma}$, then v is lower conical at $\{0\}$

$\Longrightarrow$ $f(\lambda(A^v)) \geq 1$ in B_1 viscosity sense

Caffarelli, Nirenberg L. 2013 $\qquad \uparrow$ supersolution

$\xrightarrow{\text{using } u \in C^2}$ $u > v$ in B_1

(M, g) $0 < u \in C^\infty(M)$

$$g_u := u^{\frac{4}{n-2}} g$$

$$f(\lambda(A_{g_u})) = 1, \text{ on } M$$

where

$$A_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} g\right).$$