$C^{1,1}_{\mathrm{loc}}$ regularity for the Monopolist's problem

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A simple model in economics leads to an optimization problem involving concepts from optimal transport (c-convex functions, MTW condition).

The associated PDE is difficult to write down. A complete expression has only been found in 2D when the domain is a square (McCann, Zhang 2023).

In this talk I'll present a recent $C^{1,1}_{\text{loc}}$ regularity result for the optimizers. This is joint work with Robert McCann and Kelvin Shuangjian Zhang.

Economics model

Monopolist sells products $y \in Y \subset \mathbf{R}^n$ to consumers $x \in X \subset \mathbf{R}^n$, where consumers are distributed according to density μ.

The monopolist pays price $c(y)$ and must come up with a cost for each product $v(y)$.

Consumer x obtains benefit $b(x, y)$ from product y. Thus they'll chose the product $y = Yu(x)$ which realizes the supremum in

$$u(x) := \sup_{y \in Y} b(x, y) - v(y).$$

Monopolist's problem: Find $v : Y \to \mathbf{R}$ realizing the supremum of

$$\Phi[v] := \int_X v(Yu(x)) - c(Yu(x)) \, d\mu.$$

Economics model: bilinear case

Prototypical case is bilinear benefit function $b(x, y) = x \cdot y$ and $c(y) = |y|^2/2$. In this case u defined by

$$u(x) := \sup_{y \in Y} b(x, y) - v(y) = \sup_{y \in Y} x \cdot y - v(y),$$

is the Legendre transform of v. Moreover $Yu(x)$, the y for which the supremum is obtained, is $Du(x)$ (assuming u is differentiable).

Monopolist's Problem (bilinear case). Find convex $u : X \to \mathbf{R}^+$ maximizing

$$\Phi[u] := \int_X x \cdot Du(x) - u(x) - \frac{|Du|^2}{2} \; d\mu =: \int_X F(x, u, Du) \; d\mu.$$

Form of the Euler–Lagrange equation

Consider minimizing

$$\Phi[u] := \int_X x \cdot Du(x) - u(x) - \frac{|Du|^2}{2}\, dx.$$

Without the convexity or nonnegativity constraint the Euler–Lagrange equation is

$$\Delta u = (n+1) \text{ in } X.$$

Without the convexity constraint the Euler–Lagrange equation is

$\Delta u = (n+1)\chi_{\{u>0\}}$ in X and $(Du - x) \cdot n \quad = 0$ on $\{u > 0\} \cap \partial X$.

The convexity constraint further complicates the problem. X splits into: $X_0 := \{u = 0\}$, $X_2 := \{u \text{ is strictly convex}\}$ on which $\Delta u = n + 1$ and $X_1 = X \setminus (X_0 \cup X_2)$. In general we can't write the Euler–Lagrange equation on X_1.

Results for the bilinear case

Rochet & Chonè 1998: Introduced the second form of the optimization problem involving the Legendre transform.

Carlier & Lachand-Robert 2001: $C^1(\overline{X})$ regularity when domain is convex and $\mu = f\ dx$ satisfies $f > 0$, $f \in C^0(\overline{X}) \cap W^{1,\infty}(X)$.

Caffarelli & P. L. Lions (unpublished): $C^{1,1}_{\text{loc}}$ regularity under the same hypothesis as Carlier & Lachand-Robert.

McCann & Zhang 2023: Euler-Lagrange equation in the region X_1.

Buttazzo, Ferone & Kawohl 1994: Other minimization problems over the set of convex functions: Newton's problem of minimal resistance.

General case: b-convex functions

For general benefit functions, the consumer's utility is

$$u(x) = \sup_{y \in Y} b(x, y) - u(x),$$

and they choose $y = Yu(x)$ realizing the above supremum. Using terminology from optimal transport the function u is b-convex and $Yu(x)$ is the b-exponential mapping.

Monopolist's problem (general case): Find b-convex $u : X \to \mathbf{R}^{+}$ maximizing

$$\Phi(u) = \int_X b(x, Yu(x)) - u(x) - c(Yu(x)) d\mu.$$

Background results for the general case

Carlier 2000: Existence result in terms of b-convex function.

Figalli, Kim & McCann 2011: Convexity of the problem in terms of an assumption B3, a strengthening of the MTW condition in optimal transport.

Chen 2013: $C^1(\overline{X})$ regularity of the minimizer.

Zhang 2018, McCann & Zhang 2019: Existence in an even more general setting: g-convex functions from generated Jacobian equations.

$C^{1,1}_{\text{loc}}$ Regularity result

B1. For $(x_0, y_0) \in \overline{X} \times \overline{Y}$ the following are diffeomorphisms:

$$y \in \overline{Y} \mapsto b_x(x_0, y) \text{ and } x \in \overline{X} \mapsto b_y(x, y_0).$$

B2. For $(x_0, y_0) \in \overline{X} \times \overline{Y}$, $b_x(x_0, Y)$ and $b_y(X, y_0)$ are convex.

By B1 we may define a mapping Y by

$$b_x(x, Y(x, p)) = p.$$

B3. For $\xi, \eta \in \mathbf{R}^n$ there holds

$$D_{p_k p_l} b_{x^i x^j}(x, Y(x, p))\xi^i \xi^j \eta_k \eta_l \geq 0.$$

Theorem. [McCann, R, Zhang 23] Assume b satisfies B1,B2,B3. Assume that c is uniformly b^*-convex and $\mu = f\ dx$ where $f \in C^{0,1}(X)$ satisfies $0 < \lambda \leq f(x)$. Then the solution of the Monopolist's problem, u, satisfies $u \in C^{1,1}_{\text{loc}}(X)$.

Idea behind the proof

Follows ideas from Caffarelli & Lions's result.

Similar techniques as for the Monge–Ampère equation.

The B3 condition ensures b-convexity behaves similarly enough to convexity for the key ideas to work.

No EL equation, so must work with the variational formulation.

Let u be the minimizer. Fix $x_0 \in X$ and assume $u(x_0), Du(x_0) = 0$. Put $h = \sup_{B_r(x_0)} u$. We can construct another admissible function $\tilde{u}$ such that

$$\Phi[u] - \Phi[\tilde{u}] \leq C_1 h - C_2 \frac{h^2}{r^2}.$$

However since u is the maximizer for Φ and $\tilde{u}$ is admissible $\Phi[u] - \Phi[\tilde{u}] > 0$. Thus

$$h \leq Cr^2,$$

which is known to be equivalent to $C^{1,1}_{\mathrm{loc}}$ regularity.

Idea behind the proof

In the bilinear case Caffarelli & Lions's construction of $\tilde{u}$ is based on taking a support plane $P = \frac{h}{2r}x_1$, and constructing $\tilde{u}$ as the maximum of u and this plane. For such a plane

$$\Phi(P) = \int_X x \cdot DP + P - \frac{|DP|^2}{2} = \int_X \frac{h^2}{4r^2} d\mu.$$

Similar ideas are found in Caffarelli's work on the Monge–Ampère equation.

The B3 condition ensures the theory of b-convexity behaves similarly to standard convexity. We can then perform a similar construction to Caffarelli and Lions, modifying with ideas from Figalli-Kim-McCann 2013 and Chen-Wang 2016.

Future work/Open questions

- Regularity in the more general setting of McCann & Zhang.
- Understand the structure of X_0, X_1, X_2: When can these sets be empty? What is the Euler–Lagrange equation on X_1?

Thank you!