Discrete conformality for spherical cone-metrics

Ivan Izmestiev
joint work with Roman Prosanov and Tianqi Wu

TU Wien

Applied and Computational Differential Geometry
and Geometric PDEs

BIRS, July 30 – August 4, 2023

Uniformization of Riemannian surfaces

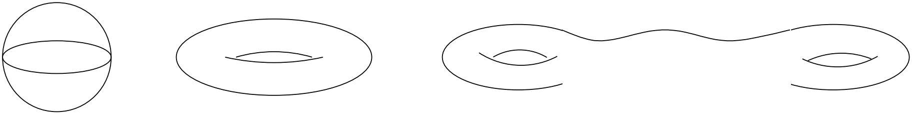
Definition

*Two Riemannian metrics g, g' are **conformal** if $g' = e^{2u} g$.*

Theorem (Koebe–Poincaré)

Every Riemannian metric on a surface S is conformal to a unique (up to scaling) metric of constant Gaussian curvature K.

$$\int_S K \, \mathrm{darea} = 2\pi\chi(S) \Rightarrow K \begin{cases} > 0, & S \text{ sphere or projective plane,} \\ = 0, & S \text{ torus or Klein bottle,} \\ < 0, & \text{otherwise.} \end{cases}$$



Metrics with prescribed curvature

Kazdan–Warner:

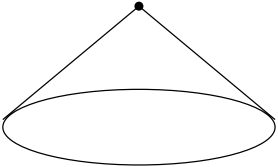
- For which functions f on a smooth manifold there is a Riemannian metric with the scalar curvature equal f?
- When there is a unique metric with scalar curvature $= f$ in a given conformal class?

Let us look at the surfaces only, scalar curvature $= K$.

First question: yes, if f satisfies the sign condition dictated by the Gauss–Bonnet theorem.
The answer to the second question is unknown for the sphere (Nirenberg problem).

Cone-metrics

- Uniformization: prescribed curvature is constant.
- Kazdan–Warner: prescribed curvature is a smooth function.
- Now, let the prescribed curvature be constant except a finite number of Dirac-type singularities, that is, cone-points.



Definition

A Euclidean/ hyperbolic/ spherical ***cone-metric*** *on a surface is locally modelled on* $\mathbb{R}^2/\mathbb{H}^2/\mathbb{S}^2$ *and on Euclidean/ hyperbolic/ spherical cones. The* ***curvature*** *of a cone-point is* 2π *minus the total angle at the point.*

Cone-metrics and polyhedra

The metric on a polyhedral surface in $\mathbb{R}^3$/ $\mathbb{H}^3$/ $\mathbb{S}^3$ is a Euclidean/ hyperbolic/ spherical cone-metric.
Convex polyhedron $\Rightarrow$ all cone-points have positive curvature.

Theorem (Alexandrov)

Every Euclidean/ hyperbolic/ spherical metric on $\mathbb{S}^2$ with cone-points of positive curvature is realized by a unique convex polyhedron.

This is a discrete analog of the Weyl problem: metric of positive Gaussian curvature is realized by a unique smooth convex surface.

Uniqueness $=$ rigidity of convex polyhedra (Alexandrov's version, strengthening of Cauchy's).

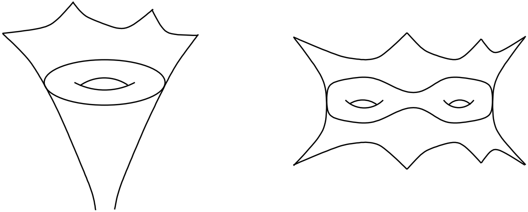
For metrics with curvature of arbitrary sign, there is no general existence and uniqueness theorem, neither in discrete, nor in smooth.

Convex cone-metrics on surfaces of higher genus

How about cone-metrics on the torus and on surfaces of higher genus?

Theorem (Fillastre, Fillastre–Izmestiev)

Every hyperbolic cone-metric on the torus or a higher-genus surface with cone-points of positive curvature is realized as a convex cusp, respectively as a convex Fuchsian polyhedron.
The realization is unique.



There is a connection between isometric realizations and the (discrete) uniformization.

Prescribed discrete curvature in a conformal class

A constant curvature metric with n cone-points determines a conformal structure on the n-punctured surface.

Theorem (Troyanov, McOwen)

Let S be a closed surface, and let $\kappa_1, \ldots, \kappa_n \in (-\infty, 2\pi)$ satisfy the Euclidean/hyperbolic Gauss–Bonnet condition on S.
Then in every conformal class on n-punctured S there is a unique Euclidean/ hyperbolic metric with cone-points of curvatures $\kappa_1, \ldots, \kappa_n$.

Generalized Gauss–Bonnet: $\int_S K \, \mathrm{darea} + \sum_i \kappa_i = 2\pi\chi(S)$.

Theorem (Luo–Tian)

Let $S = \mathbb{S}^2$, $\kappa_1, \ldots, \kappa_n \in (0, 2\pi)$ such that $\sum_i \kappa_i < 4\pi$ and $\kappa_i < \sum_{j \neq i} \kappa_j$ for all i. Then in every conformal class on S with n punctures there is a spherical metric with cone-points of curvatures $\kappa_1, \ldots, \kappa_n$.

Curvatures of arbitrary sign: possible collections of κ_i on $\mathbb{S}^2$ were described only recently by Mondello–Panov and Eremenko.
Description of conformal representatives is out of reach.

Discrete conformality for discrete curvature

A cone-surface can be triangulated, that is decomposed into a finite number of Euclidean/ hyperbolic/ spherical triangles.
$\Rightarrow$ the space of cone-metrics with n cone-points is finite-dimensional.
The dimension is $\approx 3n$, as the metric is determined by the edge lengths (in any geodesic triangulation), and $\sharp$ edges $\approx 3 \cdot \sharp$ vertices.

Is it possible to discretize the conformality relation?

Definition (First attempt)

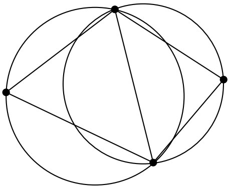
Two triangulated cone-surfaces with the same triangulation and with the edge lengths ℓ_{ij}, ℓ'_{ij} are called ***discretely conformally equivalent****, if there is a function u on the vertex set such that*

$$\ell'_{ij} = e^{u_i+u_j}\ell_{ij}, \quad \sinh\frac{\ell'_{ij}}{2} = e^{u_i+u_j}\sinh\frac{\ell_{ij}}{2}, \quad \sin\frac{\ell'_{ij}}{2} = e^{u_i+u_j}\sin\frac{\ell_{ij}}{2}$$

in the Euclidean, hyperbolic, spherical cases, respectively.

Discrete conformality, done right

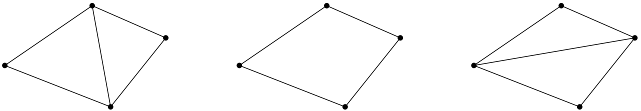
Every cone-metric has a **Delauney triangulation.**



Two cone-metrics are **discretely conformally equivalent**, if one can be transformed into the other by continuous edge scaling

$$\ell_{ij}(t) = e^{u_i(t)+u_j(t)} \quad etc.$$

and Delaunay retriangulation on the way.



Prescribed discrete curvature in a discrete conformal class

A discrete analog of the Troyanov and McOwen results:

Theorem (Gu–Guo–Luo–Sun–Wu)

Let S be a closed surface, and let $\kappa_1, \ldots, \kappa_n \in (-\infty, 2\pi)$ satisfy $\sum_i \kappa_i = 2\pi\chi(S)$, resp. $> 2\pi\chi(S)$. Then for every Euclidean, resp. hyperbolic cone-metric on S with n cone-points there is a unique cone-metric discretely conformal to it, with discrete curvatures κ_i.

A discrete analog of the Luo–Tian result:

Theorem (I.–Roman Prosanov–Tianqi Wu)

Let $\kappa_1, \ldots, \kappa_n \in (0, 2\pi)$ such that $\sum_i \kappa_i < 4\pi$ and $\kappa_i < \sum_{j \neq i} \kappa_j$ for all i. Then for every spherical cone-metric on $\mathbb{S}^2$ with n cone-points there is a unique spherical cone-metric discretely conformal to it, with discrete curvatures κ_i.

Proof: the continuity method

Recall: conformal factors correspond to the cone-points. There are n unknown conformal factors u_i and n prescribed curvatures κ_i.
This gives rise to a map

$$U \to K = \left\{ \kappa \in (0, 2\pi)^n \mid \sum_i \kappa_i < 4\pi, \kappa_i < \sum_{j \neq i} \kappa_j \right\}.$$

We want to show that this map is bijective.

- U is an open connected subset of $\mathbb{R}^n$.
- The map is a local homeomorphism.
- The map is proper (preimages of bounded subsets are bounded).

These three properties together imply that the map is a homeomorphism.

The Jacobian becomes a Hessian

- The map $U \to K$ is a local homeomorphism.

This map is actually a local diffeomorphism: the matrix $\left(\frac{\partial \kappa_i}{\partial u_j}\right)$ is non-degenerate.

To prove this, we identify $\left(\frac{\partial \kappa_i}{\partial u_j}\right)$ as the Hessian matrix of a certain functional H:

$$\frac{\partial H}{\partial u_i} = \kappa_i, \quad \frac{\partial^2 H}{\partial u_i \partial u_j} = \frac{\partial \kappa_i}{\partial u_j}$$

and show that the second variation $D^2 H$ is a non-degenerate quadratic form.

The functional H is the total curvature of a certain three-dimensional object associated with a cone-metric.

Cusped cone-polyhedra

Spherical triangles $\leftrightarrow$ $3/4$-ideal hyperbolic simplices.

Spherical cone-metrics on $\mathbb{S}^2$ $\leftrightarrow$ ideal polyhedra with singular rays from an interior point (**ideal cone-polyhedra**).

$$H(u) = -2\mathrm{vol}(P) + \sum_i u_i \kappa_i + \sum_{ij} a_{ij}(\pi - \alpha_{ij})$$

The equality $\frac{\partial H}{\partial u_i} = \kappa_i$ is a consequence of the Schläfli formula.

Discrete uniformization and isometric embedding

Similarly to Bobenko–Pinkall–Springborn,

Lemma

Two spherical cone-metrics are discretely conformally equivalent $\Leftrightarrow$ *the boundaries of corresponding cone-polyhedra are isometric.*

Theorem (Rivin)

Every hyperbolic metric on $\mathbb{S}^2$ *with cusps can be realized as the boundary of a convex ideal hyperbolic polyhedron.*

Rivin's theorem $\Leftrightarrow$ discrete uniformization of spherical cone-metrics:
spherical cone-metric $\to$ ideal cone-polyhedron $\xrightarrow{\text{Rivin}}$ isometric embedding $\to$ uniformized metric