Time periodic solutions near shear/radial flows for 2D Euler

New Trends in Fluids and Collective Dynamics

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Joint work with Ángel Castro

2D Euler

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- Global existence of classical solutions for 2D Euler ✓
- Qualitative behaviour of 2D Euler for long times ✗

2D Euler and shear/radial flows

Two important classes of steady states for 2D Euler:

Shear flows in a strip-type domain ($\mathbb{T} \times \mathbb{R}$ or $\mathbb{T} \times [0, 1]$)

$$\mathbf{v}(\mathbf{x}) = (U(y), 0), \qquad \omega(\mathbf{x}) = -U'(y),$$

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$$\mathbf{v}(\mathbf{x}) = V(|\mathbf{x}|)\frac{\mathbf{x}^\perp}{|\mathbf{x}|}, \quad \omega(|\mathbf{x}|) = \frac{V(|\mathbf{x}|)}{|\mathbf{x}|} + V'(|\mathbf{x}|),$$

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Some remarkable examples:

$$U_C(y) = y, \quad U_P(y) = y^2, \quad U_K(y) = \sin(y)$$

$$V_{TC}(|\mathbf{x}|) = |\mathbf{x}| + |\mathbf{x}|^{-1}, \quad V_G(r) = e^{-|\mathbf{x}|^2}$$

Arnold's stability

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Arnold's stability (1960's):

$$\left.\begin{aligned} -\lambda_1(D) < F'(\psi) < 0 \\ 0 < F'(\psi) < +\infty \end{aligned}\right\} \quad \Longrightarrow \quad \text{nonlinearly (Lyapunov) stable in } L^2$$

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Couette and Taylor-Couette flows are stable in L^∞.
Pouseville flow is stable in L^2.

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or equivalently

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$$\|\mathbf{u}_0\|_X \ll 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \|\mathbf{u}(t)\|_Y = ??$$

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In 1907, Orr predicted the inviscid damping for the Couette flow

`shear flow + perturbation → new shear flow`

Couette flow

Perturbation of the Couette flow in $\mathbb{T} \times \mathbb{R}$

The equation for a perturbation of the Couette flow

$$\mathbf{v}(\mathbf{x}, t) = (y, 0) + \mathbf{u}(\mathbf{x}, t), \qquad w(\mathbf{x}, t) = -1 + \omega(\mathbf{x}, t),$$

is given by

$$\partial_t \omega + \nabla^\perp \psi \cdot \nabla \omega + y \partial_x \omega = 0,$$

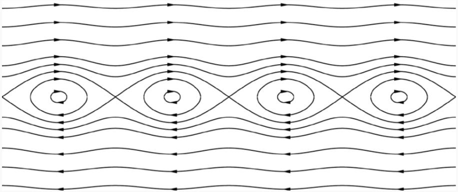
with

$$\psi = \Delta^{-1} \omega = \int_{\mathbb{T}\times\mathbb{R}} \log\left(\cosh(y - \bar{y}) - \cos(x - \bar{x})\right) \omega(\bar{x}, \bar{y}) \mathrm{d}\bar{x} \mathrm{d}\bar{y}.$$

Traveling waves and stationary states

Lin-Zeng (2011). *Inviscid dynamical structures near Couette flow*

- Existence of nontrivial and smooth **stationary states** arbitrarily close to the Couette flow in the $H^{<\frac{3}{2}}$ topology.



- Nonexistence of nontrivial smooth **traveling waves** arbitrarily close to the Couette flow in the $H^{>\frac{3}{2}}$ topology. All steady states near Couette in $H^{>\frac{3}{2}}$ are shears.

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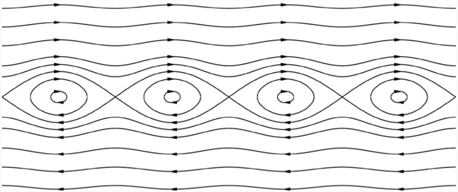

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The above results immediately implies that nonlinear inviscid damping is NOT TRUE in any H^s $(s < 3/2)$ neighborhood of Couette flow.

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ANALYTICAL $\subset$ GEVREY $\subset$ SMOOTH

Gevrey spaces $\mathcal{G}^s$:

$$|\partial^m f| \leq K^m (m!)^s \quad \forall m \geq 0.$$

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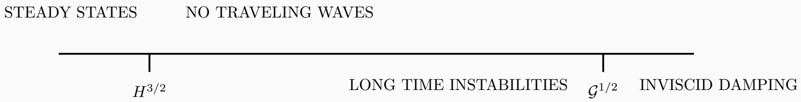

Our result

For any $0 \le s < 3/2$ and $\epsilon > 0$, the perturbed 2D Euler system admits a nontrivial smooth traveling wave solution satisfying

$$\|w + 1\|_{H^s(\mathbb{T}\times\mathbb{R})} \equiv \|\omega\|_{H^s(\mathbb{T}\times\mathbb{R})} < \epsilon.$$

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nontrivial smooth traveling wave means...

- Traveling wave:
$$\omega(x, y, t) = \widetilde{\omega}(x + \lambda t, y).$$
- Nontrivial: the dependence on x is nontrivial.
- Smooth: $\omega \in C_c^\infty$ but its H^s-norm is large for $s > 3/2$.

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The support of $\nabla\omega$ is concentrated around $y = \pm L$.
The speed of the wave satisfies

$$\lambda = L + O(\epsilon).$$

Symmetries of the system

Galilean invariance:

$$\mathbf{v}(x,y,t) \quad \rightsquigarrow \quad \overline{\mathbf{v}}(x,y,t) = \mathbf{v}(x+\lambda t, y, t) - (\lambda, 0)$$

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If $\mathbf{v}_{\text{steady}}(x,y)$ is a nontrivial Lin-Zeng stationary solution then

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From steady states to traveling waves:

$$v_1(x,y) - y = O(\epsilon) \quad \Longrightarrow \quad \overline{v}_1(x,y,t) - y = O(\epsilon) \quad \text{if} \quad \lambda = O(\epsilon)$$

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Our traveling waves satisfy $v_1(x, y) - y = O(\epsilon)$ with $\lambda = O(1)$.

Taylor-Couette flow

2D Euler in polar coordinates

$$\partial_t w + \frac{1}{r}(\partial_\theta \psi \partial_r w - \partial_r \psi \partial_\theta w) = 0, \qquad -(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)\psi = w.$$

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The velocity $\mathbf{v}(\mathbf{x}) = v^r(r,\theta)\mathbf{e}_r + v^\theta(r,\theta)\mathbf{e}_\theta$ is recovered via

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Let $0 < r_1 < r_2 < \infty$ and $\Omega_{r_1,r_2} = \{\mathbf{x} \in \mathbb{R}^2 : r_1 \le |\mathbf{x}| \le r_2\}$

$$v^\theta(r) = Ar + \frac{B}{r}, \qquad A, B \in \mathbb{R}.$$

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Time-periocic (rotanting) solutions in an annular domain near Taylor-Couette flow can be constructed using a similar strategy.

(Asymptotic) stability of a steady circular flow

- Bedrosian-Zelati-Vicol (2019): Linear inviscid damping around radially symmetric, strictly monotone decreasing vorticity.
- Ionesuc-Jia (2019): Asymptotic stability of point vortex solutions.
- Gallay-Sverak (2021): Stability of $w(r) = e^{-r^2/4}$ and $w(r) = (1 + |r|^2)^{-k}, k > 1$ for 2D Euler and NS with low regularity.

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Some previous results for the Taylor-Couette:

- Zillinger & Zelati-Zillinger (2017,2019): Linear inviscid damping around Taylor-Couette.
- An-He-Li (2021,2023): Enhanced dissipation and nonlinear asymptotic stability of Taylor-Couette for 2D NS.

Perturbation of the Taylor-Couette flow in Ω_{r_1,r_2}

The equation for a perturbation of the Taylor-Couette flow

$$\mathbf{v}(r,\theta,t) = \left(0, Ar + \frac{B}{r}\right) + \mathbf{u}(r,\theta,t), \quad w(r,\theta,t) = 2A + \omega(r,\theta,t),$$

is given by

$$\partial_t \omega + \tfrac{1}{r}(\partial_\theta \psi \partial_r \omega - \partial_r \psi \partial_\theta \omega)\omega + \left(Ar + \frac{B}{r}\right)\partial_\theta \omega = 0, \qquad on \quad \Omega_{r_1,r_2}$$

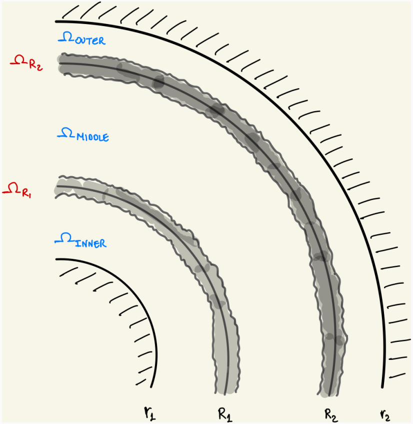
with ψ solving

$$\begin{cases} -(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)\psi &= \omega, \qquad on \quad \Omega_{r_1,r_2} \\ \psi|_{r=r_1,r_2} &= 0. \end{cases}$$

Splitting the domain Ω_{r_1,r_2}

We are looking for a solution with the following structure:

$$\omega(t) = \begin{cases} 0 & \Omega_{\text{Inner}}, \\ \text{smooth}(t) & \Omega_{R_1}, \\ \epsilon & \Omega_{\text{Middle}} \\ \text{smooth}(t) & \Omega_{R_2}, \\ 0 & \Omega_{\text{Outer}}, \end{cases}$$



with $r_1 < R_1 < R_2 < r_2$ and $|\Omega_{R_i}| = O(\epsilon)$, for $i = 1, 2$.

The dynamics of the perturbation occurs only on $\Omega_{R_1}(t) \cup \Omega_{R_2}(t)$.

2D Euler as an equation for the level curves/sets

We assume that

$$\Omega_{R_i}(t) = \{(\rho + f(\rho,\theta,t))\,(\cos\theta, \sin\theta), \rho \in [R_i - \epsilon, R_i + \epsilon], \theta \in \mathbb{T}\}.$$

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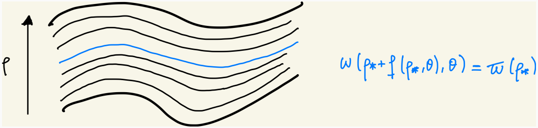
Using the transport character of the vorticity formulation:

$$\omega(\rho + f(\rho,\theta,t), \theta, t) = \varpi(\rho), \quad (\rho,\theta) \in [R_i - \epsilon, R_i + \epsilon] \times \mathbb{T} \quad \forall t \geq 0.$$



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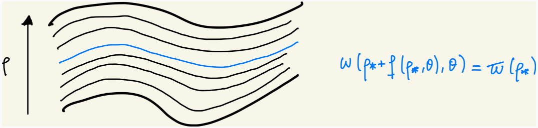

The problem reduces to study the family of graphs $(\rho + f(\rho,\theta,t), \theta)$:

$$(\rho + f(\rho,\theta,t))\partial_t f(\rho,\theta,t) = \partial_\theta \bar{\psi}[f](\rho,\theta,t),$$

with $\bar{\psi}[f](\rho,\theta,t) := \psi(\rho + f(\rho,\theta,t),\theta,t)$ and ψ solving

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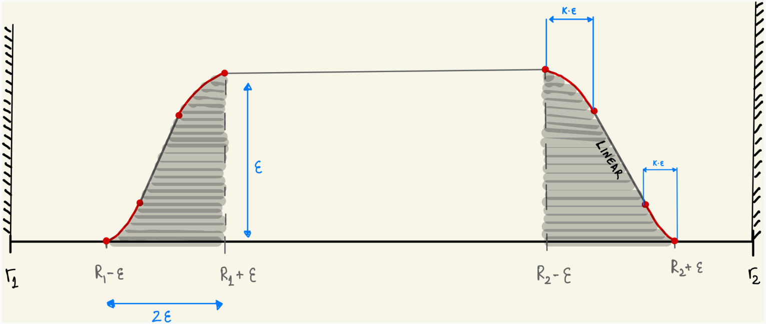
Important facts:

- Note that $F_\varpi[\lambda, 0] = 0$, for all $\lambda \in \mathbb{R}$.
- Recall that f is defined just over $\bigcup_{i=1,2}(R_i - \epsilon, R_i + \epsilon) \times \mathbb{T}$.

The function $\varpi_{\epsilon,\kappa} \in C^\infty([r_1, r_2])$

$$\varpi_{\epsilon,\kappa}(r) := \begin{cases} 0 & r_1 \le r \le R_1 - \epsilon, \\ \epsilon\varphi_\kappa\left(\frac{R_1 - r}{\epsilon}\right) & R_1 - \epsilon < r < R_1 + \epsilon, \\ \epsilon & R_1 + \epsilon \le r \le R_2 - \epsilon, \\ \epsilon\varphi_\kappa\left(\frac{r - R_2}{\epsilon}\right) & R_2 - \epsilon < r < R_2 + \epsilon, \\ 0 & R_2 + \epsilon \le r \le r_2. \end{cases}$$



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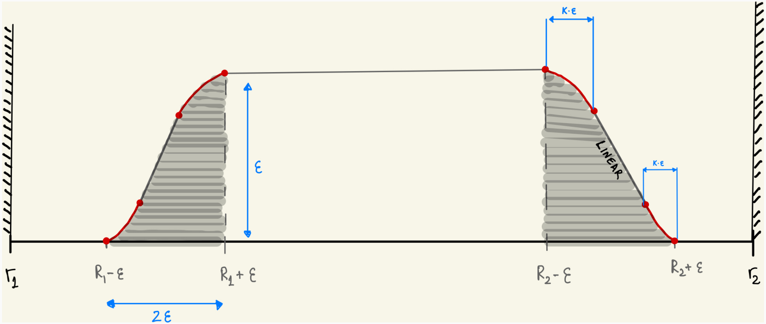

Note that $\varpi_{\epsilon,0} \in W^{1,\infty}(\mathbb{R}) \cap H^{<3/2}(\mathbb{R})$. ($\kappa$ is a regularizing parameter)

Crandall-Rabinowitz Theorem

Let X, Y be two Banach spaces, $\{0\} \in V \subset X$ and let $F : \mathbb{R} \times V \to Y$ satisfying:

1. $F[\lambda, 0] = 0$ for any $\lambda \in \mathbb{R}$.
2. The derivatives $D_\lambda F$, $D_\mathrm{f} F$ and $D^2_{\lambda,\mathrm{f}} F$ exist and are continuous.
3. $\mathcal{L}_\star = D_\mathrm{f} F[\lambda_\star, 0]$: $\mathcal{N}(\mathcal{L}_\star)$ and $Y/\mathcal{R}(\mathcal{L}_\star)$ are one-dimensional.
4. $D^2_{\lambda,\mathrm{f}} F[\lambda_\star, 0]\mathrm{h}_\star \notin \mathcal{R}(\mathcal{L}_\star)$, where $\mathcal{N}(\mathcal{L}_\star) = \mathrm{span}\{\mathrm{h}_\star\}$.

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If Z is any complement of $\mathcal{N}(\mathcal{L}_\star)$ in X, then there is a neighborhood U of $(\lambda_\star, 0)$ in $\mathbb{R} \times X$, an interval $(-\sigma_0, \sigma_0)$, and continuous functions $\varphi : (-\sigma_0, \sigma_0) \to \mathbb{R}$, $\psi : (-\sigma_0, \sigma_0) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}(0) \cap U = \Big\{ \big(\lambda_\star + \varphi(\sigma), \sigma h_\star + \sigma\psi(\sigma)\big) \,;\, |\sigma| < \sigma_0 \Big\} \cup \Big\{ (\lambda, 0) \,;\, (\lambda, 0) \in U \Big\}.$$

The linear operator

Let $h \in X$. Then, we have $h = \sum_{n\geq 1} h_n(r)\cos(n\theta)$.

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where $\mathcal{L}_n[\lambda]$ is given by

$$\mathcal{L}_n[\lambda]g(r) := \left(\Phi'_{\epsilon,\kappa}(r) + \lambda r\right) g(r)$$

$$-\frac{\mathcal{S}_n(r/r_1)}{\mathcal{S}_n(r_2/r_1)}\frac{1}{n}\int_{r_1}^{r_2} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r_2/s)g(s) + \frac{1}{n}\int_{r_1}^{r} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r/s)g(s),$$

with $\mathcal{S}_n(\cdot) = \sinh(n\log(\cdot))$ and $\Phi_{\epsilon,\kappa}$ solving

$$\begin{cases} -(\partial_r^2 + \frac{1}{r}\partial_r)\Phi_{\epsilon,\kappa} &= \varpi_{\epsilon,\kappa}, \qquad on \quad [r_1, r_2] \\ \Phi_{\epsilon,\kappa}|_{r=r_1,r_2} &= 0. \end{cases}$$

The kernel: infinity system ($n \geq 1$) $\mapsto$ m-th mode

Note that

$$\mathcal{L}[\lambda]h = 0 \iff \mathcal{L}_n[\lambda]h_n = 0 \quad \forall n \geq 1.$$

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Fixed $m \geq 1$, we consider only $h \in X$ such that

$$h_n \equiv 0 \quad \text{for} \quad n \neq m$$

and

$$h_m(r) = \begin{cases} a(r) & r \in [R_1 - \epsilon, R_1 + \epsilon] =: I_\epsilon(R_1), \\ b(r) & r \in [R_2 - \epsilon, R_2 + \epsilon] =: I_\epsilon(R_2) \end{cases}$$

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We have reduce the problem to study

$$\mathcal{L}_m[\lambda]h_m(r) \equiv \mathcal{L}_m[\lambda]\begin{pmatrix} a \\ b \end{pmatrix}(r) = 0, \qquad r \in I_\epsilon(R_1) \cup I_\epsilon(R_2).$$

The problem for λ and (a, b)

$$\left(\Phi'_{\epsilon,\kappa}(r) + \lambda r\right) a(r) + \frac{1}{n}\int_{R_1-\epsilon}^{r} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r/s)a(s)ds$$
$$- \frac{\mathcal{S}_n(r/r_1)}{\mathcal{S}_n(r_2/r_1)}\frac{1}{n}\int_{R_1-\epsilon}^{R_1+\epsilon} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r_2/s)a(s)ds$$
$$- \frac{\mathcal{S}_n(r/r_1)}{\mathcal{S}_n(r_2/r_1)}\frac{1}{n}\int_{R_2-\epsilon}^{R_2+\epsilon} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r_2/s)b(s)ds = 0, \quad r \in I_\epsilon(R_1).$$

$$\left(\Phi'_{\epsilon,\kappa}(r) + \lambda r\right) b(r) + \frac{1}{n}\int_{R_2-\epsilon}^{r} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r/s)b(s)ds$$
$$+ \frac{1}{n}\int_{R_1-\epsilon}^{R_1+\epsilon} s\varpi'_{\epsilon,\kappa}(s)\left[\mathcal{S}_n(r/s) - \frac{\mathcal{S}_n(r/r_1)}{\mathcal{S}_n(r_2/r_1)}\mathcal{S}_n(r_2/s)\right] a(s)ds$$
$$- \frac{\mathcal{S}_n(r/r_1)}{\mathcal{S}_n(r_2/r_1)}\frac{1}{n}\int_{R_2-\epsilon}^{R_2+\epsilon} s\varpi'_{\epsilon,\kappa}(s)\mathcal{S}_n(r_2/s)b(s)ds = 0, \quad r \in I_\epsilon(R_2).$$

Re-scaling

We pass from

$$[R_1 - \epsilon, R_1 + \epsilon] \cup [R_2 - \epsilon, R_2 + \epsilon] \to [-1, 1]$$

We just have to solve

$$A(s) := a(R_1 + \epsilon\, s), \qquad B(s) := b(R_2 + \epsilon\, s)$$

$$\begin{aligned}
&\left(\Phi'_{\epsilon,\kappa}(R_1 + \epsilon z) + \lambda(R_1 + \epsilon z)\right) A(z) \\
&+ \frac{\mathcal{S}_n((R_1 + \epsilon z)/r_1)}{\mathcal{S}_n(r_2/r_1)} \frac{\epsilon}{n} \int_{-1}^{+1} (R_1 + \epsilon s)\varphi'_\kappa(-s)\mathcal{S}_n(r_2/(R_1 + \epsilon s))A(s)ds \\
&- \frac{\mathcal{S}_n((R_1 + \epsilon z)/r_1)}{\mathcal{S}_n(r_2/r_1)} \frac{\epsilon}{n} \int_{-1}^{+1} (R_2 + \epsilon s)\varphi'_\kappa(s)\mathcal{S}_n(r_2/(R_2 + \epsilon s))B(s)ds \\
&\qquad - \frac{\epsilon}{n} \int_{-1}^{z} (R_1 + \epsilon s)\varphi'_\kappa(-s)\mathcal{S}_n((R_1 + \epsilon z)/(R_1 + s))A(s)ds = 0.
\end{aligned}$$

Our ansatz

We have to solve, for λ and (A, B), the system

$$\left(\Phi'_{0,\kappa}(R_1) + \lambda R_1\right) A(z) + O(\epsilon) = 0,$$
$$\left(\Phi'_{0,\kappa}(R_2) + \lambda R_2\right) B(z) + O(\epsilon) = 0.$$

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Then, we introduce the ansatz

$$A(z) = \qquad\qquad A_1(z)\,\epsilon + A_2^{\epsilon}(z)\,\epsilon^2$$
$$B(z) = B_0(z) + B_1^{\epsilon}(z)\,\epsilon$$

together with

$$\lambda = \underbrace{-\frac{\Phi'_{0,\kappa}(R_1)}{R_1}}_{U_{TC}(R_1)} + \lambda_1\,\epsilon + \lambda_2^{\epsilon}\,\epsilon^2$$

Asymptotic analysis in terms of ϵ

At order $O(1)$ we have fixed

$$\lambda_0 = U_{TC}(R_1).$$

$\Phi'_{\epsilon,\kappa}(R_i + \epsilon z) + \lambda(R_i + \epsilon z) = \alpha_0^{R_i}[\lambda_0] + \alpha_1^{R_i}[\lambda_0, \lambda_1](z)\epsilon + \alpha_{2,\epsilon}^{R_i}[\lambda_0, \lambda_1, \lambda_2^{\epsilon}](z)\epsilon^2$

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At order $O(\epsilon)$ we obtain a closed system for λ_1 and A_1, B_0:

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$$\alpha_1^{R_2}[\lambda_0, \lambda_1](z)B_0(z) - \int_{-1}^{+1} \varphi'_\kappa(s)B_0(s)ds = 0.$$

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At higher order $O(\varepsilon^2)$ we have a CONTRACTION for λ_2^ϵ and $A_2^\epsilon, B_1^\epsilon$.

$\Phi'_{\epsilon,\kappa}(R_i + \epsilon z) + \lambda(R_i + \epsilon z) = \alpha_0^{R_i}[\lambda_0] + \alpha_1^{R_i}[\lambda_0, \lambda_1](z)\epsilon + \alpha_{2,\epsilon}^{R_i}[\lambda_0, \lambda_1, \lambda_2^\epsilon](z)\epsilon^2$

Solving the system at order $O(\epsilon)$

$$\alpha_0^{R_1}[\lambda_0]A_1(z) - \int_{-1}^{+1} \varphi'_\kappa(s)B_0(s)ds = 0 \quad \Longrightarrow \quad A_1 = \frac{1}{\alpha_0^{R_1}[\lambda_0]} \int_{-1}^{+1} \varphi'_\kappa(s)B_0(s)ds$$

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Finally, one finds that

$$1 \overset{(*)}{=} \int_{-1}^{+1} \frac{\varphi'_\kappa(s)}{\alpha_1^{R_2}[\lambda_0,\lambda_1](s)} ds$$

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Fixed $\lambda_{\epsilon,\kappa,m}$, we have proved that there exists (unique modulo multiplicative constant) $h_{\epsilon,\kappa,m} \in X$ such that $\mathcal{L}[\lambda_{\epsilon,\kappa,m}]h_{\epsilon,\kappa,m} = 0$.

The condition on R_1 and R_2

Using the above argument, we get time-periodic solutions with

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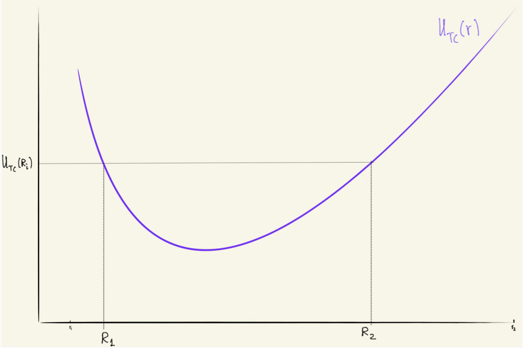
$$U_{TC}(R_1) \neq U_{TC}(R_2)$$



Note that $U_{TC}(R_1) = \lambda = U_{TC}(R_2) \implies \dim(\mathrm{Ker}(\mathcal{L}[\lambda])) > 1$.

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Then

$$\text{index} = \dim(\mathcal{N}(\mathcal{L})) - \dim(Y/\mathcal{R}(\mathcal{L}))$$

Theorem

Fixed $1 < M < \infty$. There exist $\epsilon_0(M)$, $\kappa_0(M)$ such that, for every $0 < \epsilon < \epsilon_0$, $0 < \kappa < \kappa_0$ and $m \in \mathbb{N}$, $m < M$, there exist $(\lambda^{\sigma}_{\epsilon,\kappa,m}, \mathsf{f}^{\sigma}_{\epsilon,\kappa,m})$ satisfying,

$$F_{\varpi_{\epsilon,\kappa}}[\lambda^{\sigma}_{\epsilon,\kappa,m}, \mathsf{f}^{\sigma}_{\epsilon,\kappa,m}] = 0,$$

parameterize by σ. These solutions satisfy:

1. $\mathsf{f}^{\sigma}_{\epsilon,\kappa,m}(r,\theta)$ is $\frac{2\pi}{m}$−periodic on θ.
2. The branch

$$\mathsf{f}^{\sigma}_{\epsilon,\kappa,m} = \sigma \mathsf{h}^{\sigma}_{\epsilon,\kappa,m} + o(\sigma),$$

   and the speed of the rotation is

$$\lambda^{\sigma}_{\epsilon,\kappa,m} = \lambda_{\epsilon,\kappa,m} + o(1).$$

3. $\mathsf{f}^{\sigma}_{\epsilon,\kappa,m}(r,\theta)$ depends on θ in a non-trivial way.

Theorem (continuation).

In addition, vorticity $\omega^\sigma_{\epsilon,\kappa,m}$ given implicitly by

$$\omega^\sigma_{\epsilon,\kappa,m}(\rho + \mathsf{f}^\sigma_{\epsilon,\kappa,m}(\rho,\theta),\theta) = \varpi_{\epsilon,\kappa}(\rho),$$

and extended to $[r_1, r_2] \times \mathbb{T}$ by ϵ and 0, yields a traveling way solution for 2D Euler in the sense that

$$\omega^\sigma_{\epsilon,\kappa,m}(r, \theta + \lambda^\sigma_{\epsilon,\kappa,m} t)$$

satisfies perturbed 2D Euler.
Importantly, $\omega^\sigma_{\epsilon,\kappa,m}(r,\theta)$ depends nontrivially on θ.

Pouseville flow

Work in progress

Pouseville flow $U_P(y) = y^2 - 1$ in $\mathbb{T} \times [-1, 1]$:

$$\partial_t \omega + \nabla^\perp \psi \cdot \nabla \omega + U(y)\partial_x \omega - \boxed{U''(y)\partial_x \psi} = 0, \qquad \Delta \psi = \omega.$$

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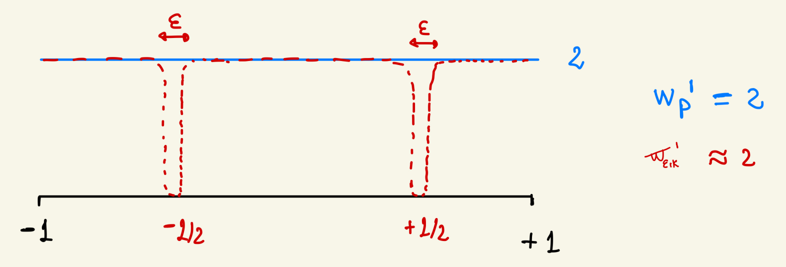
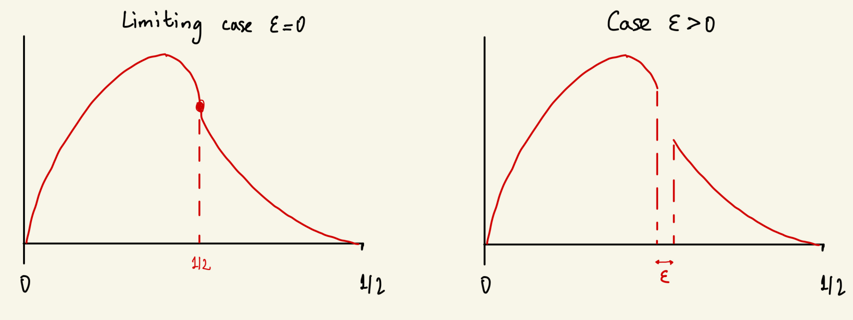


Desingularization



Thank you for your attention!