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Long memory in option pricing: A fractional discrete-time framework

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Outline

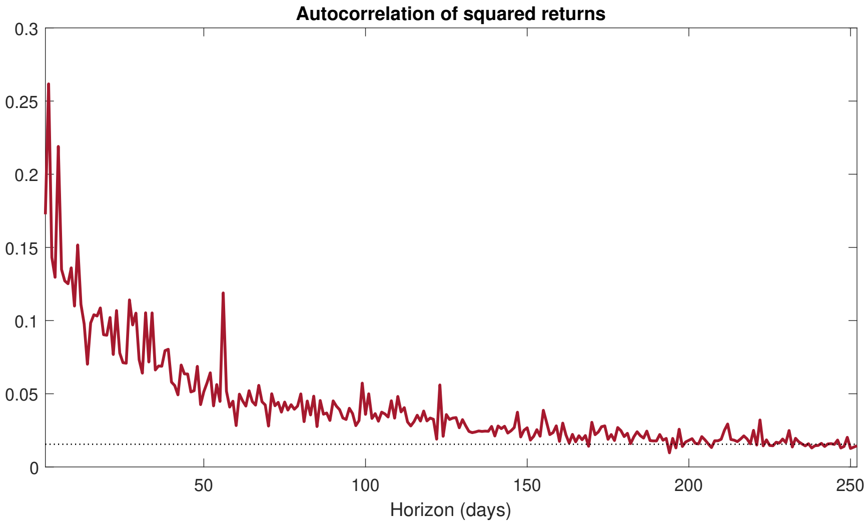
Motivation

Definition 1 (Long Memory).

A return series is said to feature long memory in volatility if the shocks to the conditional variance die out at a **slow hyperbolic rate**.

- Long memory in volatility models has been **popular** in the financial econometric literature.
- Long memory manifests itself when a time series' sample autocorrelation function (ACF) exhibits **significant autocorrelations** of squared returns over long lags (Ding et al., 1993; Ding and Granger, 1996).

Motivation, cont'd



Motivation, cont'd

- The ability of long-memory models to improve the **in-sample fit** of asset return distributions and **out-of-sample volatility forecasts** have both been widely studied (see, e.g., Baillie, 1996; Ding and Granger, 1996; Bollerslev and Mikkelsen, 1996; Mikosch and Stărică, 2004; Andersen et al., 2001; Maheu, 2005; Stărică and Granger, 2005).
- However, the impact of long memory for **option pricing** has been relatively unexplored.

Research Question.

Is long memory a relevant feature for option pricing?

Literature Review

Bollerslev and Mikkelsen (1996):

- Compared empirical performance of non-affine short- and **long-memory EGARCH models** using S&P 500 LEAPS from 1991–1993.
- Found that the prices of these option contracts are described **more accurately** when long-memory is included.

Wang (2007):

- Proposed an **affine version** of the fractional integrated model of Baillie (1996), extending Heston and Nandi (2000).
- Used S&P 500 options from 1990–1996.
- Found that a **two-component short-memory model** generates lower option RMSEs than long-memory models.

Literature Review, cont'd

Shortcoming of previous contributions:

- The proposed fractional models are **not (weakly) stationary**.
- Option prices are derived based on **monotonic pricing kernels**.
- Empirical analyses are solely based on parameters estimated using historical returns and do not incorporate the informational content from **option prices**.
- Analyses are performed over **short periods**.

Contributions

Theoretical: Development of a **discrete-time framework** for a general class of **affine component ARCH(∞) models** using a non-monotonic pricing kernel.

- → Semi-closed forms for a variety of European option payoffs.
- → Many existing GARCH option pricing models nested in the setting.
- → New (stationary) long-memory affine GARCH models by mixing short-memory and fractionally integrated processes.

Empirical: Investigation of the impact of long memory on option pricing using **joint estimation** based on returns and options on S&P 500 from 1996–2019.

- → Long-memory outperforms short-memory both in and out of sample.
- → Long-memory improvements are greater for LEAPS.

Outline

General Structure

Under the physical measure $\mathbb{P}$, the **return dynamics** are governed by

$$y_t = r + \lambda h_t + \sqrt{h_t} z_t, \quad z_t \sim \mathcal{N}(0,1),$$
$$h_t = F_\Theta \left(h_{t-1}, h_{t-2}, ..., z_{t-1}, z_{t-2}, ... \right),$$

where

- r is the constant risk-free interest rate,
- λ is the equity risk premium parameter,
- $h = \{h_t\}_{t \in \mathbb{Z}}$ is the conditional variance process,
- F_Θ is a non-linear function of the past variances and innovations, and
- Θ is set of parameters that satisfy certain non-negativity and stationarity constraints.

From Short-Memory to Fractional Models

- Starting from the **Heston and Nandi** (2000; HN hereafter) model, we have that

$$h_t = \omega + \beta h_{t-1} + \alpha \left(z_{t-1} - \gamma \sqrt{h_{t-1}} \right)^2 .$$

- Using the **lag operator notation *L***, it can be reparametrized:

$$h_t = \omega + \beta h_{t-1} + \psi^{\mathrm{HN}}(L) \left(z_t - \gamma \sqrt{h_t} \right)^2 ,$$

where

$$\psi^{\mathrm{HN}}(L) = \alpha L = \frac{1}{\gamma^2} \left(1 - \beta L - (1 - \phi L) \right) ,$$

and $\phi = \beta + \alpha\gamma^2$ measures the model persistency.

From Short-Memory to Fractional Models, cont'd

- Wang (2007) proposed a **fractional integrated version** of the HN model, called the FI model:

$$h_t = \omega + \beta h_{t-1} + \psi^{\mathrm{FI}}(L)\left(z_t - \gamma\sqrt{h_t}\right)^2, \tag{1}$$

where

$$\psi^{\mathrm{FI}}(L) = \frac{1}{\gamma^2}\left(1 - \beta L - (1 - \phi L)\,\boldsymbol{(1 - L)^d}\right) \tag{2}$$

and d is the **fractional differencing parameter** which characterizes the long memory.

Nested Case.

We obtain the HN variance dynamics if $d = 0$.

From Short-Memory to Fractional Models, cont'd

- We can rewrite the dynamics by using a Maclaurin series expansion of Equation (1):

$$h_t = \omega + \beta h_{t-1} + \sum_{j=1}^{\infty} \psi_j^{\mathrm{FI}} \left(z_{t-j} - \gamma \sqrt{h_{t-j}} \right)^2 ,$$

where

$$\psi_1^{\mathrm{FI}} = \frac{\phi - \beta + d}{\gamma^2} \quad \text{and} \quad \psi_j^{\mathrm{FI}} = \left(\frac{j-1-d}{j} - \phi \right) \delta_{j-1},$$

with

$$\delta_1 = \frac{d}{\gamma^2} \quad \text{and} \quad \delta_j = \delta_{j-1} \left(\frac{j-1-d}{j} \right), \quad j \geq 2.$$

Stationarity.

The FI model is **not** covariance stationary.

Building a Stationary Affine Fractional Model

- In the spirit of Davidson (2004), we introduce the **hyperbolic** (HY) model by mixing HN and FI:

$$\psi^{\text{HY}}(L) = (1 - \tau)\,\psi^{\text{HN}}(L) + \tau\psi^{\text{FI}}(L).$$

Stationarity.

The HY model is covariance stationary if and only if $(1 - \phi)(1 - \tau) > 0$.

ARCH(∞) Representations and Decays

- Let us assume an equivalent—but less convenient—ARCH(∞) representation:

$$h_t = \tilde{\omega} + \sum_{j=1}^{\infty} \tilde{\psi}_j \left(z_{t-j} - \gamma \sqrt{h_{t-j}} \right)^2 .$$

- The HN coefficients $\tilde{\psi}_j^{\mathrm{HN}}$ are characterized by a **geometric decay**,

$$\tilde{\psi}_j^{\mathrm{HN}} = \mathcal{O}(\beta^j).$$

- The HY and FI coefficients $\tilde{\psi}_j^{\mathrm{HY}}$ and $\tilde{\psi}_j^{\mathrm{FI}}$ are characterized by a **hyperbolic decay**

$$\tilde{\psi}_j^{\mathrm{HY}} = \mathcal{O}(j^{-1-d}) \quad \text{and} \quad \tilde{\psi}_j^{\mathrm{FI}} = \mathcal{O}(j^{-1-d}), \quad \text{for } 0 < d < 1,$$

  leading to long memory.

Affine Multi-Component Fractional Models

Multi-Component.

Although the HY is a combination of short- and long-memory models, the single volatility regime may **not be rich enough** to capture the market behaviour over different periods.

- We introduce the FI-HN model as a **mixture of HN and FI components**:

$$h_t = w_1 \sigma^2_{1,t} + w_2 \sigma^2_{2,t}, \quad w_1, w_2 \geq 0,$$
$$\sigma^2_{1,t} = \omega_1 + \beta_1 \sigma^2_{1,t-1} + \psi_1^{\mathrm{FI}}(L) \left(z_t - \gamma_1 \sqrt{h_t}\right)^2,$$
$$\sigma^2_{2,t} = \omega_2 + \beta_2 \sigma^2_{2,t-1} + \psi_2^{\mathrm{HN}}(L) \left(z_t - \gamma_2 \sqrt{h_t}\right)^2.$$

- The FI-HN model can be extended to a **HY-HN structure** by replacing $\psi_1^{\mathrm{FI}}(L)$ with $\psi_1^{\mathrm{HY}}(L)$.

Affine Multi-Component Fractional Models, cont'd

Nested Case.

We can obtain a **two-component short-memory model** by setting $d = 0$, similar to the model proposed by Christoffersen et al. (2008). It is denoted by HN-HN henceforth.

Summary of Nested Competing Models

1. **HN model**: affine GARCH(1,1) model of Heston and Nandi (2000).
2. **FI model**: the affine version of the fractionally integrated variance process proposed by Wang (2007).
3. **HY model**: a hyperbolic fractionally integrated version of the HN model, similar in spirit to Davidson (2004).
4. **HN-HN model**: a two-component model for which both components are short memory HN model, similar to Christoffersen et al. (2008).
5. **FI-HN model**: a two-component model with the first variance component given by the FI model and the second component by the HN model.
6. **HY-HN model**: a two-component model with the first variance component following the HY model and the second component the HN model.

Outline

Component Affine ARCH(∞) Models

- We develop a **pricing framework** for the valuation of European-style derivatives assuming the following $\mathbb{P}$-dynamics:

$$
\begin{aligned}
y_t &= r + \lambda h_t + \sqrt{h_t} z_t, \quad z_t \sim \mathcal{N}(0,1), \\
h_t &= \boldsymbol{w}^\top \boldsymbol{\sigma}_t^2, \quad \boldsymbol{w} = [\ w_1 \quad w_2\ ]^\top \geq \boldsymbol{0}, \\
\boldsymbol{\sigma}_t^2 &= \boldsymbol{\omega} + \boldsymbol{\beta} \odot \boldsymbol{\sigma}_{t-1}^2 + \sum_{j=1}^{\infty} \boldsymbol{\psi}_j \odot \boldsymbol{l}_{t-j},
\end{aligned}
$$

where $\boldsymbol{\sigma}_t^2 \equiv [\ \sigma_{1,t}^2 \quad \sigma_{2,t}^2\ ]^\top$ is the two-dimensional vector of conditional variance components which admit ARCH(∞) representations driven by the noise $\boldsymbol{l}_t \equiv [\ l_{1,t} \quad l_{2,t}\ ]^\top$ defined as:

$$
l_{k,t} = \left(z_t - \gamma_k \sqrt{h_t}\right)^2, \quad \text{for } k = 1, 2,
$$

where γ_k is the k^{th} component leverage parameter.

Component Affine ARCH(∞) Models, cont'd

More Component?

The pricing framework is extended to K components in the article.

Non-Monotonic Pricing Kernel

- The $\mathbb{P}$-equivalent **risk-neutral probability measure** $\mathbb{Q}$ is defined by:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \prod_{s\leq t} \exp\left(\theta_Y Y_s + \boldsymbol{\theta}_\sigma^\top \boldsymbol{\sigma}_{s+1}^2 - \mathcal{G}^{\mathbb{P}}_{(Y_s,\boldsymbol{\sigma}_{s+1}^2,\boldsymbol{l}_s)}\left(\theta_Y,\boldsymbol{\theta}_\sigma,\mathbf{0} \mid \mathcal{F}_{s-1}\right)\right),$$

  where $\mathcal{G}^{\mathbb{P}}_{(Y_s,\boldsymbol{\sigma}_{s+1}^2,\boldsymbol{l}_s)}\left(\theta_Y,\boldsymbol{\theta}_\sigma,\mathbf{0} \mid \mathcal{F}_{s-1}\right)$ is the joint cgf of Y_s, $\sigma_{1,s+1}^2$, and $\boldsymbol{l}_s$.

- Here, θ_Y and $\boldsymbol{\theta}_\sigma = [\ \theta_{1,\sigma} \ \ \theta_{2,\sigma}\ ]^\top$ represent the equity and the vector of variance component risk preference parameters, respectively, and satisfy:

$$\theta_Y = -\lambda - \frac{1}{2} + 2\left(\boldsymbol{\theta}_\sigma \odot \boldsymbol{\psi}_1\right)^\top\left(\boldsymbol{\lambda} + \boldsymbol{\gamma}\right),$$

  where $\boldsymbol{\lambda}$ is a K-dimensional vector for which all components are equal to λ.

- This is called the **no-arbitrage constraint** because it ensures that the discounted asset price is a martingale under $\mathbb{Q}$.

Valuation of European-Style Derivatives

- We use the **inverse Laplace representation** of the option payoff.
- A European call payoff with strike X admits the following representation:

$$H = f(Y_T) = \max\left[e^{Y_T} - X, 0\right] = \frac{1}{2\pi\mathrm{i}} \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} e^{zY_T} \check{f}(z)\, dz, \quad \text{for any } R > 1,$$

  where the kernel function is given by $\check{f}(z) = X^{1-z} / (z(z-1))$.
- The time-t price of an option with a maturity of $T - t$ and a strike of X is:

$$O_t^{\text{Model}}(X, T) = \frac{e^{-r(T-t)}}{2\pi\mathrm{i}} \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} \exp\left(\mathcal{G}_{Y_T}^{\mathbb{Q}}(z \mid \mathcal{F}_t)\right) \check{f}(z)\, dz, \tag{3}$$

  where $\mathcal{G}_{Y_T}^{\mathbb{Q}}(z \mid \mathcal{F}_t)$ is the risk-neutral cgf of the terminal log-price Y_T conditional on $\mathcal{F}_t$.

Valuation of European-Style Derivatives, cont'd

Proposition 1 (Log-Price Cumulant Generating Function).

For any real u and for any $t, T \in \mathbb{Z}$ with $t \leq T$, the terminal conditional cgf of the log-price $Y_T = \log S_T$ is given by

$$\mathcal{G}_{Y_T}^{\mathbb{Q}}(z \mid \mathcal{F}_t) = \mathcal{A}^*(z; t, T) + zY_t + \boldsymbol{\mathcal{B}}^*(z; t, T)^\top \boldsymbol{\sigma}_{t+1}^{*2} + \sum_{j=1}^{\infty} \boldsymbol{C}_j^*(z; t, T)^\top \boldsymbol{I}_{t+1-j}^*,$$

where

$$\boldsymbol{\sigma}_{t+1}^{*2} = \pi \boldsymbol{\sigma}_{t+1}^2, \qquad \boldsymbol{I}_{t+1-j}^* = \frac{\boldsymbol{I}_{t+1-j}}{\pi}, \quad \text{with} \quad \pi = \frac{1}{1 - 2\left(\boldsymbol{\theta}_\sigma \odot \boldsymbol{\psi}_1\right)^\top \mathbf{1}},$$

The coefficients $\mathcal{A}^*(z; t, T), \boldsymbol{\mathcal{B}}^*(z; t, T)$, and $\boldsymbol{C}_j^*(z; t, T)$ satisfy some recursions.

Outline

Data

- Daily S&P 500 index **returns** from January 1976 to December 2019 obtained from the Center for Research in Security Prices (CRSP).
  - → The estimation sample begins in 1996, and the returns from January 1976 to December 1995 are used to warm up the filter.
  - → We use a total of 6,042 daily returns in the estimation.
- Three-month Treasury bill rate from the Federal Reserve Board's H.15 report.
- OTM S&P 500 **put and call implied volatilities** on Wednesdays from January 1996 to December 2019 extracted from OptionMetrics.
  - → We use the usual filters (see, e.g., Bakshi et al., 1997; Carr and Wu, 2011; Christoffersen et al., 2012, 2013).
  - → We select the six most liquid options (based on volume) for each maturity and date, and we end up with 45,084 options.

Estimation Methodology

- Return-and option-based joint maximum likelihood estimation:

$$\ell^{\text{Joint}}(\Theta) = \frac{T+N}{2}\left(\frac{\ell^{\text{Returns}}(\Theta)}{T} + \frac{\ell^{\text{Options}}(\Theta)}{N}\right).$$

where

$$\ell^{\text{Returns}}(\Theta) = \log \prod_{t=t_0+1}^{T} \frac{1}{2\pi h_t} \exp\left(-\frac{1}{2}\frac{(y_t - r - \lambda h_t)^2}{h_t}\right),$$

$$\ell^{\text{Options}}\left(\Theta \,\middle|\, \left\{\{\mathrm{IV}_{t,i}\}_{i=1}^{n_t}\right\}_{t=t_0+1}^{T}\right) = \log \prod_{t=t_0+1}^{T} \prod_{i=1}^{n_t} \frac{1}{2\pi s_\epsilon^2} \exp\left(-\frac{1}{2}\frac{\epsilon_{t,i}^2}{s_\epsilon^2}\right),$$

and $\epsilon_{t,i} = \mathrm{IV}_{t,i} - \mathrm{IV}(O_t^{\text{Model}}(X_{t,i}, T_{t,i}))$.

Implementation

- Throughout our calculations and variance updates, infinite sums need to be truncated. We use a value of **1,000 lags**—an optimal tradeoff in terms of accuracy and computational speed.
- To begin our variance update recursions, we fix all the pre-sample terms (i.e., $t < 1$) to their **unconditional average**: we use the unconditional average level for each component and the unconditional expectation for the leveraged terms.
- Option prices are obtained by applying a simple quadrature method (i.e., **the trapezoidal rule**) to Equation (3) with 1,000 nodes.
- The risk-neutral cgf relies on a truncation of **1,000 lags**, similar to the number of lags used in the variance update calculation.

Outline

Joint Maximum Likelihood Estimates

	One-component models			Two-component models		
	HN	FI	HY	HN-HN	FI-HN	HY-HN
λ	2.14	0.00	1.77	2.98	3.15	3.16
⋮	⋮	⋮	⋮	⋮	⋮	⋮
τ	–	–	0.97	–	–	1.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮
d	–	0.45	0.49	–	0.36	0.36
⋮	⋮	⋮	⋮	⋮	⋮	⋮
π	1.06	1.06	1.08	1.19	1.04	1.04
Log-likelihood						
Return	19,496	19,531	19,499	19,501	19,512	19,512
Option	82,616	88,589	89,086	90,422	90,870	90,878
Joint	129,330	132,865	133,012	133,777	134,077	134,080
IVRMSE (%)	3.87	3.39	3.35	3.26	3.22	3.22

Out-of-Sample Study

- We focus on the **last 10 years** of our sample.
- We estimate all models jointly on returns and options using an **expanding window**.
- We then compute implied volatility on the options traded in the year **following the end of the sample**.
- The approach is similar in spirit to the out-of-sample analyses performed by Huang and Wu (2004) and Christoffersen et al. (2009).

Out-of-Sample Implied Volatility RMSEs

Panel A: Out-of-sample IVRMSEs.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
	3.77	3.03	3.04	3.09	2.97

Panel B: Out-of-sample IVRMSEs per year.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
2010	4.80	3.50**	4.12	3.89*	3.95
2011	4.19	4.18*	4.49	4.34	3.90**
2012	3.52	2.81	2.59**	2.88	2.76*
2013	2.18	1.73*	1.60**	1.93	1.97
2014	2.70	1.97	1.85*	2.18	1.82**
2015	3.16	2.41	2.17*	2.32	2.04**
2016	3.49	3.10	3.32	2.64**	2.75*
2017	3.76	3.59	2.80**	2.96*	3.14
2018	5.12	3.49**	3.67*	4.06	3.70
2019	3.62	2.64**	2.72*	2.81	2.76
Count, Best model	0	**3**	**3**	1	**3**
Count, Second-best model	0	**2**	**4**	2	**2**

Diebold–Mariano Test Statistics

Panel A: Time-series mean of the weekly IVRMSEs.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
Mean	3.41	2.70	2.69	2.76	2.61
Standard deviation	(1.56)	(1.39)	(1.47)	(1.40)	(1.44)

Panel B: DM pairwise statistics for weekly IVRMSEs.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
HN		**10.81**	**9.98**	**11.14**	**12.91**
FI			–0.62	–1.22	1.38
HY				–0.47	1.86
HN-HN					**2.88**

LEAPS: Out-of-Sample Implied Volatility RMSEs

Panel A: Out-of-sample IVRMSEs for LEAPS.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
	4.07	3.07	3.09	3.52	3.14

Panel B: Out-of-sample IVRMSEs per year for LEAPS.

	One-component models			Two-component models	
	HN	FI	HY	HN-HN	FI-HN
2010	4.81	4.30**	4.73	4.71*	4.88
2011	3.58	3.62	3.49*	3.68	3.19**
2012	4.50	4.39**	4.39*	4.77	4.53
2013	3.62	3.21	2.98*	3.48	2.92**
2014	3.69	2.21	2.11*	2.21	2.09**
2015	3.19	1.62	1.44*	2.21	1.42**
2016	3.05	2.16	2.09*	2.46	1.72**
2017	4.43	2.99	2.47**	3.20	2.65*
2018	5.13	2.73**	3.23*	4.20	3.55
2019	4.02	2.63**	2.92*	3.33	3.11
Count, Best model	0	**4**	**1**	0	**5**
Count, Second-best model	0	**0**	**8**	1	**1**

LEAPS: Diebold–Mariano Test Statistics

Panel A: Time-series mean of the weekly IVRMSEs for LEAPS.					
	One-component models			**Two-component models**	
	HN	FI	HY	HN-HN	FI-HN
Mean	3.84	2.84	2.81	3.26	2.82
Standard deviation	(1.33)	(1.31)	(1.43)	(1.43)	(1.53)
Panel B: DM pairwise statistics for weekly IVRMSEs for LEAPS.					
	One-component models			**Two-component models**	
	HN	FI	HY	HN-HN	FI-HN
HN		**11.78**	**12.63**	**8.65**	**12.60**
FI			–0.54	**–8.17**	–1.55
HY				**–11.32**	–1.32
HN-HN					**10.11**

Outline

Concluding Remarks

- We propose **new covariance-stationary long-memory models** by mixing short-memory and fractionally integrated processes.
  - → This specification leads to semi-closed forms for the **valuation of European-style derivatives** for a general class of affine multi-component ARCH(∞) volatility processes.
- Using S&P 500 option data (including LEAPS), we investigate the impact of long-memory dynamics in volatility for option pricing.
  - → Once the informational content from options is incorporated into the parameter estimation process, their **out-of-sample pricing performance stands out**.
  - → This suggests that long memory captures better the distributional properties of risk-neutral variance forecasts.

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Appendix

Maximum Likelihood Estimates

	One-component models			Two-component models		
	HN	FI	HY	HN-HN	FI-HN	HY-HN
λ	2.44	1.88	2.33	2.42	2.42	2.42
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
τ	–	–	0.87	–	–	0.99
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
d	–	0.19	0.41	–	0.23	0.23
Log-likelihood						
Return	19,630	19,663	19,668	19,688	19,689	19,689
Option	65,198	69,170	71,868	70,262	70,405	70,405
Joint	120,021	122,413	123,963	123,136	123,220	123,220
IVRMSE (%)	5.70	5.22	4.91	5.09	5.08	5.08

Annualized Volatility Forecasts