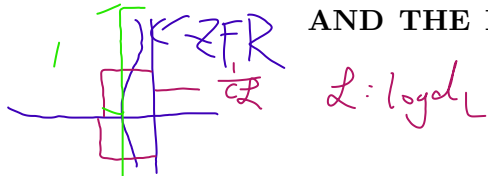*(Handwritten annotations at top:)*

$\zeta_L(s) = \sum\limits_{\mathfrak{a} \neq 0} \frac{1}{N\mathfrak{a}^s} = \prod\limits_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$

integral ideals in $\mathcal{O}_L$ prime ideals in $\mathcal{O}_L$ absol

L: no field of degree $n_L = [L:\mathbb{Q}] > 1$ and disc d_L

ZERO-FREE REGIONS OF DEDEKIND ZETA FUNCTIONS
AND THE DEURING-HEILBRONN PHENOMENON

(Handwritten left:) $)K$-ZFR $\frac{1}{c\mathcal{L}}$ $\mathcal{L} : \log d_L$

(Handwritten right:) by Hadamard's 3-4-1 trick $\Rightarrow \zeta_L(s) \neq 0$ for $\mathrm{Re}(s) = 1$

PENG-JIE WONG

To improve this note, please do not hesitate to ask for clarification and/or point out any typos or factual errors. For detailed references, please see Kadiri's note and the attached article of Kadiri-Ng-W. (2019).

1. STANDARD ZERO-FREE REGIONS OF DEDEKIND ZETA FUNCTIONS

Let $L \neq \mathbb{Q}$ be a number field of degree n_L and with absolute discriminant d_L.

- The Dedekind zeta function $\zeta_L(s)$ of admits at most one "exceptional zero" in the region

$$\mathrm{Re}(s) > 1 - \frac{1}{c \log d_L} \text{ and } |\mathrm{Im}(s)| < \frac{1}{c \log d_L};$$

such an exceptional zero, if it exists, must be real and simple, which will be denoted by β_1.
For all $L \neq \mathbb{Q}$: Stark (1974) $c = 4$; Ahn and Kwon (2019) $c = 2$.
(Note: Kadiri and W. (2021) $c = 1.7$ if β_1 exists.)

- $\zeta_L(s)$ vanishes at most once in the region

(Handwritten left:) $d_{K_g} \approx g^{\varphi(g)}$

$$\mathrm{Re}(s) > 1 - \frac{1}{r \log d_L} \text{ and } |\mathrm{Im}(s)| \leq 1.$$

For d_L sufficiently large: Kadiri (2012) $r = 12.8$; Lee (2021) $r = 12.5$.
For all $L \neq \mathbb{Q}$: Ahn and Kwon (2019) $r = 29.6(1 + 2)$.
(Here, 2 comes from Minkowski's bound $n_L \leq \frac{2 \log d_L}{\log 3}$.)

(Handwritten right:)
K_g: g^{th} cy field $= \mathbb{Q}(\zeta_g)$
$\zeta_{K_g}(s) = \prod\limits_{\chi (\text{mod } g)} L(s,\chi)$
$\mathrm{Re}(s) > 1 - \frac{1}{R \log(g \cdot |t|)}$

- For $|\mathrm{Im}(s)| > 1$, $\zeta_L(s)$ does not vanish in

$$\mathrm{Re}(s) > 1 - \frac{1}{c_1 \log d_L + c_2 n_L \log |\mathrm{Im}(s)| + c_3 n_L + c_4}.$$

For all $L \neq \mathbb{Q}$: Kadiri (2012) $c_1 = 12.6, c_2 = 9.7, c_3 = 3.1, c_4 = 58.7$; Lee (2021) $c_1 = 12.3, c_2 = 9.6, c_3 = 0.1, c_4 = 2.3$.

2. THE DEURING-HEILBRONN PHENOMENON FOR DEDEKIND ZETA FUNCTIONS

See pp. 2290-2297 (from Theorem 1.2 to the proof of Theorem 2.8) of KNW. (Note: explicit versions of the Deuring-Heilbronn phenomenon *all* number fields were obtained by Ahn-Kwon (2019) and Kadiri-W. (2021).)

3. HOMEWORK

(Handwritten: $(2.15 + 16)$)

- (Standard) Go through the proof of Lemma 2.5 of Zaman (2017).
- (Easy) Verify Eq. (2.5) and (2.6) of KNW. *(Handwritten:)* Also, prove Lemma 2.1 of KNW.
- (Tedious) Give details for the proofs of Lemma 2.7 and Theorem 2.8 of KNW.
- (Challenging) Figure out a way to extend the argument of KNW to Dirichlet L-functions.

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THE LEAST PRIME IDEAL
IN THE CHEBOTAREV DENSITY THEOREM

HABIBA KADIRI, NATHAN NG, AND PENG-JIE WONG

(Communicated by Amanda Folsom)

ABSTRACT. In this article, we prove a new bound for the least prime ideal in the Chebotarev density theorem, which improves the main theorem of Zaman [Funct. Approx. Comment. Math. 57 (2017), no. 1, 115–142] by a factor of 5/2. Our main improvement comes from a new version of Turán's power sum method. The key new idea is to use Harnack's inequality for harmonic functions to derive a superior lower bound for the generalised Fejér kernel.

1. INTRODUCTION

Let L/K be a Galois extension of number fields with Galois group $\mathcal{G}$, and let $\mathcal{C} \subset \mathcal{G}$ be a conjugacy class. Attached to each unramified prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, the ring of integers of K, is the Artin symbol $\sigma_\mathfrak{p}$, a conjugacy class in $\mathcal{G}$. The famous Chebotarev density theorem asserts that

$$\#\{\mathfrak{p} \subset \mathcal{O}_K \mid N\mathfrak{p} \leq x,\ \sigma_\mathfrak{p} = \mathcal{C}\} \sim \frac{|\mathcal{C}|}{|\mathcal{G}|}\mathrm{Li}(x)$$

as $x \to \infty$, where $\mathrm{Li}(x)$ is the usual logarithmic integral and $N = N_\mathbb{Q}^K$ is the absolute norm of K. This is a vast generalisation of both the prime number theorem and Dirichlet's theorem on primes in arithmetic progressions.

In light of Linnik's celebrated result on the least prime in an arithmetic progression, one may ask for a bound for the least prime ideal whose Artin symbol equals $\mathcal{C}$. Assuming the Generalized Riemann Hypothesis for the Dedekind zeta function of L, Lagarias and Odlyzko [10] showed that $N\mathfrak{p} \ll (\log d_L)^2$, where $d_L = |\mathrm{disc}(L/\mathbb{Q})|$ is the absolute discriminant of L (see also Bach and Sorenson [2]). The first unconditional result is due to Lagarias, Montgomery, and Odlyzko [9], who proved there exists a positive constant B such that there is a prime ideal $\mathfrak{p}$ with $\sigma_\mathfrak{p} = \mathcal{C}$ and $N\mathfrak{p} \leq d_L^B$ for $L \neq \mathbb{Q}$. Recently, Ahn and Kwon [1] established that $B = 12577$ is valid and Zaman [13] established that $B = 40$ is valid when d_L is sufficiently large.

The main result of this article is the following theorem.

Theorem 1.1. *Let L/K be a Galois extension of number fields with $\mathcal{G} = \mathrm{Gal}(L/K)$, and let $\mathcal{C} \subset \mathcal{G}$ be a conjugacy class. There exists an unramified prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, of degree one, such that $\sigma_{\mathfrak{p}} = \mathcal{C}$ and $N\mathfrak{p} \leq d_L^{16}$ when d_L is sufficiently large.*

Our main improvement is a stronger zero-repulsion theorem than in [13]. We now state our result. Let $\zeta_L(s)$ denote the Dedekind zeta function of L. It was shown in [6] that for d_L sufficiently large, $\zeta_L(s)$ has no zeros in the region

$$(1.1) \qquad \mathfrak{Re}(s) \geq 1 - \frac{1}{12.74 \log d_L} \text{ and } |\mathfrak{Im}(s)| \leq 1$$

with the exception of at most one real zero β_1. The Deuring–Heilbronn phenomenon roughly asserts that if an *exceptional zero* β_1 exists, then the zero-free region for $\zeta_L(s)$ can be enlarged. Indeed, we have the following explicit version of this phenomenon.

Theorem 1.2. *Assume $\zeta_L(s)$ admits an exceptional real zero β_1. If $\beta' + i\gamma'$ is another zero of $\zeta_L(s)$ satisfying $\beta' \geq \frac{1}{2}$ and $|\gamma'| \leq 1$, then for d_L sufficiently large,*

$$(1.2) \qquad \beta' \leq 1 - \frac{\log\left(\frac{\kappa}{(1-\beta_1)\log d_L}\right)}{14.144 \log d_L}$$

for some absolute $\kappa > 0$.

We remark that Zaman [13] established the above result with 35.8 in place of our 14.144, and that a superior Deuring–Heilbronn phenomenon was obtained by the first two authors for zeros much closer to $s = 1$ (see [7, Theorem 4]). The proof of Theorem 1.2 is a special case of the more general Theorem 2.6 below.

Also, we have a variant of the above theorem for real zeros.

Theorem 1.3. *If $\zeta_L(s)$ has a real zero β_1 and another real zero $\beta' \in (0,1)$, then there is an absolute constant $\kappa > 0$ such that for d_L sufficiently large,*

$$(1.3) \qquad \beta' \leq 1 - \frac{\log\left(\frac{\kappa}{(1-\beta_1)\log d_L}\right)}{7.071998 \log d_L}.$$

The proof of Theorem 1.3 is a special case of the more general Theorem 2.8 below. This gives an upper bound for the exceptional zero β_1 of $\zeta_L(s)$.

Corollary 1.3.1. *For d_L sufficiently large, if $\zeta_L(s)$ has a real zero β_1, then*

$$1 - \beta_1 \gg d_L^{-7.072},$$

where the implied constant is absolute and effectively computable.

Note that Theorem 1.3 and Corollary 1.3.1 improve results of Zaman, who obtained the above bounds with 16.6 in place of our 7.071998 and 7.072. The proof of Corollary 1.3.1 follows from applying Theorem 1.3 to the zero $\beta' = 1 - \beta_1$.

Remarks. We actually obtain the bound d_L^B with $B = 15.72$ in Theorem 1.1, and this likely can be slightly improved. It seems that the limit of our method is roughly the constant 14.144 which appears in the Deuring–Heilbronn type zero-free region as given in Theorem 1.2. More precisely, as can be seen in (3.19), the exponent 15.72 arises from the value $C_1 = 14.58$ in Theorem 2.6 (see equation (2.21)). In fact, our constant B is chosen to satisfy the constraints (3.19) and (3.20) below. Finally, it should be mentioned that the work of Thorner and Zaman [12] gives a significant improvement to Theorem 1.1 in the case that there is a large abelian subgroup of G intersecting C.

(handwritten at top: $1 + \cos x \geq 0$)

2. Deuring–Heilbronn via the Turán power sum method

2.1. **A new version of Turán's power sum method.** In this section, we shall prove a version of Turán's power sum method in much the same spirit as Lagarias et al. [9], who made use of properties of functions of the form

$$P(r,\theta) = \sum_{j=1}^{J} \left(1 - \frac{j}{J+1}\right) r^j \cos(j\theta),$$

where $0 \leq r \leq 1$. We have the following facts concerning $P(r,\theta)$.

(handwritten right margin: $z = re^{i\theta}$; $\longleftrightarrow (x,y) = (r\cos\theta, r\sin\theta)$)

Lemma 2.1. *Let $\theta \in \mathbb{R}$. We have*
  (i) $P(r,\theta) \geq -\frac{1}{2}$ *for* $0 \leq r \leq 1$,
  (ii) $P(1,0) = \frac{J}{2}$,
  (iii) $P(r,\theta) \geq -\frac{r}{1+r}$ *for* $0 \leq r \leq 1$.

(handwritten right margin: $1 + 2P = \mathrm{Re}\left(1 + 2\sum_{J=1}^{J}(1 - \frac{j}{J+1})z^{j}\right)$)

Proof.

  (i) Note that $1 + 2P(1,\theta)$ is Fejér's kernel and thus nonnegative. It follows that $P(1,\theta) \geq -\frac{1}{2}$. Since $P(r,\theta)$ is a harmonic function, we have $P(r,\theta) \geq -\frac{1}{2}$ for $0 \leq r < 1$.

  (ii) is a direct calculation.

  To prove (iii), we first note that the case $r = 1$ follows from (i), so we may assume that $0 \leq r < 1$. We then observe that by writing $(x,y) = (r\cos\theta, r\sin\theta)$, $F(x,y) = 1 + 2P(r,\theta)$ is the real part of the kernel $1 + 2\sum_{j=1}^{J}(1 - \frac{j}{J+1})r^j e^{ij\theta}$. Thus, F is a nonnegative harmonic function defined on the unit ball, centred at $(0,0)$, of $\mathbb{R}^2$. Recall that Harnack's inequality (see, e.g., [8, Theorem 2.1]) asserts that for any nonnegative harmonic function H defined over an open ball $\|(x,y)\|_{\mathbb{R}^2} < R$, then for any $\|(x,y)\|_{\mathbb{R}^2} = r$ with $r < R$,

(handwritten left margin: min princ.)
(handwritten right margin: (exercise))

$$\frac{R-r}{R+r}H(0,0) \leq H(x,y).$$

Applying this with $H = F$ and $R = 1$, we have $\frac{1-r}{1+r}F(0,0) \leq F(x,y)$ for any $\|(x,y)\|_{\mathbb{R}^2} = r < 1$. As $F(0,0) = 1$, we then deduce that for any $r < 1$, $\frac{1-r}{1+r} - 1 \leq 2P(r,\theta)$, which together with a simplification completes the proof. $\square$

(handwritten right margin: exercise)

Remarks.

  (i) The authors of [9] and [13] use the lower bound $P(r,\theta) \geq -\min(\frac{3r}{2}, \frac{1}{2})$ for $0 \leq r \leq 1$ to obtain a version of Turán's power sum method. The main improvements obtained in Theorems 1.1, 1.2, 1.3, and 2.2 come from our use of Lemma 2.1(iii).

  (ii) As the proof of Theorem 2.2 relies on Lemma 2.1(ii), one may try to replace $P(r,\theta)$ by a function of the form $\widetilde{P}(r,\theta) = \sum_{j=1}^{J} a_j r^j \cos(j\theta)$, which satisfies the first condition of Lemma 2.1 and $\widetilde{P}(1,0) > \frac{J}{2}$. This leads to the extremal problem of finding $\max\{a_1 + \cdots + a_J \mid \sum_{j=1}^{J} a_j \cos(j\theta) \geq -\frac{1}{2}\}$. However, $P(1,\theta)$ is, in fact, *the* solution of such a problem (see, for example, [4, Section 4]).

  Now we are in a position to prove a new version of Turán's power sum method. This refines [9, Theorem 4.2] and will help us to enlarge the zero-free region later.

Theorem 2.2. *Let $\varepsilon > 0$. For any $j \in \mathbb{N}$, set $s_j = \sum_{n \geq 1} b_n z_n^j$, and assume that*
  (i) $z_n \in \mathbb{C}$ *and* $|z_n| \leq |z_1|$ *for every* $n \geq 1$ *where* $z_1 \neq 0$,
  (ii) *each b_n is nonnegative and $b_1 > 0$.*

Set $M = b_1^{-1} \sum_{n\geq 1} \frac{b_n|z_n|}{|z_1|+|z_n|}$. *Then there exists j with $1 \leq j \leq (8+\varepsilon)M$ such that*

$$\mathfrak{Re}(s_j) \geq \frac{b_1\varepsilon}{32+4\varepsilon}|z_1|^j.$$

Proof. Let $\varepsilon > 0$. By homogeneity, we may assume $|z_1| = 1$. Write $z_n = r_n e^{i\theta_n}$ where $0 \leq r_n \leq 1$. Note that $r_1 = 1$. Observe that we have the inequality

$$(2.1) \qquad \sum_{j=1}^{J}\Big(1 - \frac{j}{J+1}\Big)\mathfrak{Re}(s_j)(1 + \cos(j\theta_1)) \leq J\Big(\max_{1\leq j\leq J}\mathfrak{Re}(s_j)\Big).$$

By definition, the sum on the left of (2.1) equals

$$(2.2) \qquad \sum_{n\geq 1} b_n\Big(P(r_n,\theta_n) + \frac{1}{2}P(r_n,\theta_n - \theta_1) + \frac{1}{2}P(r_n,\theta_n + \theta_1)\Big).$$

By Lemma 2.1(ii) and (iii), a lower bound for the $n = 1$ term is

$$(2.3) \qquad b_1\Big(-\frac{r_1}{1+r_1} + \frac{J}{4} - \frac{1}{2}\frac{r_1}{1+r_1}\Big) = b_1\Big(\frac{J+1}{4} - 2\frac{r_1}{1+r_1}\Big).$$

For the terms $n \geq 2$, Lemma 2.1(iii) bounds the remaining sum below by

$$(2.4) \qquad \sum_{n=2}^{\infty} -b_n\frac{r_n}{1+r_n}\Big(1 + \frac{1}{2} + \frac{1}{2}\Big) = -2\sum_{n=2}^{\infty}\frac{b_n r_n}{1+r_n}.$$

Combining (2.3) and (2.4) it follows that a lower bound for (2.2) is

$$b_1\Big(\frac{J+1}{4} - 2M\Big) \geq b_1\frac{\varepsilon M}{4} \text{ for } J = \lfloor(8+\varepsilon)M\rfloor.$$

In particular, there exists $1 \leq j \leq (8+\varepsilon)M$ such that $\mathfrak{Re}(s_j) \geq \frac{b_1\varepsilon M}{4J} \geq \frac{b_1\varepsilon}{4(8+\varepsilon)}$. $\qquad\square$

Remarks.

(i) Here we only apply this with $b_n \equiv 1$, which was also the case for [13]. In addition, thanks to Lemma 2.1 we reduce to 8 and, respectively, to 32 the constants 12 and 48 appearing in [13, Theorem 2.3].

(ii) This theorem can be used to improve the main theorem in [1] and some of the theorems in [12]. This is currently a work in progress.

2.2. **A quantitative version of the Deuring–Heilbronn phenomenon.** In this section, we will employ our version of Turán's power sum method to prove a quantitative version of the Deuring-Heilbronn phenomenon. We require a result of Odlyzko on discriminants of number fields. (We shall note that the bound (2.5) below is, in fact, a weak form of [11, Theorem 1, Eq. (4)].)

Theorem 2.3 (Odlyzko, [11]). *Let L be a number field of degree n_L and discriminant d_L. Let $r_1 = r_1(L)$ and $2r_2 = 2r_2(L)$ denote the number of real and complex embeddings of L, respectively. For d_L sufficiently large,*

$$(2.5) \qquad (\log 60)\cdot r_1 + (\log 22)\cdot 2r_2 \leq \log d_L.$$

We shall also borrow a lemma from Zaman, which bounds certain sums of zeros of Dedekind zeta functions. For $\alpha \geq 1$ and $t \in \mathbb{R}$, we define

$$(2.6) \qquad S_L(\alpha, t) = \sum_{\rho}\Big(\frac{1}{|\alpha + 1 - \rho|^2} + \frac{1}{|\alpha + 1 + it - \rho|^2}\Big),$$

where ρ ranges through nontrivial zeros of $\zeta_L(s)$. We further define the functions

$$G_1(\alpha; t) := \frac{\Delta(\alpha + 1, 0) + \Delta(\alpha + 1, t)}{2} - \log \pi,$$

$$G_2(\alpha; t) := \frac{\Delta(\alpha + 1, 0) + \Delta(\alpha + 2, 0) + \Delta(\alpha + 1, t) + \Delta(\alpha + 2, t)}{4} - \log \pi,$$

and $\Delta(x, y) := \mathfrak{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{x + iy}{2}\right)\right).$

In [13, Lemma 2.5 and Eq. (2.10)], Zaman showed the following estimate.

Lemma 2.4. *Let L be a number field with r_1 and $2r_2$, the number of real and complex embeddings, respectively. Let $\mathrm{L} = \log d_L$. For $\alpha \geq 1$ and $t \in \mathbb{R}$, we have*

$$(2.7) \qquad \mathrm{S}_L(\alpha, t) \leq \frac{1}{\alpha}\mathrm{L} + \frac{G_1(\alpha; |t|)}{\alpha} \cdot r_1 + \frac{G_2(\alpha; |t|)}{\alpha} \cdot 2r_2 + \frac{2}{\alpha^2} + \frac{2}{\alpha + \alpha^2}.$$

We now use Theorem 2.2 to derive an inequality for a power sum associated to the exceptional zero β_1 of $\zeta_L(s)$. Let the set of nontrivial zeros (resp., trivial zeros) of $\zeta_L(s)$ be denoted by $\mathcal{S}$ (resp., $\mathcal{T}$), and set $\mathcal{S}(T) = \{\rho \in \mathcal{S} \mid |\mathfrak{Im}(\rho)| \leq T\}$. For $j \in \mathbb{N}$ and $c \geq 2$, define

$$(2.8) \qquad S_j = \sum_{\rho \in \mathcal{S}\backslash\{\beta_1\}} \frac{1}{(c - \rho)^{2j}} + \sum_{\rho \in \mathcal{S}\backslash\{\beta_1\}} \frac{1}{(c + it - \rho)^{2j}}.$$

Lemma 2.5. *Let $c \geq 2$, let $T \geq 1$, let $0 < \eta < 1$, and let $0 < \varepsilon < 1$. Let β_1 be an exceptional zero satisfying $\beta_1 > 1 - \frac{1}{12.74\mathrm{L}}$, and let $\rho' = \beta' + it \in \mathcal{S}(T)\backslash\{\beta_1\}$ be such that β' is maximal and $|t|$ is chosen minimally. If $1 - \eta \leq \beta' < 1$, then for d_L sufficiently large there exist $a > 0$ and $\delta > 0$ such that*

$$(2.9) \qquad \mathfrak{Re}(S_j) \geq \delta(c - \beta')^{-2j}$$

for some $1 \leq j \leq (8 + 2\varepsilon)(c - 1 + \eta)^2 a\mathrm{L}$, where

$$(2.10) \qquad a := a(c, \eta, T) = \frac{1}{d - 1} + \max_{|t| \leq T}\left\{\frac{G_1(d - 1; |t|)}{(d - 1)\log 60}, \frac{G_2(d - 1; |t|)}{(d - 1)\log 22}, 0\right\}$$

and

$$(2.11) \qquad d := d(c, \eta) = \sqrt{2c^2 + (1 - \eta)^2 - 2c(1 - \eta)}.$$

Proof. Let $0 < \varepsilon < 1$ be a positive parameter. Throughout this proof a nontrivial zero of $\zeta_L(s)$ is denoted $\rho = \beta + i\gamma$. Without loss of generality, we may assume $t \geq 0$. Let $m = \min\{|c - \rho|, |c + it - \rho| \mid \rho \in \mathcal{S}\}$. By definition we have that $m \leq |c + it - \rho'| = c - \beta'$. Since we assume $\beta' \geq 1 - \eta$, it follows that $m \leq c - (1 - \eta) = c - 1 + \eta$. It shall be convenient to define

$$(2.12) \qquad \mathcal{A} := \mathcal{A}(c, \eta) = (c - 1 + \eta)^2.$$

Observe that we have $m^2 \leq \mathcal{A}$ and also $\mathcal{A} \geq (c - 1)^2$. Now write

$$z_\rho = \frac{m^2}{(c - \rho)^2}, \quad w_\rho = \frac{m^2}{(c + it - \rho)^2}, \quad \text{and} \quad S'_j = \sum_{\rho \in \mathcal{S}\backslash\{\beta_1\}} z_\rho^j + \sum_{\rho \in \mathcal{S}\backslash\{\beta_1\}} w_\rho^j.$$

Observe that

$$(2.13) \qquad S_j = m^{-2j}S'_j \geq (c - \beta')^{-2j}S'_j,$$

where we recall that S_j is defined by (2.8). The normalization guarantees that $|z_\rho|, |w_\rho| \leq 1$. Let $z_\rho = r_\rho e^{i\theta_\rho}$, and let $w_\rho = s_\rho e^{i\alpha_\rho}$. We then apply Theorem 2.2, in the special case $b_n = 1$, to the multiset $\{z_\rho, w_\rho : \rho \neq \beta_1\}$. In this case we have

$$M = \sum_{\rho \in \mathcal{S} \setminus \{\beta_1\}} \frac{r_\rho}{1 + r_\rho} + \sum_{\rho \in \mathcal{S} \setminus \{\beta_1\}} \frac{s_\rho}{1 + s_\rho}, \quad J = \lfloor (8 + \varepsilon) M \rfloor,$$

and there exists j satisfying $1 \leq j \leq J$ such that

$$(2.14) \qquad \qquad \mathfrak{Re}(S_j') \geq \frac{\varepsilon}{32 + 4\varepsilon}.$$

We now provide an upper bound for J. To do this, we must bound M. As

$$r_\rho \leq \frac{\mathcal{A}}{(c - \beta)^2 + \gamma^2} \quad \text{and} \quad s_\rho \leq \frac{\mathcal{A}}{(c - \beta)^2 + (\gamma - t)^2},$$

it follows that

$$(2.15) \qquad \frac{r_\rho}{1 + r_\rho} \leq \frac{\mathcal{A}}{(c - \beta)^2 + \gamma^2 + \mathcal{A}} \quad \text{and} \quad \frac{s_\rho}{1 + s_\rho} \leq \frac{\mathcal{A}}{(c - \beta)^2 + (\gamma - t)^2 + \mathcal{A}}.$$

Thus

$$M \leq \mathcal{A} \sum_{\rho \in \mathcal{S} \setminus \{\beta_1\}} \left(\frac{1}{(c - \beta)^2 + \gamma^2 + \mathcal{A}} + \frac{1}{(c - \beta)^2 + (\gamma - t)^2 + \mathcal{A}} \right).$$

By (2.12) and the fact that $c \geq 2$, it may be checked that for $\beta \in (0, 1)$,

$$(2.16) \qquad (c - \beta)^2 + \mathcal{A} = (c - \beta)^2 + (c - 1 + \eta)^2 \geq (d - \beta)^2,$$

where $d := d(c, \eta) = \sqrt{c^2 + \mathcal{A}} = \sqrt{2c^2 + (1 - \eta)^2 - 2c(1 - \eta)}$ agrees with (2.11). Thus $M \leq \mathcal{A}S_L(d - 1, t)$, where we recall that S_L is defined in (2.6). We are now able to apply the explicit inequality from Lemma 2.4 to bound M:

$$M \leq \mathcal{A}\left(\frac{L}{d - 1} + \frac{G_1(d - 1; |t|)}{d - 1} \cdot r_1 + \frac{G_2(d - 1; |t|)}{d - 1} \cdot 2r_2 + \frac{2}{(d - 1)^2} + \frac{2}{d - 1 + (d - 1)^2} \right).$$

Now let

$$(2.17) \qquad \mathfrak{M}(T; \eta) = \max_{|t| \leq T} \left\{ \frac{G_1(d - 1; |t|)}{(d - 1) \log 60}, \frac{G_2(d - 1; |t|)}{(d - 1) \log 22}, 0 \right\}.$$

For d_L sufficiently large, we have the bound (2.5) of Theorem 2.3, and thus

$$(2.18) \qquad M \leq \mathcal{A}\left(\frac{1}{d - 1} + \mathfrak{M}(T; \eta) + \frac{2}{(d - 1)^2 L} + \frac{2}{(d - 1 + (d - 1)^2)L} \right) L.$$

Thus if $0 < \varepsilon < 1$ and d_L is sufficiently large in terms of ε, η, and c, $M \leq L\mathcal{A}a(1 + \frac{\varepsilon}{9})$. It follows that there exists j satisfying $1 \leq j \leq (8 + \varepsilon)M \leq (8 + \varepsilon)(1 + \frac{\varepsilon}{9})\mathcal{A}aL \leq (8 + 2\varepsilon)\mathcal{A}aL$ such that (2.14) holds. Combining this with (2.13) we obtain (2.9) and we complete the proof. $\qquad \square$

From the last lemma, we deduce the following version of the Deuring–Heilbronn phenomenon.

Theorem 2.6. *Let $c \geq 2$, let $\eta \in (0, 1)$, let $T \geq 1$, and let $0 < \varepsilon < 1$. Suppose that $\beta_1 = 1 - \frac{\lambda_1}{L}$ is an exceptional zero of $\zeta_L(s)$ and $\rho' = \beta' + i\gamma'$ is another zero*

$L = \log d_L$

of $\zeta_L(s)$ with $|\gamma'| \le T$. Write $\beta' = 1 - \frac{\lambda'}{L}$. Then either $\beta' \le 1 - \eta$ or there exist positive effectively computable constants C and κ such that

(2.19) $$\lambda' \ge \frac{1}{C} \log(\kappa \lambda_1^{-1}), \ C = C(c, \eta, T, \varepsilon) = \frac{\mathcal{A}a(16 + 4\varepsilon)}{c - 1},$$

where a and A are defined in (2.10) and (2.12).

From this theorem we obtain the following values for $C = C(c, \eta, T, \varepsilon)$:

(2.20) $$C(5, 0.5, 1, \tilde{\varepsilon}) = 14.1439\ldots, \ C(5.2, 0.1, 1, \tilde{\varepsilon}) = 12.2618\ldots,$$

(2.21)
$$C(4.6, \tfrac{1}{2}, 4.6, \tilde{\varepsilon}) = 14.58\ldots, C(5.8, \tfrac{1}{2}, 10, \tilde{\varepsilon}) = 15.50\ldots, C(10.3, \tfrac{1}{2}, 130, \tilde{\varepsilon}) = 20.21\ldots,$$

where $\tilde{\varepsilon} = 10^{-11}$. Observe that Theorem 1.2 follows from the first entry of (2.20).

Proof. As explained in [9, p. 289], there are α and r, depending on L, such that

L hadamard fact.

$$-\frac{\zeta_L'}{\zeta_L}(s) = \frac{1}{s-1} - \alpha - \sum_{\omega \in \mathcal{S} \cup \mathcal{T} \setminus \{0\}} \left(\frac{1}{s - \omega} + \frac{1}{\omega} \right) - \frac{r}{s} = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} (\log N\mathfrak{p})(N\mathfrak{p})^{-ms}$$

$\rho' = \beta' + i\gamma'$

for $\Re(s) > 1$. Differentiating the above equation $2j - 1$ times yields
(2.22)
$$\frac{1}{(2j-1)!} \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} (\log N\mathfrak{p})(\log N\mathfrak{p}^m)^{2j-1}(N\mathfrak{p})^{-ms} = \frac{1}{(s-1)^{2j}} - \sum_{\omega \in \mathcal{S} \cup \mathcal{T}} \frac{1}{(s-\omega)^{2j}}.$$

For $c > 1$, we apply this identity with $s = c$ and $s = c + i\gamma'$ to obtain

$$\frac{1}{(2j-1)!} \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} (\log N\mathfrak{p})(\log N\mathfrak{p}^m)^{2j-1}(N\mathfrak{p})^{-mc}(1 + (N\mathfrak{p}^m)^{-i\gamma'})$$

(2.23)
$$= \frac{1}{(c-1)^{2j}} + \frac{1}{(c+i\gamma'-1)^{2j}} - \frac{1}{(c-\beta_1)^{2j}} - \frac{1}{(c+i\gamma'-\beta_1)^{2j}}$$
$$- \sum_{\omega \in \mathcal{S} \cup \mathcal{T} \setminus \{\beta_1\}} \left(\frac{1}{(c-\omega)^{2j}} + \frac{1}{(c+i\gamma'-\omega)^{2j}} \right).$$

As $1 + \cos(\gamma' m \log N\mathfrak{p}) \ge 0$, the real part on the left is positive, which gives

$$\frac{1}{(c-1)^{2j}} + \Re\frac{1}{(c+i\gamma'-1)^{2j}} - \frac{1}{(c-\beta_1)^{2j}} - \Re\frac{1}{(c+i\gamma'-\beta_1)^{2j}}$$
$$- \sum_{\omega \in \mathcal{S} \cup \mathcal{T} \setminus \{\beta_1\}} \Re\left(\frac{1}{(c-\omega)^{2j}} + \frac{1}{(c+i\gamma'-\omega)^{2j}} \right) \ge 0. \quad (*1)$$

Observe that for $t' \in \mathbb{R}$, we have the bound
(2.24)
$$\left| \frac{1}{(c+it'-1)^{2j}} - \frac{1}{(c+it'-\beta_1)^{2j}} \right| = 2j \left| \int_{\beta_1}^{1} (c+it'-x)^{-2j-1} dx \right| \le \frac{2j(1-\beta_1)}{(c-1)^{2j+1}}.$$

Applying this with $t' = 0$ and $t' = \gamma'$, we derive

$$\frac{4j(1-\beta_1)}{(c-1)^{2j+1}} \ge \sum_{\omega \in \mathcal{S} \cup \mathcal{T} \setminus \{\beta_1\}} \Re\left(\frac{1}{(c-\omega)^{2j}} + \frac{1}{(c+i\gamma'-\omega)^{2j}} \right). \quad (*2)$$

Since $-\mathfrak{Re}(z) \leq |z|$, for any $\omega \in \mathbb{R}$, one has

$$(2.25) \quad -\mathfrak{Re} \frac{1}{(c+i\gamma'-\omega)^{2j}} \leq \frac{1}{|c+i\gamma'-\omega|^{2j}} = \frac{1}{((c-\omega)^2+\gamma'^2)^j} \leq \mathfrak{Re} \frac{1}{(c-\omega)^{2j}}.$$

It follows that for the trivial zeros we have

$$\sum_{\omega \in \mathcal{T}} \mathfrak{Re} \left(\frac{1}{(c-\omega)^{2j}} + \frac{1}{(c+i\gamma'-\omega)^{2j}} \right) \geq 0. \qquad \langle *3 \rangle$$

By Lemma 2.5, there exist $a > 0, \delta > 0$, and $1 \leq j \leq (8+2\varepsilon)\mathcal{A}aL$ so that

$$*2 - *4 \qquad \sum_{\omega \in \mathcal{S} \backslash \{\beta_1\}} \mathfrak{Re} \left(\frac{1}{(c-\omega)^{2j}} + \frac{1}{(c+i\gamma'-\omega)^{2j}} \right) \geq \delta(c-\beta')^{-2j}. \quad \langle *4 \rangle$$

Combining these estimates, we then deduce that $\frac{4j(1-\beta_1)}{(c-1)^{2j+1}} \geq \delta(c-\beta')^{-2j}$, and thus

$$\frac{4j(1-\beta_1)}{c-1} \geq \delta \left(\frac{c-1}{c-\beta'} \right)^{2j} = \delta \exp \left(-2j \log \left(\frac{c-\beta'}{c-1} \right) \right).$$

Observe that

$$\log \left(\frac{c-\beta'}{c-1} \right) = \log \left(1 + \frac{1-\beta'}{c-1} \right) \leq \frac{1-\beta'}{c-1}$$

since $\log(1+x) \leq x$ for $x \geq 0$. As we have $j \leq (8+2\varepsilon)\mathcal{A}aL$, it follows that

$$\frac{4 \cdot (8+2\varepsilon)\mathcal{A}aL(1-\beta_1)}{c-1} \geq \delta \exp \left(-\frac{2 \cdot (8+2\varepsilon)\mathcal{A}aL(1-\beta')}{c-1} \right). \quad \Longleftarrow$$

Recalling $\beta_1 = 1 - \frac{\lambda_1}{L}$ and $\beta' = 1 - \frac{\lambda'}{L}$, we see that this is equivalent to

$$exercise) \qquad (32+8\varepsilon)\mathcal{A}a\lambda_1 \geq \delta(c-1)\exp \left(-\frac{(16+4\varepsilon)\mathcal{A}a\lambda'}{c-1} \right).$$

Rearranging this implies the inequality in (2.19). $\qquad \square$

2.3. **Location of other real zeros.** In this section, we derive a version of the Deuring–Heilbronn phenomenon that *only* concerns real zeros of $\zeta_L(s)$. To do this, we shall keep using the notation introduced in the previous sections, but we now instead consider, for fixed $c \geq 2$ and $j \in \mathbb{N}$,

$$\tilde{S}_j = \sum_{\rho \in \mathcal{S} \backslash \{\beta_1\}} \frac{1}{(c-\rho)^{2j}},$$

where $\mathcal{S}$ again stands for the set of nontrivial zeros of $\zeta_L(s)$. As in the previous section, we first require the following version of power sum inequality for $\tilde{S}_j$'s.

Lemma 2.7. *Let $0 < \eta < 1$, and let $0 < \varepsilon < 1$. Let β_1 be an exceptional zero satisfying $\beta_1 > 1 - \frac{1}{12.74L}$, and let $\rho' = \beta'$ be another real zero with maximal β'. If $1 - \eta \leq \beta' < 1$, then there exist $a' > 0$ and $\delta' > 0$ such that*

$$\mathfrak{Re}(\tilde{S}_j) \geq \delta'(c-\beta')^{-2j}$$

for some $1 \leq j \leq 4(c-1+\eta)^2 a'(1+2\varepsilon)L$ where

$$(2.26) \qquad a' := a'(c,\eta) = \frac{1}{d-1} + \mathfrak{M}(0;\eta),$$

and we recall that d is defined in (2.11) and that $\mathfrak{M}(T;\eta)$ is defined in (2.17).

Proof. As the proof is almost the same as the proof of Lemma 2.5 with $t = T = 0$, here we shall just give a sketch of the proof and outline some differences. Now we let $m = \min_{\rho \in \mathcal{S}} |c - \rho|$. Clearly, we have that $m \leq c - \beta'$. As $\beta' \geq 1 - \eta$, it again follows that $m^2 \leq \mathcal{A} := (c - 1 + \eta)^2$. Now write

$$\tilde{S}_j = m^{-2j} \tilde{S}'_j = m^{-2j} \sum_{\rho \in \mathcal{S} \setminus \{\beta_1\}} z_\rho^j$$

and $z_\rho = \frac{m^2}{(c-\rho)^2}$. By following the same argument as in Theorem 2.2 we establish

$$(2.27) \qquad J\left(\max_{1 \leq j \leq J} \mathfrak{Re}(\tilde{S}'_j) \right) \geq \frac{J+1}{4} - 2M,$$

where $M = \sum_{\rho \in \mathcal{S} \setminus \{\beta_1\}} \frac{|z_\rho|}{1 + |z_\rho|}$. By (2.15) and (2.16) we have

$$\frac{|z_\rho|}{1 + |z_\rho|} \leq \frac{\mathcal{A}}{(c - \beta)^2 + \gamma^2 + \mathcal{A}} \leq \frac{\mathcal{A}}{(d - \beta)^2 + \gamma^2},$$

where d and $\mathcal{A}$ are given by (2.11) and (2.12). Therefore $2M \leq \mathcal{A} \mathrm{S}_L(d - 1, 0)$, where S_L is given by (2.6). Furthermore, by (2.7), $\mathrm{S}_L(d - 1, 0) \leq a'(1 + \varepsilon)\mathrm{L}$ for d_L sufficiently large. Combining the last two inequalities with (2.27) we have

$$J \max_{1 \leq j \leq J} \mathfrak{Re}(\tilde{S}'_j) \geq \frac{J+1}{4} - \mathcal{A}a'(1 + \varepsilon)\mathrm{L} \geq \mathcal{A}a'\varepsilon\mathrm{L}$$

by the choice $J = \lfloor 4\mathcal{A}a'(1 + 2\varepsilon)\mathrm{L} \rfloor$. Therefore, it follows that there exists $1 \leq j \leq J$ such that $\tilde{S}'_j \geq \frac{\mathcal{A}a'\varepsilon\mathrm{L}}{J} \geq \delta' = \frac{\varepsilon}{4(1+2\varepsilon)}$, and thus $\tilde{S}_j \geq (c - \beta)^{-2j} \tilde{S}'_j \geq \delta'(c - \beta)^{-2j}$ as desired. $\square$

We now deduce a version of the Deuring–Heilbronn phenomenon for real zeros of $\zeta_L(s)$.

Theorem 2.8. *Let $c \geq 2$, let $\eta \in (0, 1)$, and let $0 < \varepsilon < 1$. If $\beta_1 = 1 - \frac{\lambda_1}{\mathrm{L}}$ is an exceptional zero of $\zeta_L(s)$ and $\beta' = 1 - \frac{\lambda'}{\mathrm{L}}$ is another real zero of $\zeta_L(s)$, then either $\beta' \leq 1 - \eta$ or there exist positive effectively computable constants C' and κ' such that*

$$(2.28) \qquad \lambda' \geq \frac{1}{C'} \log(\kappa' \lambda_1^{-1}), \; C' = C'(c, \eta, \varepsilon) = \frac{8\mathcal{A}a'(1 + 4\varepsilon)}{c - 1}.$$

Note that Theorem 1.3 follows from this since $C'(5, \frac{1}{2}, \frac{1}{10^{11}}) = 7.07199744\ldots$.

Proof. By applying (2.22) with $s = c$ and then taking real parts and using nonnegativity, we have

$$\frac{1}{(c - 1)^{2j}} - \frac{1}{(c - \beta_1)^{2j}} - \sum_{\omega \in \mathcal{S} \cup \mathcal{T} \setminus \{\beta_1\}} \mathfrak{Re} \frac{1}{(c - \omega)^{2j}} \geq 0.$$

This, together with the bound (2.24) and inequality (2.25) (with $\gamma' = 0$), yields

$$\frac{2j(1 - \beta_1)}{(c - 1)^{2j+1}} \geq \sum_{\omega \in \mathcal{S} \setminus \{\beta_1\}} \mathfrak{Re} \frac{1}{(c - \omega)^{2j}},$$

which, by Lemma 2.7, is greater than $\delta'(c - \beta')^{-2j}$ for a' given by (2.26), some $\delta' > 0$, and $1 \leq j \leq 4\mathcal{A}a'(1 + 2\varepsilon)\mathrm{L}$. Finally, using the exact same argument as in the last few lines of the proof of Theorem 2.6, we then deduce (2.28). $\square$

3. The least prime ideal

To prove Theorem 1.1, we shall use the method based on Heath-Brown's proof of Linnik's theorem [5] as adapted to the number field setting by Zaman [13]. Let us consider $\mathrm{P} = \{\mathfrak{p} \subset \mathcal{O}_K \mid \mathfrak{p} \text{ is unramified}, \mathfrak{p} \text{ is of degree } 1, \text{ and } \sigma_{\mathfrak{p}} = \mathcal{C}\}$ and

$$(3.1) \qquad S = \sum_{\mathfrak{p} \in \mathrm{P}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}} f\left(\frac{\log N\mathfrak{p}}{\mathrm{L}}\right),$$

where f is a compactly supported function.

We shall evaluate S with the weights $f = f_{\ell,A,B}(t)$, described in the following lemma. This family of weights was introduced by Zaman [13, Lemma 2.6], who generalised the weights used by Heath-Brown [5].

Lemma 3.1. *For $A, B > 0$ and $\ell \in \mathbb{Z}^+$ satisfying $B > 2A\ell$, there exists a real variable function $f(t) = f_{\ell,A,B}(t)$ satisfying the following properties:*

   (i) $0 \leq f(t) \leq A^{-1}$ *for all $t \in \mathbb{R}$.*
   (ii) *The support of f is contained in $[B - 2A\ell, B]$.*
   (iii) *The Laplace transform $F(z) = F_{\ell,A,B}(z) = \int_{\mathbb{R}} f_{\ell,A,B}(t)e^{-zt}dt$ is*

$$(3.2) \qquad F(z) = e^{-(B-2A\ell)z}\left(\frac{1 - e^{-Az}}{Az}\right)^{2\ell}.$$

   (iv) *Let $\mathrm{L} \geq 1$ be arbitrary. Suppose $s = \sigma + it \in \mathbb{C}$ satisfies $\sigma < 1$ and $t \in \mathbb{R}$. Write $\sigma = 1 - \frac{x}{\mathrm{L}}$ and $t = \frac{y}{\mathrm{L}}$. If $0 \leq \alpha \leq 2\ell$, then*

$$(3.3)$$
$$|F((1-s)\mathrm{L})| \leq e^{-(B-2A\ell)x}\left(\frac{2}{A\sqrt{x^2 + y^2}}\right)^{\alpha} = e^{-(B-2A\ell)(1-\sigma)\mathrm{L}}\left(\frac{2}{A|1-s|\mathrm{L}}\right)^{\alpha}.$$

Furthermore,

$$(3.4) \qquad |F((1-s)\mathrm{L})| \leq e^{-(B-2A\ell)x} \text{ and } F(0) = 1.$$

In our argument, we shall show that (3.1) is positive for functions $f = f_{\ell,A,B}(t)$ for various choices of ℓ, A, and B. In such cases, it follows that there is a prime $\mathfrak{p} \in \mathrm{P}$ so that $N\mathfrak{p} \leq d_L^B$. We now summarise the arguments of [9] and [13] that relate S to the low lying zeros of $\zeta_L(s)$; we first consider the function

$$\tilde{\Psi}_{\mathcal{C}}(s) = \sum_{\mathfrak{p} \subseteq \mathcal{O}_K} \sum_{m \geq 1} \sum_{\sigma_{\mathfrak{p}}^m = \mathcal{C}} \frac{\log N\mathfrak{p}}{(N\mathfrak{p}^m)^s} \text{ for } \mathfrak{Re}(s) > 1.$$

Let $g_{\mathcal{C}}$ be a representative of $\mathcal{C}$, and let E be its fixed field. Then Deuring's reduction [3] ensures that $\tilde{\Psi}_{\mathcal{C}}(s)$ equals $\Psi_{\mathcal{C}}(s)$ plus an error arising from the ramified primes where

$$\Psi_{\mathcal{C}}(s) = -\frac{|\mathcal{C}|}{|\mathcal{G}|} \sum_{\chi} \overline{\chi}(g_{\mathcal{C}}) \frac{L'}{L}(s, \chi, L/E),$$

and the sum is over the irreducible characters of $\mathrm{Gal}(L/E)$. By [13, Lemma 4.1],

$$\mathrm{L}^{-1}S = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Psi_{\mathcal{C}}(s)F((1-s)\mathrm{L})ds + \mathcal{E}_1,$$

where $\mathcal{E}_1$ arises from the contribution of the ramified prime ideals, the prime ideals $\mathfrak{p}$ with $N\mathfrak{p}$ nonrational prime, and the powers of prime ideals

$$(3.5) \qquad \mathcal{E}_1 \ll A^{-1}\mathrm{L}^2 e^{-(B-2A\ell)\mathrm{L}/2}.$$

Now as Artin reciprocity yields that $L(s, \chi, L/E)$ is a Hecke L-function of E, the classical analytic machinery can be invoked. We have from [13, Lemma 4.2] that

$$\left| \frac{|\mathcal{G}|}{|\mathcal{C}|} \mathrm{L}^{-1} S - F(0) \right| \leq \sum_{\substack{\rho \\ |\gamma| \leq T^*}} |F((1-\rho)\mathrm{L})| + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4,$$

where $\mathcal{E}_2$ arises from the zeros with $|\gamma| > T^*$, $\mathcal{E}_3$ is from the zero at $s = 0$, and $\mathcal{E}_4$ comes from the integral along the line $\mathfrak{Re}(s) = -1/2$:

$$(3.6) \qquad \mathcal{E}_2 \ll \mathrm{L} \left(\frac{2}{AT^*\mathrm{L}} \right)^{2\ell},$$

$$(3.7) \qquad \mathcal{E}_3 \ll \mathrm{L} \left(\frac{1}{A\mathrm{L}} \right)^{2\ell} e^{-(B-2A\ell)\mathrm{L}},$$

$$(3.8) \qquad \mathcal{E}_4 \ll \mathrm{L} \left(\frac{2}{A\mathrm{L}} \right)^{2\ell} e^{-3(B-2A\ell)\mathrm{L}/2}.$$

Finally, [13, Lemma 4.3] removes the zeros outside of various zero-free regions. Let $J \geq 1$ be given, and let $T^* \geq 1$ be fixed. Suppose

$$1 \leq R_1 \leq R_2 \leq \cdots \leq R_J \leq \mathrm{L}, \ 0 = T_0 < T_1 \leq T_2 \leq \cdots \leq T_J = T^*.$$

We define $\sum'$ as the sum over the zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ satisfying

$$\beta > 1 - \frac{R_j}{\mathrm{L}}, \ T_{j-1} \leq |\gamma| < T_j \text{ for some } 1 \leq j \leq J.$$

Then

$$(3.9) \qquad \sum_{\substack{\rho \\ |\gamma| \leq T^*}} |F((1-\rho)\mathrm{L})| = \sum_{\rho}' |F((1-\rho)\mathrm{L})| + \mathcal{E}_5 + \mathcal{E}_6,$$

where $\mathcal{E}_5$ arises from the zeros in the region $\beta \leq 1 - \frac{R_1}{\mathrm{L}}, |\gamma| < T_1$, and $\mathcal{E}_6$ is from the zeros in the union over $j = 2, \ldots, J$ of the regions $\beta \leq 1 - \frac{R_j}{\mathrm{L}}, T_{j-1} \leq |\gamma| < T_j$:

$$(3.10) \qquad \mathcal{E}_5 \ll \min \left(\left(\frac{2}{A} \right)^{2\ell}, \mathrm{L} \right) e^{-(B-2A\ell)R_1},$$

$$(3.11) \qquad \mathcal{E}_6 \ll \sum_{j=2}^{J} \mathrm{L} \left(\frac{2}{AT_{j-1}\mathrm{L}} \right)^{2\ell} e^{-(B-2A\ell)R_j}.$$

Note that if $J = 1$, then $\mathcal{E}_6$ vanishes. Recognizing $F(0) = 1$, we summarise all of the above in the following lemma.

Lemma 3.2. *Let $J \geq 1$ be given, and let $T^* \geq 1$ be fixed. Suppose*

$$1 \leq R_1 \leq R_2 \leq \cdots \leq R_J \leq \mathrm{L}, \ 0 = T_0 < T_1 \leq T_2 \leq \cdots \leq T_J = T^*.$$

Then

$$(3.12) \qquad \frac{|\mathcal{G}|}{|\mathcal{C}|} \mathrm{L}^{-1} S \geq 1 - \sum_{\rho}' |F((1-\rho)\mathrm{L})| + \mathcal{O} \left(\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6 \right),$$

where the $\mathcal{E}_i$'s are defined in (3.5), (3.6), (3.7), (3.8), (3.10), *and* (3.11).

This is the key formula allowing us to show that S is positive in various cases. There are four cases to consider. Let $\tilde{\eta} \in (0, \frac{1}{12.74})$ be a sufficiently small absolute positive constant.

1. **The nonexceptional case:**
   Assume all zeros $\beta + i\gamma$ satisfy $\beta = 1 - \frac{\lambda}{L}$ with $\lambda \geq \frac{1}{12.74}$.
2. **The exceptional cases:**
   We assume there is an exceptional zero $\beta_1 = 1 - \frac{\lambda_1}{L}$ with $\lambda_1 < \frac{1}{12.74}$.
   (i) λ_1 *small:* $\tilde{\eta} \leq \lambda_1 < \frac{1}{12.74}$ (with $\tilde{\eta}$ fixed parameter $0 < \tilde{\eta} < 1$).
   (ii) λ_1 *very small:* $L^{-200} \leq \lambda_1 < \tilde{\eta}$.
   (iii) λ_1 *extremely small:* $\lambda_1 < L^{-200}$.

We shall not consider cases 1 and 2(i); for the proofs we refer to [13, pp. 136–138]. Note that the proofs in these cases make use of a number of results from [6] and [7]. Namely, the zero-free region for $\zeta_L(s)$ (1.1) [6, Theorem 1.1], the Deuring–Heilbronn phenomenon for zeros close to $s = 1$ [7, Theorem 4], and a smoothed explicit local formula for zeros of the Dedekind zeta function [7, Lemma 7]. Instead, we shall improve cases 2(ii) and (iii). In the following table, in column 2, we record Zaman's values of B for which it can be shown that S is positive for $f = f_{\ell,A,B}(t)$ (for some ℓ and A), in each case. Column 3 lists our improvements to Zaman's values.

Cases	B (Zaman)	B
1. The nonexceptional case	7.41	–
2(i) λ_1 *small*	2.63	–
2(ii) λ_1 *very small*	36.5	12.48
2(iii) λ_1 *extremely small*	39.5	15.72

This table shows that, in each case, there exists a prime ideal $\mathfrak{p} \in P$ such that $N\mathfrak{p} \leq d_L^B$, where B is the listed constant(s). It follows that we can unconditionally choose $B = 15.72$ and this establishes Theorem 1.1. Note that we do not attempt to improve cases 1 and 2(i) as we currently are unable to reduce the constants in cases 2(ii) and (iii) below 7.41. We now treat the last two cases.

3.1. **Very small case:** $L^{-200} \leq \lambda_1 < \tilde{\eta}$. We shall select the weight function $f = f_{\ell,A,B}$ with parameters $\ell = 101$ and A and B to be determined. Note that for $\eta = 0.1$, Theorem 2.6 holds with $C = 12.262$ (see (2.20)). For now we assume that $B - 2A\ell > C = 12.262$. The selection of these choices shall be explained shortly. By (3.5), (3.6), (3.7), and (3.8),

$$\mathcal{E}_1 \ll A^{-1}L^2 e^{-(B-2A\ell)L/2} \ll L^{-2\ell+1}, \quad \mathcal{E}_2 \ll L\left(\frac{2}{AT^*L}\right)^{2\ell} \ll L^{-2\ell+1},$$

$$\mathcal{E}_3 \ll L\left(\frac{1}{AL}\right)^{2\ell} e^{-(B-2A\ell)L} \ll L^{-2\ell+1},$$

$$\mathcal{E}_4 \ll L\left(\frac{2}{AL}\right)^{2\ell} e^{-3(B-2A\ell)L/2} \ll L^{-2\ell+1}.$$

We apply Lemma 3.2 with $J = 1, T^* = 1$, and $R_1 = \frac{\log(\kappa\lambda_1^{-1})}{C}$, for some $\kappa > 0$, to conclude that there is no error term $\mathcal{E}_6$ and that

$$\mathcal{E}_5 \ll \min\left(\left(\frac{2}{A}\right)^{2\ell}, L\right) e^{-(B-2A\ell)R_1} \ll e^{-(B-2A\ell)R_1} \ll \lambda_1^{\frac{B-2A\ell}{C}}.$$

Note that $\sum'$ is over the zeros $\rho = \beta + i\gamma$ with $\beta > 1 - \frac{R_1}{L}$ and $|\gamma| \leq 1$, and that for any fixed $\eta \in (0,1)$, as $L^{-2\ell+2} \leq \lambda_1$, we have $R_1 \leq \frac{1}{12.26}(\log\kappa + (2\ell - 2)\log L) = o(L)$. Thus, for L sufficiently large, $1 - \eta < 1 - \frac{R_1}{L}$. From this, Theorem 2.6 ensures that any other zero $\rho = \beta + i\gamma \neq \beta_1$, with $|\gamma| \leq 1$, has to satisfy $\beta \leq 1 - \frac{R_1}{L}$, and

hence $\sum'$ only contains the exceptional zero β_1. Thus,

$$\frac{|\mathcal{G}|}{|\mathcal{C}|}\mathrm{L}^{-1}S \geq 1 - |F((1-\beta_1)\mathrm{L})| + \mathcal{O}\left(\mathrm{L}^{-2\ell+1}\right) + \mathcal{O}\left(\lambda_1^{\frac{B-2A\ell}{C}}\right).$$

Note that by (3.4), we have $|F((1-\beta_1)\mathrm{L})| \leq e^{-(B-2A\ell)\lambda_1}$. Thus,

$$\frac{|\mathcal{G}|}{|\mathcal{C}|}\mathrm{L}^{-1}S \geq 1 - e^{-(B-2A\ell)\lambda_1} + \mathcal{O}\left(\mathrm{L}^{-2\ell+1}\right) + \mathcal{O}\left(\lambda_1^{\frac{B-2A\ell}{C}}\right).$$

Using $1 - e^{-x} > x - \frac{x^2}{2} \geq \frac{3x}{4}$ for $0 < x \leq \frac{1}{2}$, the above becomes positive whenever

$$\tfrac{3}{4}(B-2A\ell)\lambda_1 + \mathcal{O}\left(\mathrm{L}^{-2\ell+1}\right) + \mathcal{O}\left(\lambda_1^{\frac{B-2A\ell}{C}}\right) > 0,$$

which happens if

$$(B-2A\ell)\lambda_1 \gg \mathrm{L}^{-2\ell+1} \text{ and } (B-2A\ell)\lambda_1 \gg \lambda_1^{\frac{B-2A\ell}{C}}.$$

The first condition is true under our assumption $\mathrm{L}^{-200} = \mathrm{L}^{-2\ell+2} \leq \lambda_1$. Since $\lambda_1 < 1$, the second condition is true if $\frac{B-2A\ell}{C} > 1$. Thus, we must choose B as small as possible such that $B > C + 2A\ell = 12.262 + 2A\ell$. By a computer calculation, we obtain

(3.13) $\ell = 101, \ B = 12.48, \ A = \dfrac{1}{970}, \ \text{ and } B - 2A\ell = 12.271\ldots.$

3.2. **Extremely small case: $\lambda_1 < \mathrm{L}^{-200}$.** We shall assume that there are positive increasing parameters T_j for $0 \leq j \leq J$ for some $J \geq 1$. Associated to each j are positive constants C_j and κ_j such that if $\beta' + i\gamma'$ is another zero of $\zeta_L(s)$ with $\beta' \geq \frac{1}{2}$ and $|\gamma'| \leq T_j$, then

(3.14) $$\beta' \leq 1 - \frac{\log\left(\frac{\kappa_j}{(1-\beta_1)\mathrm{L}}\right)}{C_j\mathrm{L}}.$$

The parameters C_j shall increase with j and we set $T_0 = 1$, $C_0 = 14.144$. Note that

(3.15) $$\mathrm{L} < \lambda_1^{-1/200},$$

and from Corollary 1.3.1, $1 - \beta_1 \gg d_L^{-\frac{C_0}{2}}$ where $C_0 = 14.144$, which implies that

(3.16) $$\mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}} \ll \lambda_1 < \mathrm{L}^{-200}.$$

We define $0 < u < 1 < v$ to be positive parameters. We choose $\ell = \lceil v\mathrm{L} \rceil \in [v\mathrm{L}, v\mathrm{L}+1)$, $A = \frac{u}{\mathrm{L}}$ so that $A\mathrm{L} = u \ll 1$ and $uv \leq A\ell < uv + O(\mathrm{L}^{-1}) = uv + o(1)$. Throughout this section, we write $o(1)$ to denote a term which approaches 0 as $d_L \to \infty$. By (3.5), (3.6), (3.7), and (3.8), we find

$$\mathcal{E}_1 \ll A^{-1}\mathrm{L}^2 e^{-(B-2A\ell)\mathrm{L}/2} \ll \mathrm{L}^3 e^{-\frac{B-2uv+o(1)}{2}\mathrm{L}},$$

$$\mathcal{E}_2 \ll \mathrm{L}\left(\frac{2}{AT^*\mathrm{L}}\right)^{2\ell} \ll \mathrm{L}e^{-2v\log(\frac{uT^*}{2})\mathrm{L}},$$

$$\mathcal{E}_3 \ll \mathrm{L}\left(\frac{1}{A\mathrm{L}}\right)^{2\ell} e^{-(B-2A\ell)\mathrm{L}} \ll \mathrm{L}e^{-(B-2uv-2v\log(1/u)+o(1))\mathrm{L}},$$

$$\mathcal{E}_4 \ll \mathrm{L}\left(\frac{2}{A\mathrm{L}}\right)^{2\ell} e^{-3(B-2A\ell)\mathrm{L}/2} \ll \mathrm{L}e^{-(\frac{3B}{2}-3uv-2v\log(2/u)+o(1))\mathrm{L}},$$

and thus

(3.17) $$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 \ll \mathrm{L}^3 e^{-\frac{B-2uv+o(1)}{2}\mathrm{L}} + \mathrm{L}e^{-2v\log(\frac{uT^*}{2})\mathrm{L}}.$$

Using (3.4) and (3.16), and assuming $B - 2uv > 0$, we have

$$(3.18) \quad 1 - |F((1-\beta_1)\mathrm{L})| > \tfrac{3}{4}(B - 2uv + o(1))\lambda_1 \gg (B - 2uv + o(1))\mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}.$$

We assume $\frac{B-2uv}{2} - \frac{C_0}{2} > 0$ and $\frac{C_0}{2} - 2v\log(\frac{uT^*}{2}) < 0$ so that

$$\mathrm{L}^3 e^{-\frac{B-2uv+o(1)}{2}\mathrm{L}} = \mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}e^{-(\frac{B-2uv+o(1)}{2} - \frac{C_0}{2} - \frac{2\log\mathrm{L}}{\mathrm{L}})\mathrm{L}} = o\Big(\mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}\Big) \text{ and}$$

$$\mathrm{L}e^{-2v\log(\frac{uT^*}{2})\mathrm{L}} = \mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}e^{(\frac{C_0}{2} - 2v\log(\frac{uT^*}{2}))\mathrm{L}} = o\Big(\mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}\Big).$$

By choosing B and T^* with $B > C_0 + 2uv$ and $T^* > \frac{2}{u}e^{\frac{C_0}{4v}}$, we then have

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 = o(1 - |F((1-\beta_1)\mathrm{L})|).$$

We now apply Lemma 3.2 with $J \geq 1$, $T^* = T_J > \frac{2}{u}e^{\frac{C_0}{4v}}$, and $R_j = \frac{\log(\kappa_j\lambda_1^{-1})}{C_j}$ for each $j \in \{1, \ldots, J\}$. Now by (3.14), for any nontrivial zero $\rho = \beta + i\gamma \neq \beta_1$ with $|\gamma| \leq T_j$, either $\beta \leq 1 - \frac{1}{2}$ or $\beta \leq 1 - \frac{R_j}{\mathrm{L}}$. From the symmetry of the nontrivial zeros of $\zeta_L(s)$ with respect to $\mathfrak{Re}(s) = \frac{1}{2}$, it follows that $\beta \leq 1 - \frac{R_j}{\mathrm{L}}$, and thus the sum $\sum'$ contains at most β_1 and $1 - \beta_1$, where the latter only contributes a negligible error. By our choice of $R_1 = \frac{\log(\kappa_1\lambda_1^{-1})}{C_1}$ and (3.15), we have

$$\mathcal{E}_5 \ll \mathrm{L}e^{-(B-2A\ell)R_1} \ll \mathrm{L}\left(\kappa_1^{-1}\lambda_1\right)^{\frac{(B-2uv)}{C_1}} \ll \lambda_1^{\frac{(B-2uv)}{C_1} - \frac{1}{200}} = o(\lambda_1)$$

if we assume $B > \left(1 + \frac{1}{200}\right)C_1 + 2uv$. Finally, we analyse (3.11)

$$\mathcal{E}_6 \ll \sum_{j=2}^{J} \mathrm{L}\Big(\frac{2}{AT_{j-1}\mathrm{L}}\Big)^{2\ell} e^{-(B-2A\ell)R_j} \ll \mathrm{L}\sum_{j=2}^{J} e^{-2v\mathrm{L}\log\left(\frac{uT_{j-1}}{2}\right)}\lambda_1^{\frac{B-2uv}{C_j}}.$$

Note that choosing $\frac{B-2uv}{C_j} \geq 1$ for $2 \leq j \leq J$ would immediately give $\mathcal{E}_6 = o(\lambda_1)$ as long as $T_1 > \frac{2}{u}$. However, this would force us to take $B \geq C_J + 2uv$. As we are trying to minimize B, we instead assume $\frac{B-2uv}{C_j} < 1$ for $2 \leq j \leq J$. This implies $2uv < B < C_2 + 2uv$; we also have

$$\mathcal{E}_6 \ll \lambda_1 \mathrm{L}\sum_{j=2}^{J} e^{-2v\mathrm{L}\log\left(\frac{uT_{j-1}}{2}\right)}\Big(\mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}}\Big)^{\frac{B-2uv}{C_j} - 1} \quad (\text{since } \lambda_1 \gg \mathrm{L}e^{-\frac{C_0}{2}\mathrm{L}})$$

$$\ll \lambda_1 \sum_{j=2}^{J} \mathrm{L}^{\frac{B-2uv}{C_j}} e^{-\left(2v\log\left(\frac{uT_{j-1}}{2}\right) - \frac{C_0}{2}\left(1 - \frac{B-2uv}{C_j}\right)\right)\mathrm{L}}.$$

It follows that $E_6 = o(\lambda_1)$ if $B > \left(1 - \frac{4v}{C_0}\log\left(\frac{uT_{j-1}}{2}\right)\right)C_j + 2uv$ for all $2 \leq j \leq J$. To conclude, the desired inequality $\frac{|\mathcal{G}|}{|\mathcal{C}|}\mathrm{L}^{-1}S > 0$ follows from the assumptions

$$(3.19) \qquad T^* = T_J > \frac{2}{u}e^{\frac{C_0}{4v}}, \quad (1 + \tfrac{1}{200})C_1 + 2uv < B, \text{ and}$$

$$(3.20) \qquad \max_{2 \leq j \leq J}\left(\left(1 - \tfrac{4v}{C_0}\log\left(\tfrac{uT_{j-1}}{2}\right)\right)C_j\right) + 2uv < B < C_2 + 2uv.$$

(Note that the condition $C_0 + 2uv < B$ is dropped since $C_0 < C_1$.) We employ a computer search to determine admissible parameters T_j, C_j, B, u, and v. We now

apply Theorem 2.6 to obtain $C_j = C(c, \frac{1}{2}, T_j, \varepsilon)$ for numerically optimal choices of c and ε. We use the parameters

(3.21)
$$T_1 = 4.6, \ C_1 = 14.58\ldots, \ T_2 = 10, \ C_2 = 15.50\ldots, \ T_3 = 130, \ ,C_3 = 20.21\ldots$$

(see (2.21) just after Theorem 2.6) and

(3.22) $\quad\quad B = 15.72, \ u = 0.53, \ v = 1.001, \ A = u\mathrm{L}^{-1}, \ \ell = \lceil v\mathrm{L} \rceil, \ J = 3.$

These satisfy conditions (3.19) and (3.20), and thus complete our proof.

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