Basis properties of the eigensystem of non-self-adjoint operators

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1 Overview of the Field

Our Research in Teams within non-self-adjoint spectral theory in Hilbert spaces focused on the properties of the eigensystem (eigenvectors and root vectors) of operators T with compact resolvent. In the classical case of self-adjoint or normal T, the eigenvectors form an orthonormal basis. Nevertheless, the basis properties of the eigensystem are a delicate problem in general, even for perturbations of self-adjoint operators. To succeed in applications in differential operators, tools from operator theory are usually combined with technical and precise results on specific eigenfunctions and eigenvalues of the unperturbed operator (e.g. analytic properties of eigenfunctions in the complex plane, their asymptotics or concentration for a large spectral parameter).

As model problems, consider Schrödinger operators in $L^2(\mathbb{R})$ with complex potentials

$$T = -\tfrac{\mathrm{d}^2}{\mathrm{d}x^2} + |x|^\beta + iV(x), \quad \beta > 1, \tag{1}$$

having the form $T = A + B$ with $B = iV$ and $A = A^*$, where the perturbation $V : \mathbb{R} \to \mathbb{R}$ is assumed to be "small" in a suitable sense (although typically unbounded or singular). It is known that the eigensystem of T contains a Riesz basis for instance if

$$\exists C > 0,\ \exists \gamma < \frac{\beta - 2}{2}\ :\ |V(x)| \leq C(1 + x^2)^{\frac{\gamma}{2}}, \quad x \in \mathbb{R}; \tag{2}$$

singular perturbations $V \in L^p(\mathbb{R})$ with a suitable $p \in [1, \infty)$ can be included as well, see [9].

This claim is based on the abstract Riesz basis test relying on the form version of the local subordination, cf. [1, 9, 13], where the size of the gaps between eigenvalues of the unperturbed operator and the decay of the "matrix elements" of the perturbation are compared. In more detail, let the eigenvalues $\{\mu_k\} \subset \mathbb{R}_+$ of a self-adjoint operator A with compact resolvent be simple and satisfy

$$\exists \gamma, \kappa > 0\ :\ \mu_{k+1} - \mu_k \geq \kappa k^{\gamma - 1}, \quad k \in \mathbb{N}. \tag{3}$$

If a perturbation B satisfies that

$$\exists M > 0,\ \exists \alpha > \frac{1 - \gamma}{2}\ :\ |(B\psi_m, \psi_n)| \leq \frac{M}{(mn)^\alpha}, \quad m, n \in \mathbb{N}, \tag{4}$$

where $\{\psi_k\}$ are normalized eigenvectors of A related to the eigenvalues $\{\mu_k\}$, then the eigensystem of $T = A + B$ contains a Riesz basis, see [9]. It is also known that the condition on α and γ cannot be weakened to $2\alpha = 1 - \gamma$ in general.

The goals of our stay at BIRS were to work on several open problems (see details in Section 2) and start a preparation of a monograph covering the progress in the topic in approximately last two decades.

2 Recent Developments and Open Problems

During the program, we focused mainly on the following topics and open problems.

2.1 Local subordination

While the local form-subordination (4) allowed for a progress in Schrödinger operators (1), it appears not to cover the optimal range of perturbations in general. For instance, only a limited range of singular potentials is included comparing to the results on one-dimensional Hill and Dirac operators obtained by other strategies of the proof in e.g. [2, 4, 12, 8]. Moreover, it requires a very regular (power-like) behavior of the eigenvalues' gaps (3) and "matrix elements" of the perturbation (4).

To simplify the presentation in the following, we formulate open problems and our related observations mostly for the special case of the regular gaps behavior with $\gamma = 1$ in (3), i.e. perturbations of the harmonic oscillator, $\beta = 2$ in (1), in applications.

Less regular behavior

It is open if assumptions on the eigenvalues' gaps (3) and "matrix elements" (4) can be relaxed, see also [13, Sec. 7]. For instance in our special case (the harmonic oscillator), can the condition on the perturbation in (4) be replaced by

$$|(B\psi_m, \psi_n)| \leq \omega_m \omega_n, \tag{5}$$

where $\{\omega_n\}_n$ satisfies some summability condition instead of $\omega_n \leq n^{-\alpha}$ with $\alpha > 0$? Our analysis of the existing proofs suggests that such improvement is possible in the final step (and seemingly the main one), where the Schur test is employed to show the boundedness of the operator in $\ell^2(\mathbb{N})$ introduced through its matrix elements

$$\frac{(B\psi_m, \psi_n)}{m - n}, \quad m \neq n, \tag{6}$$

see [9, 13]. An applicable tool are the Schur multipliers, see [6, Sec. 4]. Nevertheless, the previous (normally simpler) steps in the proofs in [9, 13] still seem to require the regular behavior.

Highly singular perturbations

The form local subordination condition (4) is satisfied for the harmonic oscillator perturbed by a compactly supported $V \in H^{-s}(\mathbb{R})$, $s < 1/2$, see [9]. It is not known if the case $s = 1/2$ is extreme (i.e. if the Riesz basis is absent for $s > 1/2$ in general). The earlier results on $-\partial_x^2 + V$ on $(-\pi, \pi)$ and subject to Dirichlet or periodic boundary conditions do not rely on the local subordination and suggest that a different type of estimate is needed (the extreme case here seems to be $V \in H^{-1}$). In more detail, it is crucial to use the decay of the remainder of the H^{-1}-norm of V (which can expressed as a series) and also to employ the algebraic property of the unperturbed system (like $\psi_m \psi_n = \psi_{m+n}$ for $\psi_n(x) = \exp(inx)$), see e.g. [2]. Hence, to find a new abstract Riesz basis test, which would include the results on $-\partial_x^2$ as well as the perturbations of the harmonic oscillator in case $s = 1/2$, the condition in the form of the product bound in (4) or (5) should be replaced by a more general bound capturing possible Toeplitz/Hankel-type structure of the perturbations. For Schrödigner operators, the exact algebraic property of the exponentials is not present, nonetheless, it can suffice to use a replacement for large m, n employing the oscillatory behavior of the eigenfunctions in the region between the turning points.

Critical case

It an open question whether the restriction on the perturbation (2), obtained from the local subordination, is optimal. For the special case of harmonic oscillator, other methods ([1] based on Katsnelsons theorem or [11]) allow to include also bounded perturbations with a natural restriction on the perturbation's norm. On the other hand, the lower resolvent estimates in [7] for perturbations $iV(x) = i \operatorname{sgn} x|x|^\delta$ with $\delta > 0$ would exclude the Riesz basis property if the spectrum of the perturbed operator was localized more precisely (which is an ongoing project). Thus (2) seems to be close to optimal, nonetheless, to treat the extreme cases, an improvement of the local subordination like in (5) or for the singular potentials would be essential.

2.2 Absence of basis properties

The mechanisms behind the loss of the basis properties are only partially understood, in particular for differential operators.

Simple eigenvalues

The existing abstract constructions, see [1, 9, 11], provide examples of perturbations of self-adjoint operators with simple eigenvalues for which the eigensystem of the perturbed operators lacks the basis property. Nonetheless, the geometric understanding of this mechanism seems missing. Unlike for perturbations of multiple eigenvalues, where perturbations of arbitrarily small norm can lead to the absence of the basis property, see e.g. [5, 3], the perturbation in case of simple eigenvalues needs to be sufficiently large and of an appropriate form.

In case of the harmonic oscillator (or more general Schrödinger operators like in (1)), specific perturbations arising from a conjugation cause the basis property absence, see [10]. However, the effect of more general perturbations (in particular in potentials) is not known.

Multiple eigenvalues

The existing examples with double eigenvalues suggest that the structure of the perturbation, not its norm, is crucial to loose the basis property, see e.g. [3]. This observation and the result on the Riesz property of the block decomposition for the bounded perturbations in [11] is the first step in a construction of a potential perturbation of multi-dimensional Schrödinger operators (e.g. of the two-dimensional harmonic oscillator) without the basis property. On an abstract level, it seems possible to generalize results on a suitable structure of the perturbation from multiplicity two to an arbitrary one. Nevertheless, in potential perturbations of Schrödinger operators, the parameters of the perturbations are no longer directly accessible and they cannot be chosen (almost) freely.

2.3 Other related topics

Other discussed related topics include the basis properties of eigenfunctions of Schrödinger operators in $L^p(\mathbb{R})$ for $p \neq 2$, weighted L^p-estimates of eigenfunctions of Schrödinger operators, estimates on the number of non-real eigenvalues for perturbations with a symmetry (like $\mathcal{PT}$) or completeness of eigensystem of Schrödinger operators with imaginary potentials.

3 Outcome of the Meeting

Although the meeting was hybrid eventually (one participant in Banff, one in USA), the possibility to fully focus on the planned research led to very fruitful exchanges. The environment and support of BIRS are exceptional and contributed greatly to this post-covid restart of our collaboration.

Several new ideas how to proceed in the open questions arose in our discussions, in particular, in the improvements of the local subordination, see Section 2.1. The work on these questions is ongoing and it is expected to lead to a new paper.

Regarding the preparation of the monograph, we collected and discussed a number of relevant papers and drafted a detailed outline. The anticipated main chapters comprise: Bases in Hilbert spaces; Operational Calculus and geometric Riesz basis criteria; Perturbations of harmonic and anharmonic oscillators; Perturbations of multiple eigenvalues and Hill operators; Analysis beyond geometry (conjugations, imaginary potentials, completeness of eigensystem).

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