Transportation along Langevin semigroups

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Joint work with Dan Mikulincer

How to transport μ to ν?

Let $(Y_t)_{t\geq 0}$ be the Langevin dynamics:

$$dY_t = \nabla \log\left(\frac{d\nu}{dx}\right)(Y_t)dt + dB_t, \quad Y_0 \sim \mu,$$

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with $(B_t)_{t\geq 0}$ a Brownian motion in $\mathbb{R}^d$. Let (Q_t) be the Langevin semigroup: $Q_t\eta(x) = E[\eta(Y_t)|Y_0 = x]$, and let $\rho_t := Q_t\left(\frac{d\mu}{d\nu}\right)d\nu$ so that the path of measures $(\rho_t)_{t\geq 0}$ interpolates between $\rho_0 = \mu$ to $\rho_\infty = \nu$.

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The Langevin path $(\rho_t)_{t\geq 0}$ satisfies the continuity equation

$$\partial_t \rho_t + \nabla(V_t \rho_t) = 0,$$

where

$$V_t(x) = -\nabla \log\left(\frac{d\rho_t}{d\nu}\right)(x) = -\nabla \log Q_t\left(\frac{d\mu}{d\nu}\right)(x)$$

(because $\partial_t Q_t \eta = \Delta Q_t \eta + \langle \nabla Q_t \eta, \nabla \log\left(\frac{d\nu}{dx}\right)\rangle$).

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The transport maps along Langevin semigroups are defined as

$$S_{\mathrm{LVN}} := \lim_{t \to \infty} S_t \quad \text{transports } \mu = \rho_0 \text{ to } \rho_\infty = \nu,$$

$$T_{\mathrm{LVN}} := \lim_{t \to \infty} T_t \quad \text{transports } \nu = \rho_\infty \text{ to } \rho_0 = \mu.$$

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Theorem (Mikulincer, S)

- *If $\nu = \gamma_d$ and $\mu = \kappa$-log-concave (i.e., $-\nabla^2 \log\left(\frac{d\mu}{dx}(x)\right) \succeq \kappa I_d$), for $\kappa > 0$, then T_{LVN} is $\frac{1}{\sqrt{\kappa}}$-Lipschitz. The result is sharp and already follows from results of Kim and E. Milman.*

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The theorem parallels the analogous results for the optimal transport map.

Semi-log-concave measures with bounded support

Theorem (Mikulincer, S)

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- *In particular, if μ is log-concave (so $\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$, then T_{LVN} is $e^{1/2}R$-Lipschitz. The order of the Lipschitz constant is sharp.*

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The question (due to Kolesnikov) of whether the optimal transport map from γ_d to μ which is log-concave with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$ is $O(R)$-Lipschitz, is open.

Gaussian mixtures with bounded mixing measure

Theorem (Mikulincer, S)

If $\nu = \gamma_d$ and $\mu = \gamma_d \star m$ with $\mathrm{diam}(\mathrm{supp}(m)) \leq R$, then T_{LVN} is $e^{\frac{R^2}{2}}$-Lipschitz. The order of the Lipschitz constant is sharp.

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The theorem leads to improved log-Sobolev inequalities for mixtures of Gaussians.

Further Lipschitz properties

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If $\nu = \mu \star \gamma_d$ and μ = log-concave, then T_{LVN} is 1-Lipschitz.

Theorem (Neeman)

If $\nu = \gamma_d$ and $\mu = e^{-U} d\gamma_d$ where $U^* \leq U \leq U^* + c$ with U^* convex and c a constant, then T_{LVN} is e^c-Lipschitz.

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- Since the transport map along the Langevin semigroup is a finite-dimensional map it can be used to prove *dimensional* functional inequalities (e.g. eigenvalues comparison, see later), unlike the Brownian transport map.
- There are some similarities in the proof techniques for both transport maps.

Transport of functional inequalities

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Then μ satisfies a Poincaré inequality with constant L^2:

$$\begin{aligned}&\mathrm{Var}_\mu(f) = \mathrm{Var}_{\gamma_d}(f \circ T) \leq \mathbb{E}_{\gamma_d}\left[|\nabla(f \circ T)|^2\right] \\ &\leq \mathbb{E}_{\gamma_d}\left[|DT|^2|\nabla f(T)|^2\right] \leq L^2\mathbb{E}_{\gamma_d}\left[|\nabla f(T)|^2\right] = L^2\mathbb{E}_\mu\left[|\nabla f|^2\right].\end{aligned}$$

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Denote by $\lambda_i(\mu)$ and $\lambda_i(\gamma_d)$ the eigenvalues of the Langevin semigroup operators $\Delta + \left\langle \nabla, \nabla \log \frac{d\mu}{dx} \right\rangle$ and $\Delta + \left\langle \nabla, \nabla \log \frac{d\gamma_d}{dx} \right\rangle$, respectively.

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Corollary

- *If* $\nu = \gamma_d$ *and* μ *is* κ*-log-concave with* $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$*, and* $\kappa R^2 < 1$*, then*

$$\lambda_i(\mu) \geq \frac{1}{e^{1-\kappa R^2} R^2} \lambda_i(\gamma_d).$$

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Isoperimetric inequalities

Gaussian isoperimetric inequality: For any $\epsilon \geq 0$,

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Corollary

If μ is log-concave with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$, then

$$\mu(K + \epsilon B_d) \geq \Phi\left(\Phi^{-1}(\gamma_d(K)) + \frac{\epsilon}{e^{1/2} R}\right).$$

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Gromov: Let $1 \leq \ell \leq d$ and $f : \mathbb{R}^d \to \mathbb{R}^\ell$ continuous. There exists $t \in \mathbb{R}^\ell$ such that, for all $\epsilon > 0$, $\gamma_d(f^{-1}(t) + \epsilon B_d) \geq \gamma_\ell(\epsilon B_\ell)$.

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Klartag (localization technique): If $K \subset \mathbb{R}^d$ convex body and $f : \mathbb{R}^d \to \mathbb{R}^\ell$ continuous, then

$$\sup_{t \in \mathbb{R}^\ell} \mathrm{Vol}_{n-\ell}(f^{-1}(t)) \ge \frac{\mathrm{Vol}_n(K)}{\sup_{E : \dim E = \ell} \mathrm{Vol}_\ell(K \cap E)}.$$

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Combining **transport method** (due to Klartag) with our above Lipschitz properties, we can show the weaker result

$$\sup_{t \in \mathbb{R}^\ell} \mathrm{Vol}_{n-\ell}(f^{-1}(t)) \geq \frac{1}{c^\ell} \frac{\mathrm{Vol}_n(K)}{\mathrm{diam}(K)^\ell}.$$

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The above argument, due to Kim and Milman, shows that in dimension 1, the transport map along Langevin semigroups is identical to the optimal transport map.

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Back to high level idea of proofs

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Lemma

If $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$ *then*

$$\sup_x \lambda_{\max}(-\nabla V_t(x)) \leq F(t, R).$$

If μ *is* κ*-log-concave*

$$\sup_x \lambda_{\max}(-\nabla V_t(x)) \leq G(t, \kappa) \begin{cases} \textit{for all } t \in [0, 1] & \textit{if } \kappa \geq 0, \\ \textit{for all } t \in \left[0, \frac{1}{2}\log\left(\frac{\kappa - 1}{\kappa}\right)\right] & \textit{if } \kappa \geq 0. \end{cases}$$

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- When t is small, F is bad but G is good. When t is large, F is good and G is bad.
- To prove Lemma represent $\nabla V_t(x)$ as covariance matrix and then use Brascamp-Lieb inequality.

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Lemma

If μ is β-semi-log-convex then

$$\inf_x \lambda_{\min}(-\nabla V_t(x))$$
$$\geq \textit{corresponding term when } \mu \textit{ is a Gaussian measure}$$
$$\textit{with covariance } \frac{1}{\beta} I_d.$$

Thank You