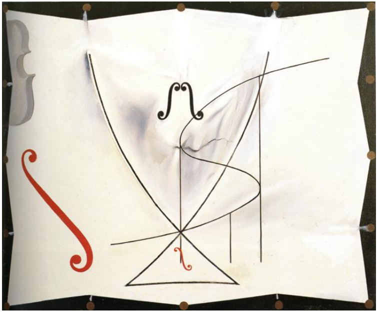
Cluster Structures and Legendrian Links

BIRS Workshop – Interactions of gauge theory with contact and symplectic topology in dimensions 3 and 4

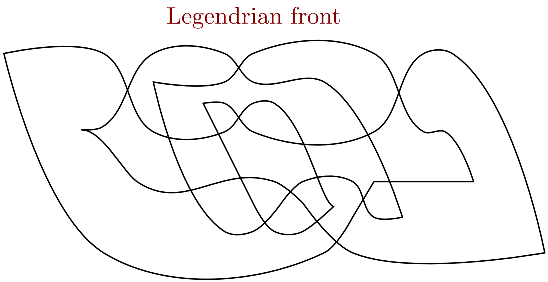


Roger Casals (UC Davis)

March 7th 2022

Simplified Main Result

Main goal: Construction of **quasi-cluster A-structures** on the moduli $\mathfrak{M}(\Lambda)$ of sheaves with singular support in a Legendrian link $\Lambda \subset (\mathbb{R}^3, \xi_{st})$.



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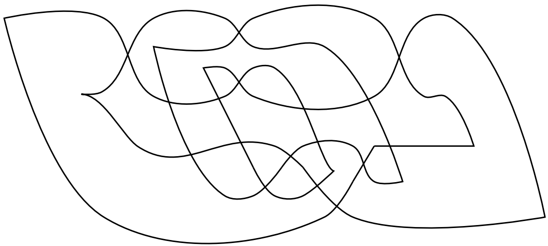

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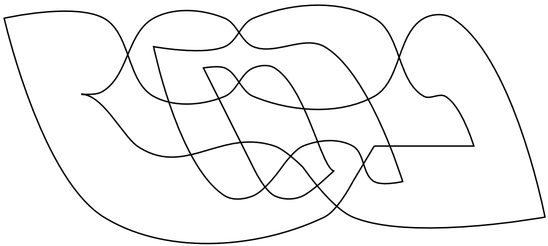

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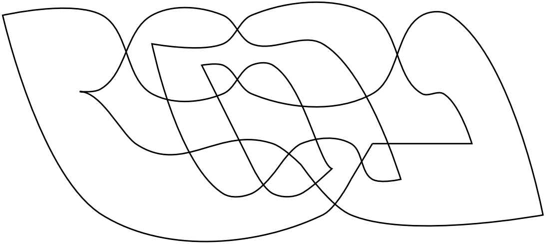

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(iii) Why is it useful to have cluster *A*-structures?

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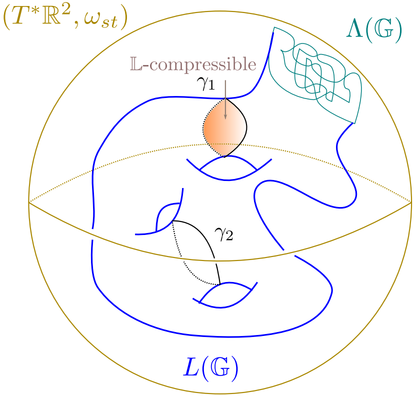

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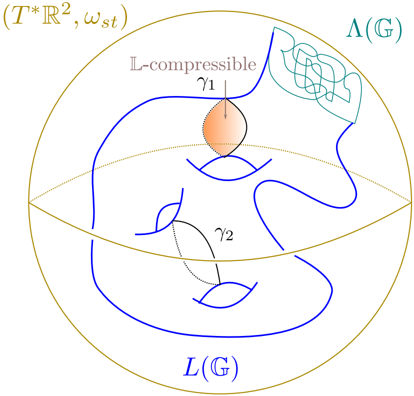

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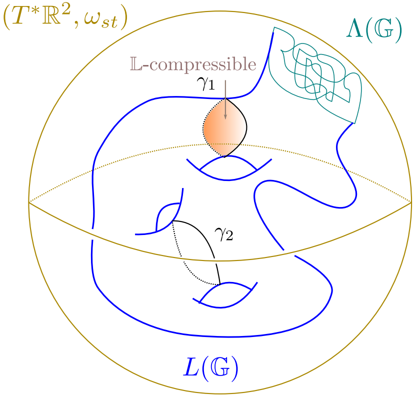

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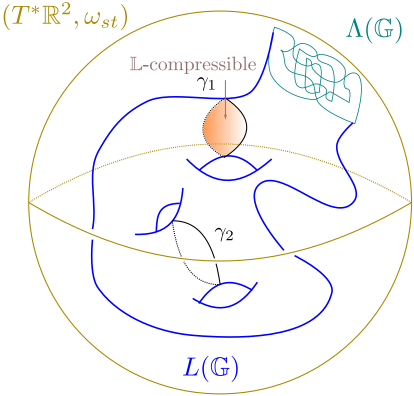
- Lagrangian filling gives $(\mathbb{C}^*)^{b_1(L(\mathbb{G}))} \subset \mathfrak{M}(\Lambda)$ chart.



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2. $\exists$ conjectural classification for positive braids:

Conjecture (ADE Classification of Lagrangian Fillings)

Let $\Lambda \subset (\mathbb{R}^3, \xi_{st})$ be the Legendrian closure of a positive braid. Then:

(A) *If Λ is link of the A_n-singularity, then Λ has precisely $\frac{1}{n+2}\binom{2n+2}{n+1}$ fillings.*

(D) *If Λ is link of the D_n-singularity, then Λ has precisely $\frac{3n-2}{n}\binom{2n-2}{n-1}$ fillings.*

(E) *If E_6, E_7, E_8-singularities, then precisely 833, 4160, and 25080 fillings.*

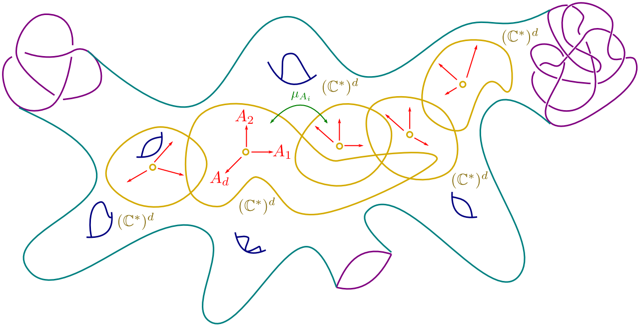
Else Λ *has* **infinitely** *many exact Lagrangian fillings.*

The ∞-many fillings above can be conjecturally parametrized using the *cluster algebras*. ($\Longrightarrow$ App. #1: Distinguishing fillings.)

The intuition for cluster varieties

Definition

A **cluster A-variety** $\mathfrak{M}$ is a union $\mathfrak{M} \overset{(cd.2)}{=} \bigcup_{s\in S} T_s$, $T_s \cong (\mathbb{C}^*)^d$ algebraic tori, with a **given identification** Spec $T_s \cong \mathbb{C}[A_{s,1}^{\pm 1}, \ldots, A_{s,d}^{\pm 1}]$ such that, in these identifications, **the transition functions are A-mutations** $\mu_{A_{s,i}}$.



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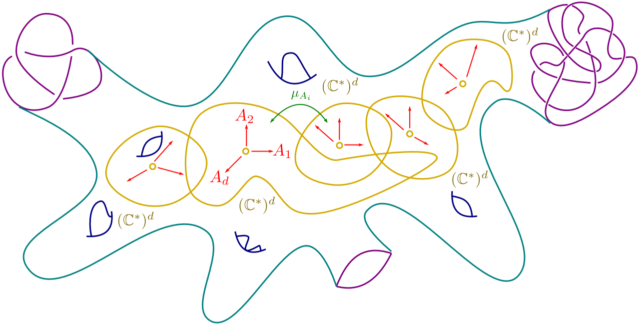

Input to define all $\mu_{A_{s,i}}$ is a *quiver*, or lattice basis with intersection form.

Properties and Examples

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- *Outstanding geometry*: computation of **singular cohomology**, with mixed Hodge structure, existence of **holomorphic symplectic** form, with curious Lefschetz, $\mathbb{F}_q$-**point counts**, any more. (E.g. $H^*(\mathfrak{M}(\Lambda_{8_{19}}), \mathbb{C})$.)

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- **Trefoil Example**: Then $\mathfrak{M}(\Lambda_{3_1}) = \{z_1 + z_3 + z_1z_2z_3 + 1 = 0\} \subset \mathbb{C}^3$, quiver is $\bullet \to \bullet$ and we have *five* algebraic tori:

$$T_1 = \mathrm{Spec}\{z_1^{\pm1}, (1+z_1z_2)^{\pm1}\},\quad T_2 = \mathrm{Spec}\{z_3^{\pm1}, (1+z_3z_2)^{\pm1}\},\quad T_3 = \mathrm{Spec}\{z_1^{\pm1}, z_3^{\pm1}\},$$

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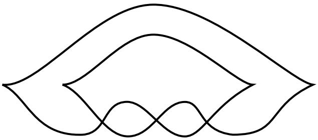
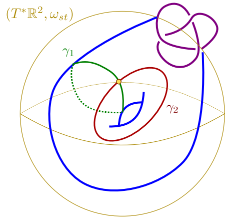




The Main Result

Main Theorem: For $\Lambda = \Lambda(\mathbb{G})$, the moduli variety $\mathfrak{M}(\Lambda(\mathbb{G}), T)$ is a **(quasi)cluster A-variety**. In fact, the quiver is $Q(\mathbb{G}, B)$ and the mutable vertices are $\mathbb{L}$-compressible in a canonical filling $L(\mathbb{G})$.

Theorem (C.-Weng - Coming Soon)

Let $\mathbb{G} \subset \mathbb{R}^2$ be an admissible grid plabic graph, $\Lambda = \Lambda(\mathbb{G})$ its associated Legendrian link and $T \subset \Lambda$ π_0-surjective marked points. **Then**, *there exists a canonical embedded exact Lagrangian filling $L = L(\mathbb{G})$ of Λ and a basis $B = \{\eta_1, \dots, \eta_s\}$ of the relative homology group $H_1(L \setminus T, \Lambda \setminus T; \mathbb{Z})$, indexed by Faces$(\mathbb{G})$ and T, such that:*

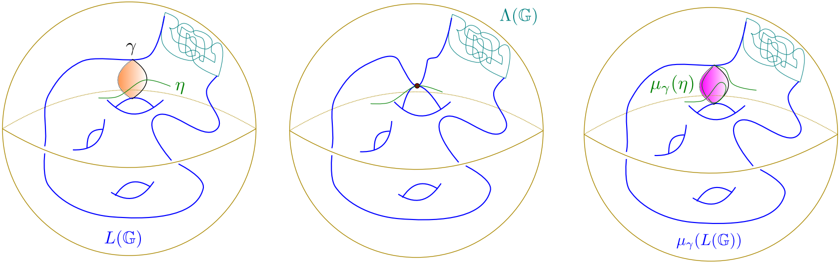
(i) The microlocal merodromies associated to the cycles η_i in L, $i \in [1, s]$, are global regular functions on the moduli variety $\mathfrak{M}(\Lambda, T)$. In addition, the construction of the basis B dictates which microlocal merodromies are globally non-vanishing.

(ii) For each sugar-free hull of $\mathbb{G}$, there exists a unique relative cycle $\eta \in B$ that is Poincaré dual to an $\mathbb{L}$-compressible absolute cycle $\gamma \in H_1(L, \mathbb{Z})$, bounding an embedded Lagrangian disk $D(\gamma)$, and a canonical relative cycle $\mu(\eta, D(\gamma))$ in $H_1(\mu(L, D(\gamma)) \setminus T, \Lambda \setminus T; \mathbb{Z})$ such that the microlocal merodromy along $\mu(\eta, D(\gamma))$ is a global regular function on the moduli variety $\mathfrak{M}(\Lambda, T)$.

(iii) The new microlocal merodromy $\mu(\eta, D(\gamma))$ is a cluster A-mutation of the initial microlocal merodromy of η with quiver $Q(\mathbb{G}, B)$, the intersection quiver of the basis B.

Lagrangian Disk Surgeries

A symplectic fact towards cluster algebras: **Lagrangian surgery**.



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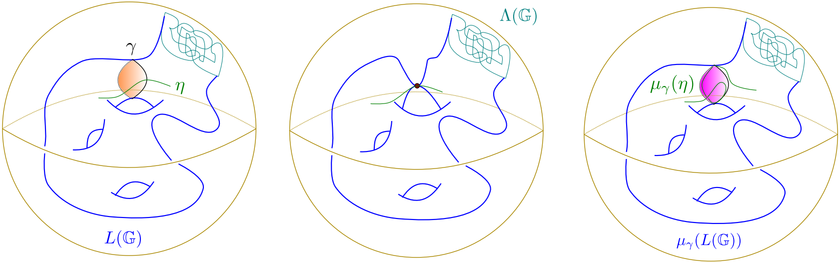

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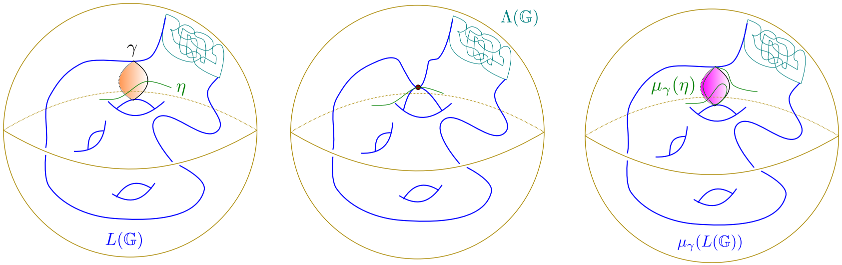

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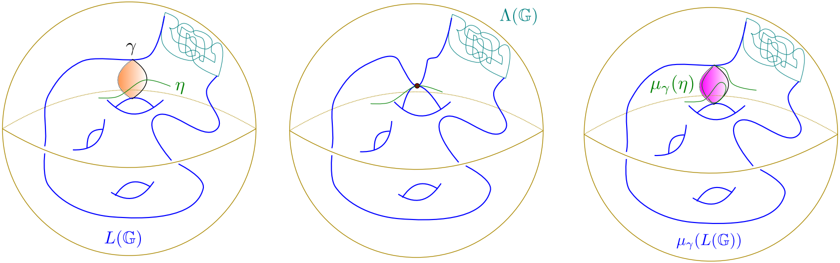

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(iii) How do you find these? $\longrightarrow$ **Legendrian weaves**.
Calculus in *Geom.&Top.* '22, ∞-fillings in *Ann. Math.* '22 + more

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Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{st}) \rightsquigarrow D^-$-stack $\mathfrak{M}(\Lambda)$ of objects in $\mathrm{Sh}_\Lambda(\mathbb{R}^2)$.

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(I) Need Λ such that D^--stack $\mathfrak{M}(\Lambda)$ is accessible, e.g. affine variety or algebraic quotient thereof, so cluster structures make sense:

$\rightsquigarrow$ Legendrian links Λ obtained from *grid plabic graph* $\mathbb{G}$

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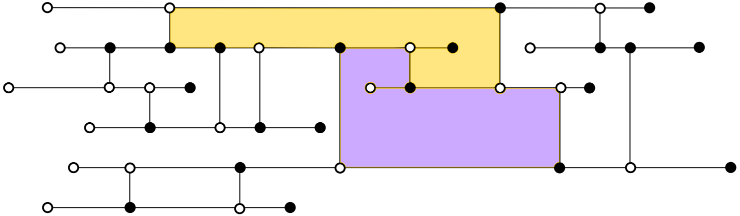

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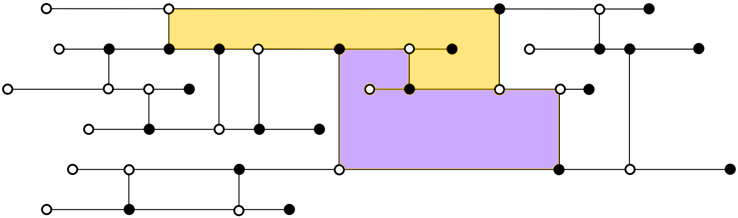

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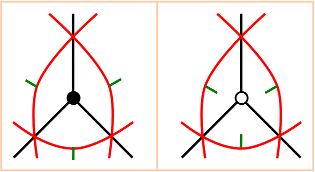

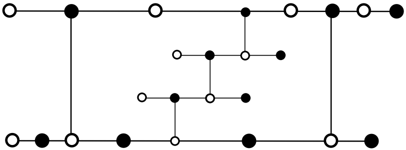

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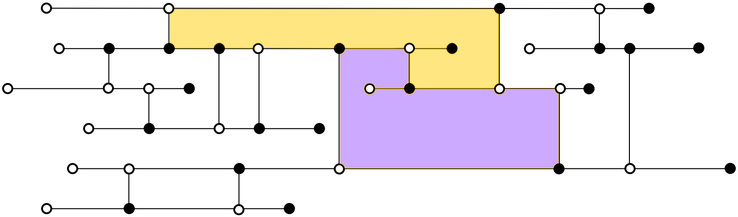

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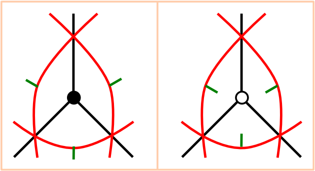

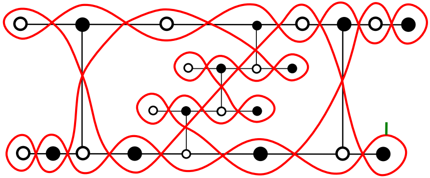

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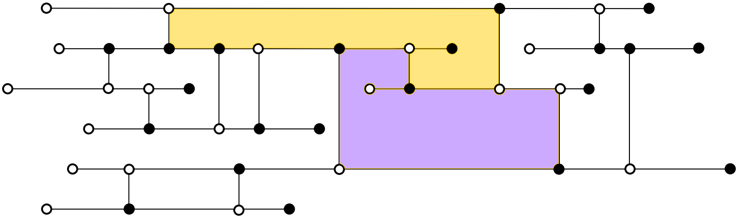

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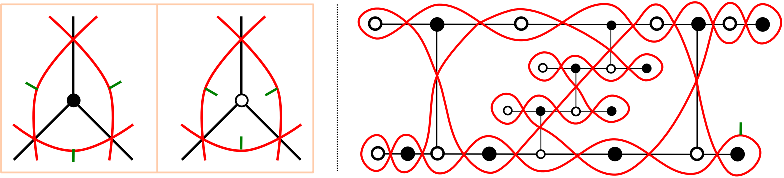
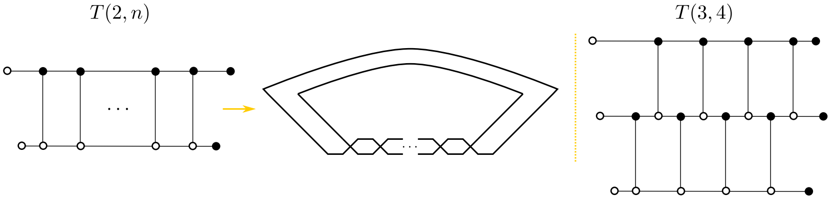


Then, $\Lambda(\mathbb{G}) \subset (\mathbb{R}^3, \xi_{st})$ **is the Legendrian link associated this front**, after satelliting the Legendrian S^1-fiber of $T^*_\infty\mathbb{R}^2$ to the standard unknot.

Examples of $\Lambda(\mathbb{G})$

Positive braid closures via plabic fences:



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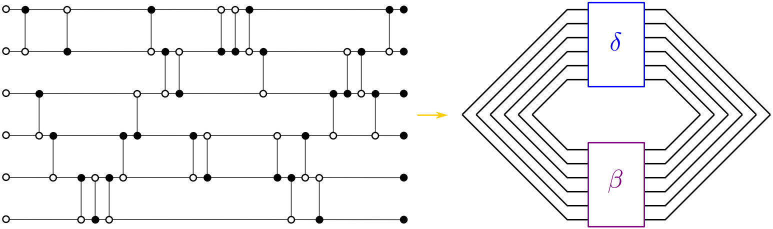

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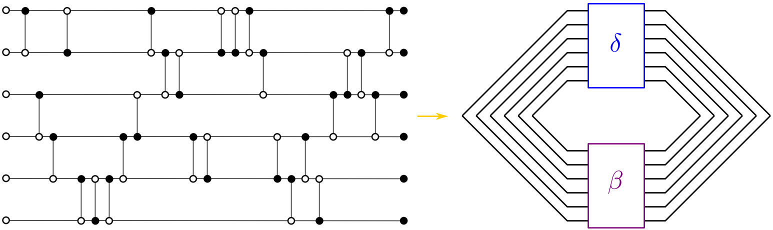
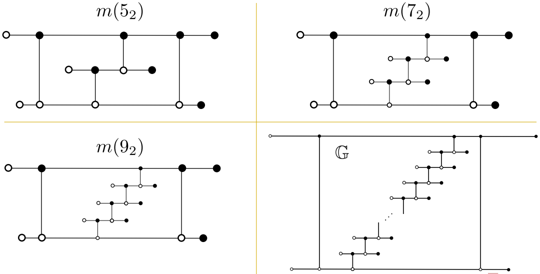


Legendrian Twist Knots:



The Lagrangian filling $L(\mathbb{G})$ and its basis

A fundamental property of $\Lambda(\mathbb{G})$ is given by the following result:

Theorem (Construction of weave Lagrangian filling with basis)

There exists a **canonical weave** $\mathfrak{w}(\mathbb{G})$ *representing an embedded Lagrangian filling* $L(\mathbb{G})$ *of* $\Lambda(\mathbb{G})$. *(Algorithmically from* $\mathbb{G}$*.)*

In addition, $\exists$ **basis of Y-cycles** *for* $H_1(L(\mathbb{G});\mathbb{Z})$ *from Hasse diagram of sugar-free hulls. In there, sugar-free cycles are* $\mathbb{L}$**-compressible** *and the rest, in bijection with some faces, are* **immersed**.

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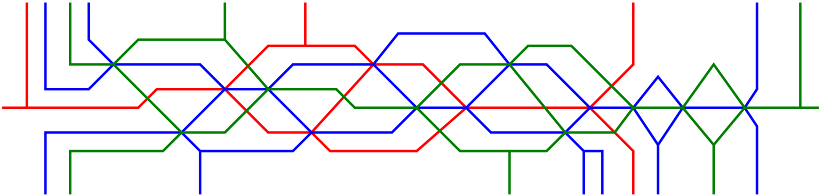
Calculus of weaves allows for a *planar diagrammatic manipulation* of **Lagrangian fillings in** $(\mathbb{R}^4, \lambda_{st})$ + study of their $\mathbb{L}$**-compressible cycles**.



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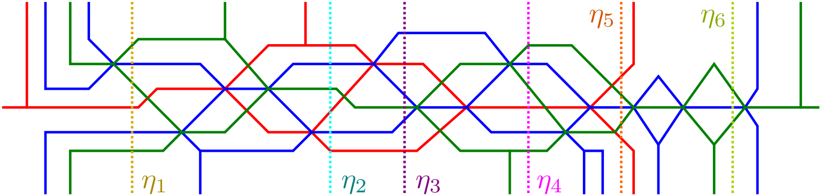
Calculus of weaves allows for a *planar diagrammatic manipulation* of **Lagrangian fillings in** $(\mathbb{R}^4, \lambda_{st})$ + study of their $\mathbb{L}$**-compressible cycles**.



The moduli $\mathfrak{M}(\Lambda)$

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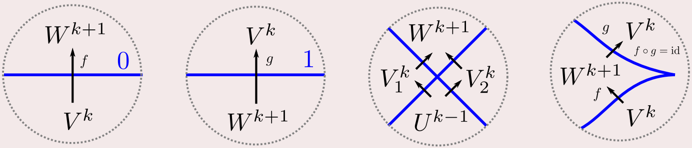

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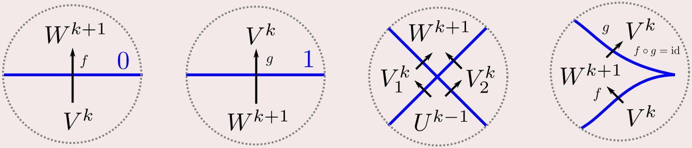

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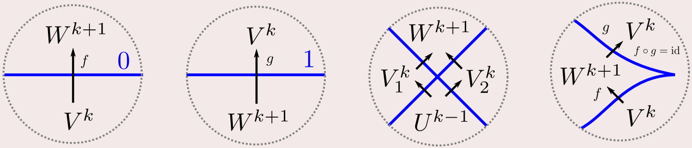

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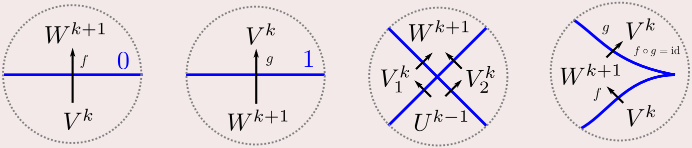

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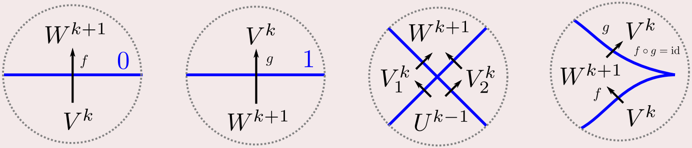

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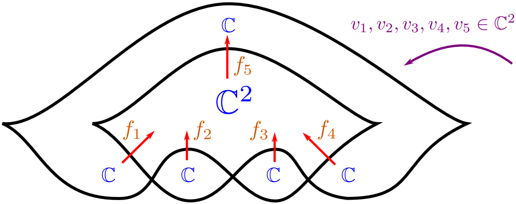
(iv) *Marked points* allow framing to be rescaled.

Examples of $\mathfrak{M}(\Lambda(\mathbb{G}))$ – Part I

Example Trefoil: Consider the plabic fence $\mathbb{G}$ for $\beta = \sigma_1^3 \in \mathrm{Br}_2^+$. Then

$$\mathfrak{M}(\Lambda(\mathbb{G})) = \{(v_1, v_2, v_3, v_4, v_5) : v_i \in \mathbb{C}^2, \det(v_i, v_{i+1}) = 1, i \in \mathbb{Z}_5\}/\mathrm{PGL}_2(\mathbb{C})$$



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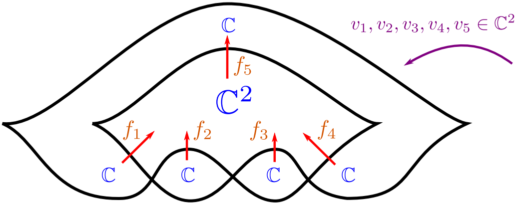

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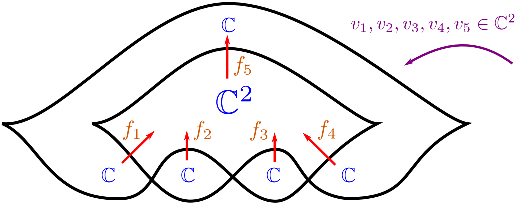

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- In this variety, $T_1 = \{z_1 \neq 0, z_3 \neq 0\} = \{v_1 \nparallel v_3, v_1 \nparallel v_4\}$ gives a toric chart $(\mathbb{C}^*)^2 \subset \mathfrak{M}(\Lambda(\mathbb{G}))$, and z_1 and z_3 basis.

How do we choose a basis? ($\{v_1 \nparallel v_3, v_2 \nparallel v_4\}$ does **not** work.)

Examples of $\mathfrak{M}(\Lambda(\mathbb{G}))$ – Part II

Positive braids: $\mathbb{G}$ plabic fence for $\beta = \sigma_{i_1} \ldots \sigma_{i_s} \in \mathrm{Br}_n^+$. Then $\mathfrak{M}(\Lambda(\mathbb{G}))$ is the moduli of tuples of affine flags in $(GL_n/U)^{s+n(n-1)}$ with F_j, F_{j+1} in s_{i_j}-relative position, with a Δ_n^2, plus framing conditions. ([CGGS 1&2])

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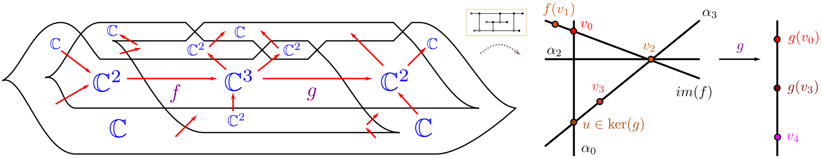

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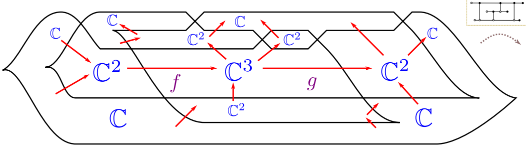

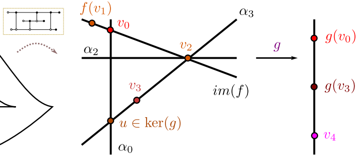

Some degenerations allowed, but **some not**!

The key points at this stage

- **Theorem A**: Let $\mathbb{G}$ be a GP-graph. Then

$$\exists \mathfrak{w}(\mathbb{G}) \textbf{ weave} \overset{s.t.}{\rightsquigarrow} \textbf{embedded Lagrangian filling } L(\mathbb{G}) + \textbf{basis of } \mathrm{Y}\textbf{-cycles}$$

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- **Next: Theorem C**. Need to introduce the basis of regular functions:

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  In addition, this **basis** $\mathbb{C}[T_{\mathfrak{w}(\mathbb{G})}]$ must change according to cluster A-mutation for $Q(B(\mathbb{G}))$ when **Lagrangian surgery is performed**.

The microlocal local system on $L(\mathbb{G})$ and $\Lambda(\mathbb{G})$

Define candidate A-**variables** with Guillermou-Kashiwara-Schapira maps:

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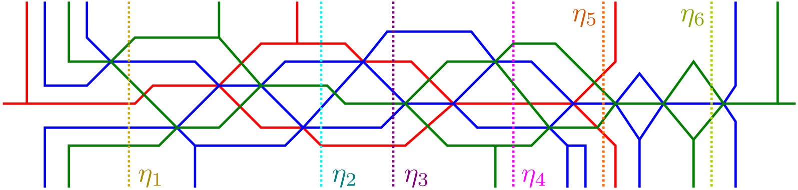
(2) **Theorem**: This parallel transport can be computed by using cones in the braid slice of a weave: *ratios of wedges of decorations*.



Microlocal Merodromies

Definition (Key new concept)

Let $\mathbb{G}$ be a GP-graph and $B(\mathbb{G})$ the **dual relative basis** of Y-cycles of the weave $\mathfrak{w}(\mathbb{G})$. The **microlocal merodromy** along $\eta \in B(\mathbb{G})$ is

$$A_\eta : \mathfrak{M}(\mathbb{G}) \longrightarrow \mathbb{C}$$

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These properties are **not** true unless η belongs to $B(\mathbb{G})$!

The resulting cluster A-structure

Finally, after developing these results, we can conclude:

Theorem (Simplified Upshot)

The moduli $\mathfrak{M}(\mathbb{G})$ admits an **upper (quasi)cluster A-structure** *in its coordinate ring, with initial cluster seed* **as symplectically described**.

The crucial step is showing that the inclusion of the upper bound into $\mathfrak{M}(\mathbb{G})$ is an isomorphism, up to codimension 2. This is done by applying "*Technical Properties*" and an argument with *immersed* weaves.

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- The stronger theorem being proved is in great part symplectic geometric: ability to define cluster A-coordinate symplectically via **merodromies** on:

Lagrangian fillings and a **basis of dually $\mathbb{L}$-compressible relative cycles**

The end

Thanks a lot!

Comparison: A-structure ('22) vs. partial X-structure ('16)

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Our construction of cluster A-structures always accesses **all tori**, even if infinitely many, and always **open tori** ($s = d$).