TRANSVERSE TORI IN ENGEL MANIFOLDS

ROBERT E. GOMPF

Very recent research (after AIM 2017).
Assume everything is compatibly oriented.

1) Contact structures: Hyperplane fields ξ in odd dimensional manifolds M. Maximally nonintegrable: $[\xi, \xi] = TM$. Gives nondegenerate form $[\,\cdot\,,\,\cdot\,]\colon \xi \times \xi \to TM/\xi$ extending to $d\alpha$ for a contact form α (up to sign).
   a) all the same locally b) open condition
   Part of a larger family of *topologically stable* distributions satisfying (a,b) (Cartan):
2) Line fields $\mathcal{W}$ (every dimension)
3) Even contact structures: Hyperplane fields $\mathcal{E}$ in even dimensional manifolds M. $[\mathcal{E}, \mathcal{E}] = TM$. Gives maximal rank form $[\,\cdot\,,\,\cdot\,]\colon \mathcal{E} \times \mathcal{E} \to TM/\mathcal{E}$. This has a 1-dimensional kernel $\mathcal{W} \subset \mathcal{E}$ and extends to $d\alpha$ as before. Every hypersurface transverse to $\mathcal{W}$ inherits a contact structure, preserved by any flow tangent to $\mathcal{W}$. Classified up to homotopy through even contact structures by homotopy-theoretic data (h-principle).
4) Engel structures: 2-plane fields $\mathcal{D}$ on 4-manifolds M. $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$, $[\mathcal{E}, \mathcal{E}] = TM$. Get a flag field $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$. Not known if h-principle applies to closed Engel manifolds (i.e. whether "tight" Engel structures exist.)

Main Example: Prolongation $\mathcal{P}N$ of a contact 3-manifold (N, ξ). $\mathcal{P}N$ is the circle bundle in $\mathfrak{p}\colon \xi \to N$. $\mathcal{W}$ tangent to fibers, $\mathcal{D}_p$ projects to the given line $\mathcal{L}_{\mathfrak{p}(p)}$, $\mathcal{E}$ projects to ξ. Line fields $\mathcal{L}$ in ξ correspond to sections of the bundle. Every $\mathcal{W}$-transverse N^3 in an Engel manifold has a neighborhood identified with one in $\mathcal{P}N$. Locally every Engel manifold looks like $\mathcal{P}\mathbb{R}^3$ with $\alpha = dz + xdy$ whose kernel is $\mathcal{E}$. Then $\mathcal{D} = \ker\alpha \cap \ker\beta$ where $\beta = \sin(w)dx + \cos(w)dy$. Note β is a contact structure on each plane of constant z.

Knots in contact 3-manifolds can always be made transverse to ξ. If they are nullhomologous, the self-linking number $l(K)$ distinguishes infinitely many transverse isotopy classes. The importance of such knots suggests:

Question 0.1. *(Eliashberg) What can be said about making closed surfaces transverse to $\mathcal{D}$ in Engel manifolds?*

Easy observation: A surface Σ with $\Sigma \pitchfork \mathcal{D}$ must be a torus with trivial normal bundle since $\nu\Sigma \cong \mathcal{D}|\Sigma$ and $T\Sigma \cong (TM/\mathcal{D})|\Sigma$ are trivialized by the Engel flag.

Theorem 0.2. *Every torus with trivial normal bundle is C^0-small isotopic to a transverse torus.*

Theorem 0.3. *a) If the torus is trivial in $H_2(M;\mathbb{Z})$ and in $H_1(M;\mathbb{Q})$ then it realizes infinitely many transverse isotopy classes.*
b) *There are many examples isotopic to infinitely many transverse regular homotopy classes, each of which realizes infinitely many transverse isotopy classes.*

These are distinguished by invariants $\Delta_T, \Delta_\nu \in H^1(\Sigma;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ comparing the Engel trivializations with topologically defined trivializations (so Δ_ν is analogous to $l(K)$).

What about surfaces (necessarily tori) analogous to Legendrian knots? Surfaces tangent to $\mathcal{D}$ everywhere exist, but not common. In $\mathcal{P}N$ they are precisely given by the circle bundle restricted to Legendrian knots. *Half-Legendrian* tori are more interesting: $\dim T\Sigma \cap \mathcal{D} = 1$ everywhere.

Insight: Can construct transverse tori by first constructing half-Legendrian tori and pushing off.

A generic surface Σ in $(M, \mathcal{D})$ has finitely many $\mathcal{W}$-tangencies. If $e(\nu\Sigma) = 0$, these cancel in pairs. Then can extend to $N \approx \Sigma \times \mathbb{R}$ transverse to $\mathcal{W}$. This inherits a contact structure. Can assume $\Sigma \subset N$ is convex. Want to simplify the dividing set. Need to find bypass disks.

Lemma 0.4. *In an Engel manifold, bypass disks always exist!*

Proof. Every bypass arc $C \subset \Sigma \subset N$ has a local model

$$[-\pi, \pi] \times 0 \times 0 \subset xy\text{−plane} \subset (\mathbb{R}^3, \cos(x)dy - \sin(x)dz)$$

The plane is convex with respect to the vector field $(0, 0, 1)$, with dividing set $x = n\pi$, which C hits in 3 points. Legendrian arc L (front projection, left figure) and C together bound a disk; $\mathrm{tb} = -1$. Looks like a bypass disk, but hits Σ. Eliminate extra intersection by perturbing along $\mathcal{W}$. □

Now can simplify $\Sigma = T^2$ to a standard model (right figure) with 2 dividing curves, 2 closed leaves, and all leaves running in the same direction (no Reeb components). Transverse to ξ, hence $\mathcal{E}$ in M, but still not transverse to $\mathcal{D}$ since $\mathcal{L}$ is tangent to Σ along a complicated 1-manifold. Control this by carefully twisting along arcs transverse to ξ. Arrange $\mathcal{L}$ tangent to Σ only on parallel red circles transverse to the foliation. The leaves are transverse to $\mathcal{D}$ in between. In a local model, they must be transverse in wxy-space relative to β. By standard contact topology, make them β-Legendrian. □