Classical wave methods and modern gauge transforms:

Spectral asymptotics in the one dimensional case

Joint with L. Parnovski and R. Shterenberg

High energy spectral asymptotics: the origins

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Conjecture (Sommerfeld–Lorentz, 1910)

Let

$$N(\lambda) := \#\{j : \lambda_j \leq \lambda\}.$$

Then,

$$N(\lambda) = \frac{\mathrm{vol}_g(M)\mathrm{vol}_{\mathbb{R}^d}(B_1)}{(2\pi)^d}\lambda^d + o(\lambda^d).$$

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Theorem (Weyl, 1911 (slightly modified setting))

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Let (M, g) be a smooth, compact Riemannian manifold of dimension d. Then, there are $\{a_j\}_{j=1}^{\infty}$ such that for all N,

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Let $N(\lambda) := \#\{j : \lambda_j \leq \lambda\}$. Then, there are $\{b_j\}_{j=1}^{\infty}$ such that for all N

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The naive conjecture is obviously false

- Let $(M, g) = (\mathbb{S}^2, g_{\text{round}})$.
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\begin{aligned}
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If there are 'very' few periodic geodesics, then $E(\lambda, g, V) = O(\lambda^{d-1}/\log\lambda)$.

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All based on Levitan's wave method (to be explained later).

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- New problem!: $N(\lambda, g, V)$ does not make sense here.

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If there are few loops from x to itself, then

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If the geodesics through x are all periodic with the same time,

$$\left|e(-\Delta_g + V, \lambda)(x) - (2\pi)^{-d}\mathrm{vol}_{\mathbb{R}^d}(B_1)\lambda^d\right| \neq o(\lambda^{d-1}).$$

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- Still a problem $V = |x|^2$.

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We say $V \in C_b^\infty(\mathbb{R}^d)$ if $V \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$, there are $C_\alpha > 0$ such that

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Suppose $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $\{a_j(x)\}_{j=0}^\infty$ such that for any $N > 0$,

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Suppose $V_1, V_2 \in C_b^\infty(\mathbb{R}^d)$. Then, if $V_1 = V_2$ in a neighborhood of x, for any $N > 0$, we have

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compactly supported +periodic on $\mathbb{R}$	wave method + GT	[G 2020]

The conjecture is true in 1 dimension

Theorem (G – Parnovski – Shterenberg 2022)

Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$*. Then there are* $\{a_j(x)\}_{j=0}^\infty$ *such that for all* $N > 0$*, there is* $C_N > 0$ *satisfying*

$$\left| e(-\Delta_{\mathbb{R}} + V, \lambda)(x) - \sum_{j=0}^{N-1} a_j(x)\lambda^{1-2j} \right| \leq C_N \lambda^{1-2N}.$$

Moreover $a_j(x)$ *can be determined from a finite (*j*-dependent) number of derivatives of* V *at* x*.*

Corollaries of the theorem: Spectral Gaps

Corollary (G – Parnovski – Shterenberg 2022)

Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$*. Then for all* $N > 0$*, there is* $C_N > 0$ *such that for all* $\lambda \geq 1$ *and* $\epsilon > 0$*, if*

$$\mathrm{spec}(-\Delta_{\mathbb{R}} + V) \cap [\lambda - \epsilon, \lambda + \epsilon] = \emptyset,$$

then

$$\epsilon \leq C_N \lambda^{-N}.$$

Corollaries of the theorem: Almost plane waves

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Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$*. Then for all* $N > 0$ *there are* $c_N > 0$ *and* $C > 0$ *such that for any* $\lambda > 1$ *and any solution of*

$$(-\Delta_{\mathbb{R}} + V - \lambda^2)u = 0,$$

and any $x_1, x_2 \in \mathbb{R}$ *with* $|x_1 - x_2| < c_N \lambda^N$*,*

$$|u(x_1)|^2 + \lambda^{-2}|u'(x_1)|^2 \leq e^{C\lambda^{-1}}(|u(x_2)|^2 + \lambda^{-2}|u'(x_2)|^2)$$

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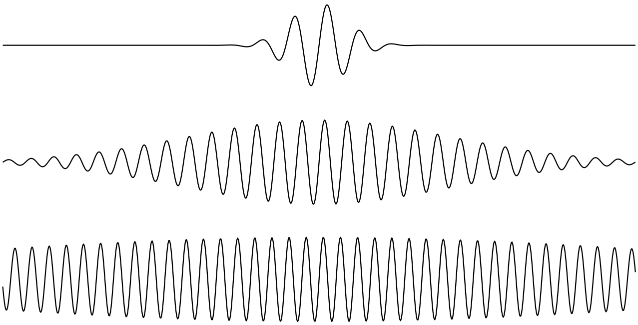

Corollaries of the theorem: Lyapunov exponents

Corollary (G – Parnovski – Shterenberg 2022, (see also Delyon–Foulon 1986))

Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. If the Lyapunov exponent, $\Lambda(\lambda)$, makes sense, then $\Lambda(\lambda) \le C_N \lambda^{-N}$.

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Heuristic message

The spectrum WANTS to be absolutely continuous.

Ideas from the proof: Wave method

- Use the Fourier transform to write:

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- Tauberian methods or scattering theory allow us to compare smoothed with unsmoothed.

Ideas from the proof: Gauge transform

- Use Moser averaging to reduce a periodic problem to a constant coefficient problem: Find Φ so that

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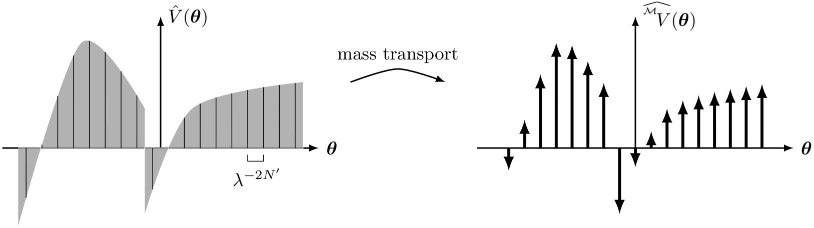

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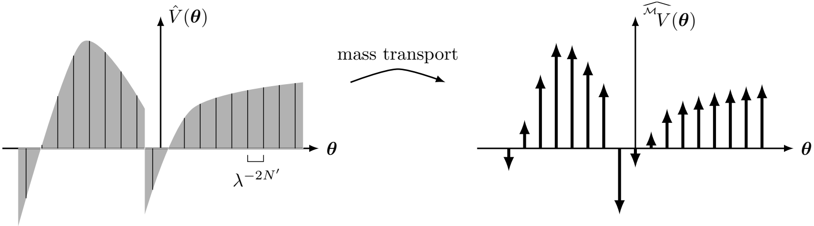

- Crucial new feature – the periodic lattice is huge! ($\gg \lambda^N$).

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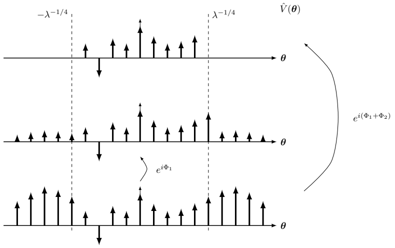
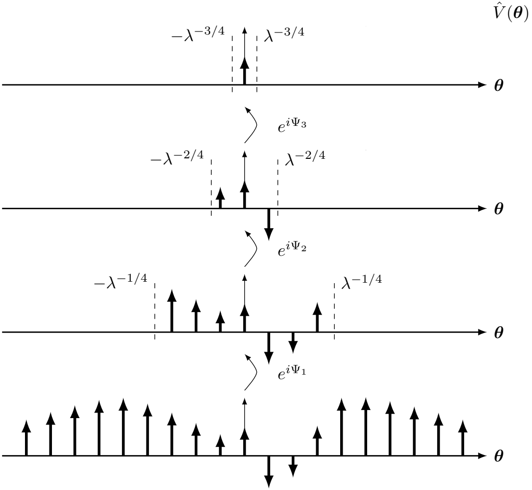


Peeling successive layers



Thank you!