Rough Invariant Imbedding

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New interfaces of Stochastic Analysis and Rough Paths

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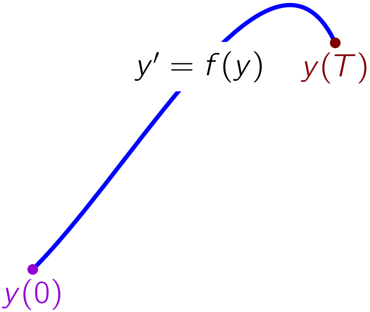
2-point affine boundary condition (BC)



Constraint: $H_0 y(0) + H_1 y(T) = v$

2-point affine boundary condition (BC)

We consider extending to the “rough context” for H_0, H_1 bounded, linear operators the problem

$$\begin{cases} y(v,t,T) = y(v,0,T) + \int_0^t f(y(v,s,T))\,\mathrm{d}s, \ t \in [0,T], \\ H_0 y(v,0,T) + H_1 y(v,T,T) = v. \end{cases}$$

Origin: transport problem, radiative transfer, biology, waves in disordered media, …

Our approach relies on the invariant imbedding[1], which is “considered as a concept and not as a technique”.

Reference: R. Bellman & G.M. Wing (1975, reprint 1987)

[1]V.A. Ambarzumyan, S. Chandrasekhar, R. Bellman *et al.*, …

Step 1: Differentiate the flow

$$\text{With } y(v,t,T) = y(v,0,T) + \int_0^t f(y(v,s,T))\,\mathrm{d}s,$$

$$\partial_v y(v,t,T) = \partial_v y(v,0,T) + \int_0^t \nabla f(y(v,s,T))\partial_v y(v,s,T)\,\mathrm{d}s,$$

$$\partial_T y(v,t,T) = \partial_T y(v,0,T) + \int_0^t \nabla f(y(v,s,T))\partial_T y(v,s,T)\,\mathrm{d}s.$$

Let Z be the unique solution to the linear eq.

$$Z(t) = \mathrm{Id} + \int_0^t \nabla f(y(v,s,T))Z(s)\,\mathrm{d}s.$$

Thus,

$$\partial_v y(v,t,T) = Z(t)\partial_v y(v,0,T),$$
$$\partial_T y(v,t,T) = Z(t)\partial_T y(v,0,T).$$

Step 2: Differentiate the boundary condition

$$H_0 y(v,0,T) + H_1 y(v,T,T) = v,$$

$$\partial_v \implies \qquad (H_0 + H_1 Z(T))\partial_v y(v,0,T) = \mathrm{Id}, \qquad (\clubsuit)$$

$$(H_0 + H_1 Z(T))\partial_T y(v,0,T) + H_1 \nabla f(y(v,T,T)) = 0. \qquad (\spadesuit)$$

As long as $H_0 + H_1 Z(T)$ is invertible,

$$\partial_v y(v,0,T) H_1 \nabla f(y(v,T,T)) + \partial_T y(v,0,T) = 0. \qquad (\heartsuit)$$

With the method of characteristics[2]

$$y(v(T),0,T) = y(v,0,0)$$

$$\text{with } v(t) = v + \int_0^t H_1 \nabla f(y(v,s,s))\,\mathrm{d}s$$

[2]Just differentiate $t \mapsto y(v(t),0,t)$ and use $(\heartsuit)$

Step 3: Use the boundary condition with $T = 0$

Since

$$H_0 y(v, 0, T) + H_1 y(v, T, T) = v,$$

setting $T = 0$,

$$H_0 y(v, 0, 0) + H_1 y(v, 0, 0) = v.$$

Provided that $H_0 + H_1$ is invertible,

$$y(v, 0, 0) = (H_0 + H_1)^{-1} v.$$

Conclusion:

$$y(v(T), 0, T) = (H_0 + H_1)^{-1} v$$

We have the initial condition, but not for the right value of v.

Step 4: Use the flow, Luke

$$y(v,0,T) = (H_0 + H_1)^{-1}\left(v - H_1 \int_0^T f(y(v,s,s))\,\mathrm{d}s\right).$$

But $y(v,s,s) = \mathcal{I}[y(v,0,s)](s)$ where

$$\mathcal{I}[a](t) = a + \int_0^t f(\mathcal{I}[a](s))\,\mathrm{d}s,\ \ t \geqslant 0$$

is the flow of the ODE $y' = f(y)$. Provided that $f \in \mathcal{C}^1$, $\mathcal{I}$ is well defined and is $\mathcal{C}^1$ in space and time.

Thus, the starting point is solution to the fixed point problem

$$y(v,0,T) = (H_0+H_1)^{-1}\left(v - H_1 \int_0^T f(\mathcal{I}[y(v,0,T)](s))\,\mathrm{d}s\right)$$
$$= (H_0 + H_1)^{-1}\big(v - H_1(\mathcal{I}[y(v,0,T)](T) - y(v,0,T))\big).$$

Invariant imbedding: To summarize

If there is a solution to the problem

$$y'(t) = f(y(t)) \text{ with } H_0 y(0) + H_1 y(T) = v \qquad (\clubsuit)$$

then $a := y(0)$ solves the non-linear problem

$$a = (H_0 + H_1)^{-1}(v - H_1(\mathcal{I}[a](T) - a)) \qquad (\spadesuit)$$

provided that

- $H_0 + H_1$ is invertible (excludes periodic BC)
- $y' = f(y)$ has a $\mathcal{C}^1$ flow $(a, t) \mapsto \mathcal{I}[a](t)$
- $H_0 + H_1 \partial_a \mathcal{I}[a](T)$ is invertible for any starting point a.

The converse is (obviously) true: if a solves ($\spadesuit$), then $t \mapsto \mathcal{I}[a](t)$ solves ($\clubsuit$)

Invariant imbedding: extension to RDE

Let x be a Young/rough path. Consider the RDE with 2-point affine BC:

$$\begin{cases} y(t) = y(0) + \int_0^t f(y(s))\,\mathrm{d}x(s), \\ H_0 y(0) + H_1 y(T) = v \end{cases}$$

Why?

- arises (driven by BM or fBM) as limits of ODE with highly-oscillating coefficients
- convenient for a wide range of stochastic drivers
- extend what is known about SDE[3]
- avoid considerations on anticipative stochastic calculus
- for the fun

[3] Occone & Pardoux (1989), Donati-Martin (1991), Nualart & Pardoux (1991), Fouque & Merzbach (1994), Garnier (1995), ...

Flow property of RDE (1/2)

The Young/Rough Differential Equation

$$y(t) = a + \int_0^t f(y(s))\,\mathrm{d}x(s) \qquad (\bigstar)$$

enjoys "similar" properties to ODE provided that f is regular enough[4]:

- Classical ($x(t) = t$): If $f \in \mathcal{C}^k$, $k \geqslant 1$, then the solution to ($\bigstar$) is unique and is locally $\mathcal{C}^k$ (sup-norm).
- Young ($x \in \mathcal{C}^\alpha$, $\alpha > 1/2$): if $f \in \mathcal{C}^{k+\gamma}$, $\alpha(1+\gamma) > 1$, then the solution to ($\bigstar$) is unique and is locally $\mathcal{C}^{k+\gamma-\epsilon}$ with respect to (a, x, f) (Hölder norm)
- Rough ($x \in \mathcal{C}^\alpha$, $1/3 < \alpha \leqslant 1/2$): if $f \in \mathcal{C}^{k+1+\gamma}$, $\alpha(2+\gamma) > 1$, then the solution to ($\bigstar$) is unique and is locally $\mathcal{C}^{k+\gamma-\epsilon}$ with respect to (a, x, f) (Hölder norm).

[4] Long history, started from Lyons & Qian, Bailleul, Lyons & Li, Friz & Victoir, Y. Inahama & H. Kawabi, and Coutin & L.

Flow property of RDE (2/2)

In all cases, provided that f is regular enough, the solution to

$$y(t,a) = a + \int_0^t f(y(s,a))\,\mathrm{d}x(s) \qquad (\bigstar)$$

is well defined and $a \mapsto y(t,a)$ defines a $\mathcal{C}^1$-diffeomorphism for any $t \geqslant 0$. If f is bounded, $a \mapsto y(t,a)$ is globally Lipschitz.

We denote by $\mathcal{I}$ the Itô map

$$\mathcal{I}[a,x,f] = t \mapsto y(t,a) \text{ where } y \text{ solves } (\bigstar).$$

In particular, $\mathrm{D}_a\mathcal{I}[a,x,f]$ solves the linear RDE

$$\mathrm{D}_a\mathcal{I}[a,x,f](t) = \mathrm{Id} + \int_0^t \nabla f(y(s)) \cdot \mathrm{D}_a\mathcal{I}[a,x,f](s)\,\mathrm{d}x(s).$$

Invariant imbedding for RDE

Set $H = H_0 + H_1$ and assume H invertible.
We consider the 2-points affine BC

$$\begin{cases} y(t) = \mathcal{I}[y(0), x, f](t), \\ H_0 y(0) + H_1 y(T) = v \end{cases} \qquad (\spadesuit)$$

as well as the non-linear fixed-point problem

$$H^{-1}v - H^{-1}H_1(\mathcal{I}[a, x, f](T) - a) = a. \qquad (\clubsuit)$$

Existence in short time (Marty & L)

If $|(x, f)| \leqslant M$ for a given constant, f is bounded and $T^\alpha M|H^{-1}| < 1$ then there exists a unique solution a to ($\clubsuit$) (Banach fixed-point) and to ($\spadesuit$) (by setting $y_t(a) = \mathcal{I}[a, x, f](t)$).

Continuity of the solution

Of course, the solution inherits from the regularity of the Itô map thanks to the Implicit Function Theorem.

Continuity (Marty & L)

Under the conditions of existence and uniqueness, the map giving the solution to the 2-point affine BC is Lipschitz continuous when x remains in a ball (the time horizon and the radius of the ball are linked).

Applications: random noise

$$\begin{cases} \dfrac{dY^n(t)}{dt} = \sqrt{n} \displaystyle\sum_{j=1}^{d} \xi_{j,\lfloor nt \rfloor} f_j(Y^n(t)), \\ H_0 Y^n(0) + H_1 Y^n(T) = v \end{cases}$$

where $\{\xi_{k,j}\}$, iid, mean 0, variance 1, finite moments. Define

$$W_j^n = \text{linear interpolation of } \frac{1}{\sqrt{n}} \sum \xi_{j,\cdot}$$

and $\mathbf{W}^n$ its enhanced version. By Donsker, $\mathbf{W}^n$ converges to $\mathbf{W}$ (enhanced Brownian motion).

The process Y^n converges, restricted to the event $\{|\mathbf{W}^n| \leqslant M\}$, to

$$\mathrm{d}Y(t) = f(Y(t)) \circ \mathrm{d}\mathbf{W}(t) \text{ and } H_0 Y(0) + H_1 Y(T) = v.$$

Similar results hold for other kinds of noise, with fBM in the limit.

Global existence

We have proved existence and uniqueness for small time.

What about global existence?

If f is not bounded, no global solution may exists.

We used the Banach fixed point theorem, which holds thanks to controls on the Lipschitz norm of

$$a \mapsto H^{-1}v - H^{-1}H_1(\mathcal{I}[a, x, f](T) - a).$$

This control holds for "short time", as $\mathcal{I}[a, x, f](T)$ is close to a.

Brouwer's degree

Consider

- O open subset
- $\Phi : \overline{O} \to \mathbb{R}^d$ continuous
- $y \in \mathbb{R}^d$, $y \notin \Phi(\partial O)$

The Brouwer's degree is

$$\deg(\Phi, O, y) = \sum_{x \in \Phi^{-1}(\{y\})} \operatorname{sgn} \det \operatorname{Jac}[\Phi](x)$$

whenever y is a regular value, that is $\operatorname{Jac}[\Phi](x) \neq 0$ for all $x \in \Phi^{-1}(\{y\})$.

The degree has a lot of nice properties, including

- If $\deg(\Phi, O, y) \neq 0$, then there exists a least one solution to $\Phi(x) = y$ for $y \notin \partial O$.
- It is stable by homotopy (continuous deformation) $\rightsquigarrow$ practical computation

Existence for large time

Existence for large time (R. Marty & L)

In a finite-dimensional state space, if $H_0 + H_1$ is invertible, there exists a solution to the 2-points affine BC.
Besides, for almost every v, the number of solutions is finite.

We use the degree theory for $a \mapsto a + H^{-1}H_1(\mathcal{I}[a, x, f](T) - a)$ (continuous deformation from $a \mapsto a$ in T).

The finiteness of the number of solutions is a consequence of the Sard theorem.