On the stability of laminar flows between plates

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Joint work with: Bernard Helffer

Braude, Nantes

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Spectral formulation

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Navier-Stokes Equations

$$\begin{cases} \partial_t \mathbf{v} - \epsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{b} & \text{in } \mathbb{R}_+ \times D \\ \mathbf{v} = 0 & \text{on } \mathbb{R}_+ \times \partial D \end{cases}$$

$\mathbf{v}$: velocity $\quad$ p : pressure $\quad$ $\mathbf{b} = b(x_2)\hat{i}_1$: body force

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$$(\mathfrak{T} - \Lambda)\mathbf{u} - \nabla q = \mathbf{F}\,.$$

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Restriction to $[0, 1]$ – even modes

$$D(\mathcal{B}_{\lambda,\alpha,\beta}) = \{u \in H^4(0,1)\,,\; u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\}.$$

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Grenier, Guo, &Nguyen (2016) Rigorous analysis

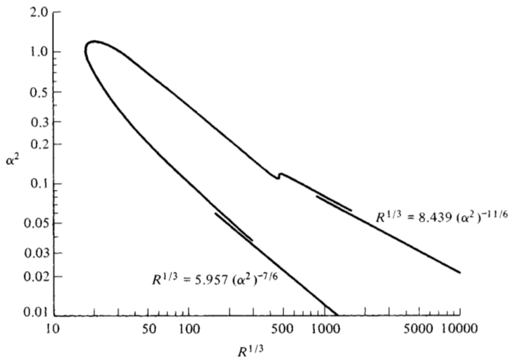
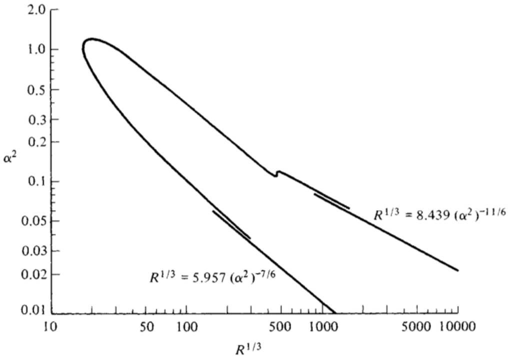


Fig. 4.11. The curve of marginal stability for plane Poiseuille flow based on equation (28.33). (From Reid 1965.)



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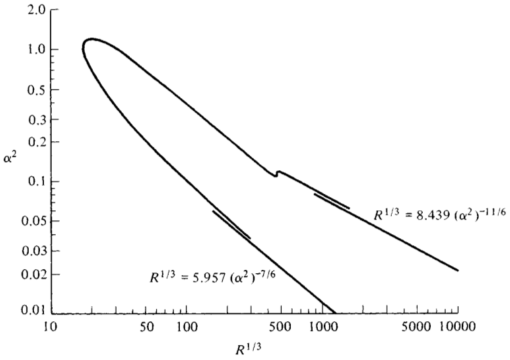

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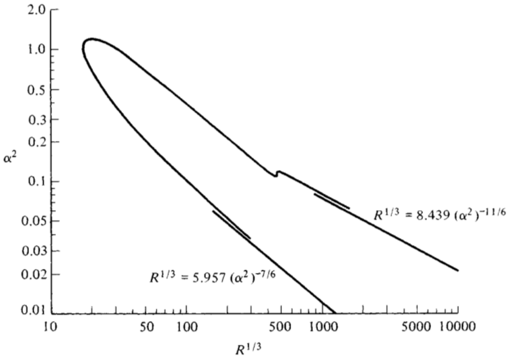

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Theorem

Let $\delta > 0$ *and* $U \in C^4([0,1])$ *satisfy*

$$U(1) = 0 \quad ; \quad \max_{x\in[-1,1]} U''(x) < 0 \quad ; \quad U'(0) = U^{(3)}(0) = 0$$

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Then $\exists(\alpha_L, C, \Upsilon) \in \mathbb{R}^3_+$, *and* $\beta_0 > 1$*:* $\forall \beta > \beta_0$ *it holds that*

$$\sup_{\substack{0\leq\alpha\leq\alpha_L\beta^{-1/6} \\ \Re\lambda<\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-1/2}\,.$$

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Proof outlines

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Alternative formulation of $\mathcal{B}_{\lambda,\alpha,\beta}\phi=f$

$$\begin{cases}(\mathcal{L}-\beta\lambda)v_{\mathfrak{D}}=g\\ v_{\mathfrak{D}}(1)=v_{\mathfrak{D}}'(0)=0\end{cases}\qquad \mathcal{L}=-\frac{d^2}{dx^2}+i\beta U$$

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$$\int_0^1 \mathcal{A}_{0,0}\phi\,dx = 0 \quad \phi(1)=0 \Rightarrow \mathcal{A}_{0,0}\phi\big|_{x=1} = 0\,.$$

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$$\begin{cases} \left(-\frac{d^2}{dx^2} + i\beta U\right)v = Uf + 2U'\phi^{(3)} + U''\phi'' - (U''\phi)'' & \text{in } [0,1] \\ v(1) = v'(0) = 0 \end{cases}$$

$$\|v\|_2 \leq C\Big(\beta^{-2/3}\big(\|\phi\|_{1,2} + \|\phi''\|_2 + \|\phi^{(3)}\|_2\big) + \beta^{-1}\|f\|_2\Big)$$

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$$\|v\|_2 \leq C\Big(\beta^{-2/3}(\|\phi\|_{1,2} + \|\phi''\|_2 + \|\phi^{(3)}\|_2) + \beta^{-1}\|f\|_2\Big)$$

$$\|\phi''\|_2 \leq C\left(\beta^{-2/3}\|f\|_2 + \beta^{1/6}\|\phi\|_{1,2}\right)$$

$$\|\phi^{(3)}\|_2 \leq C\left(\beta^{-1/3}\|f\|_2 + \beta^{1/2}\|\phi\|_{1,2}\right)$$

$$\|\phi\|_{1,2} \leq C\|v\|_2 \leq C\Big(\beta^{-1/6}\|\phi\|_{1,2} + \beta^{-1}\|f\|_2\Big)$$

Proposition

Let $U \in C^4([0,1])$*:* $U(1) = U'(0) = U^3(0) = 0$*;* $U'' < 0$*. Then,* $\exists C > 0,\ \beta_0 > 0$*:* $\forall \beta \geq \beta_0$

$$\left\|(\mathcal{B}_{0,0,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{0,0,\beta})^{-1}\right\| + \beta^{-1/3}\left\|\frac{d^2}{dx^2}(\mathcal{B}_{0,0,\beta})^{-1}\right\| \leq C\beta^{-1}\,.$$

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}^{\mathfrak{D}}_{0,0,\beta}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}^{\mathfrak{D}}_{0,0,\beta}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

Corollary

Let $U \in C^4([0,1])$*:* $U(1) = U'(0) = U^3(0) = 0$*;* $U'' < 0$*. Then,*

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$$

$$\Updownarrow$$

$$\mathcal{B}^{\mathfrak{D}}_{0,0,\beta}\phi = f + \beta\lambda(\phi'' - \alpha^2\phi) - \alpha^2\phi'' + i\alpha^2\beta U\phi$$

Corollary

Let $U \in C^4([0,1])$*:* $U(1) = U'(0) = U^3(0) = 0$*;* $U'' < 0$*. Then,*

$$\exists C > 0,\ \beta_0 > 0,\ \alpha_0 > 0,\ \lambda_0 > 0 : \forall \beta \geq \beta_0$$

$$\sup_{\substack{0\leq\alpha\leq\alpha_0 \\ |\lambda|<\lambda_0\beta^{-1/3}}} \left\|(\mathcal{B}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-1}\,.$$