Some non-self-adjoint problems in the theory of composites

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Three topics:

- Conduction in composites in the presence of a magnetic field
- Conversion to a self-adjoint problem
- PT symmetry in space-time field patterns

Conductivity equation: $\nabla \cdot \boldsymbol{\sigma} \nabla V = 0$,

In a isotropic material $\boldsymbol{\sigma}(\mathbf{x}) = \sigma(\mathbf{x})\mathbf{I}$.

Set $\psi = \sigma^{1/2} V, \quad q = \frac{\Delta \sigma^{1/2}}{\sigma^{1/2}}$

Schrödinger equation: $\Delta \psi = q\psi$

In a magnetic field $\boldsymbol{\sigma}(\mathbf{x})$ is not symmetric (Hall effect)

$$\mathbf{j} = \boldsymbol{\sigma}\mathbf{e}, \quad \nabla \cdot \mathbf{j} = 0, \quad \mathbf{e} = -\nabla V$$

$$\mathbf{j} = \boldsymbol{\sigma}_0 \mathbf{e} + (\mathbf{S}\mathbf{b}) \times \mathbf{e}, \quad \mathbf{e} = \boldsymbol{\rho}_0 \mathbf{j} + (\mathbf{A}_H \mathbf{b}) \times \mathbf{j}$$

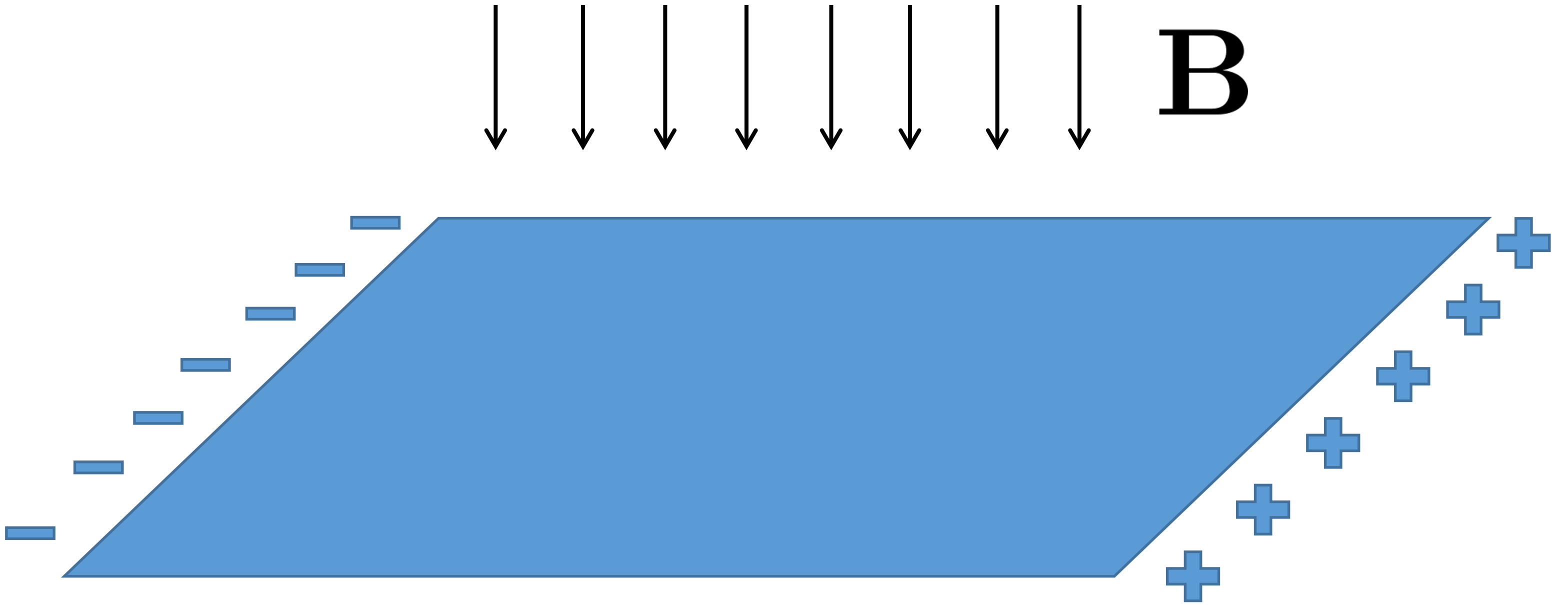
For an isotropic material $\mathbf{A}_H = R_H \mathbf{I}$, ($R_H$ - Hall coefficient)

Achieving reversal of the sign of the Hall coefficient

With Marc Briane, Christian Kern, Muamer Kadic,
Martin Wegener, Dylon Whyte,

An example of surprising properties of composites:

Reversal of the Hall-effect coefficient



Hall Voltage

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

However there is a counterexample!

Mathematically: Find a conducting periodic composite with say cubic symmetry, where the matrix-valued electric field has negative trace of its cofactor matrix in some regions.

Geometry suggested by artist Dylon Whyte

Picture
Courtesy
Dylon Whyte

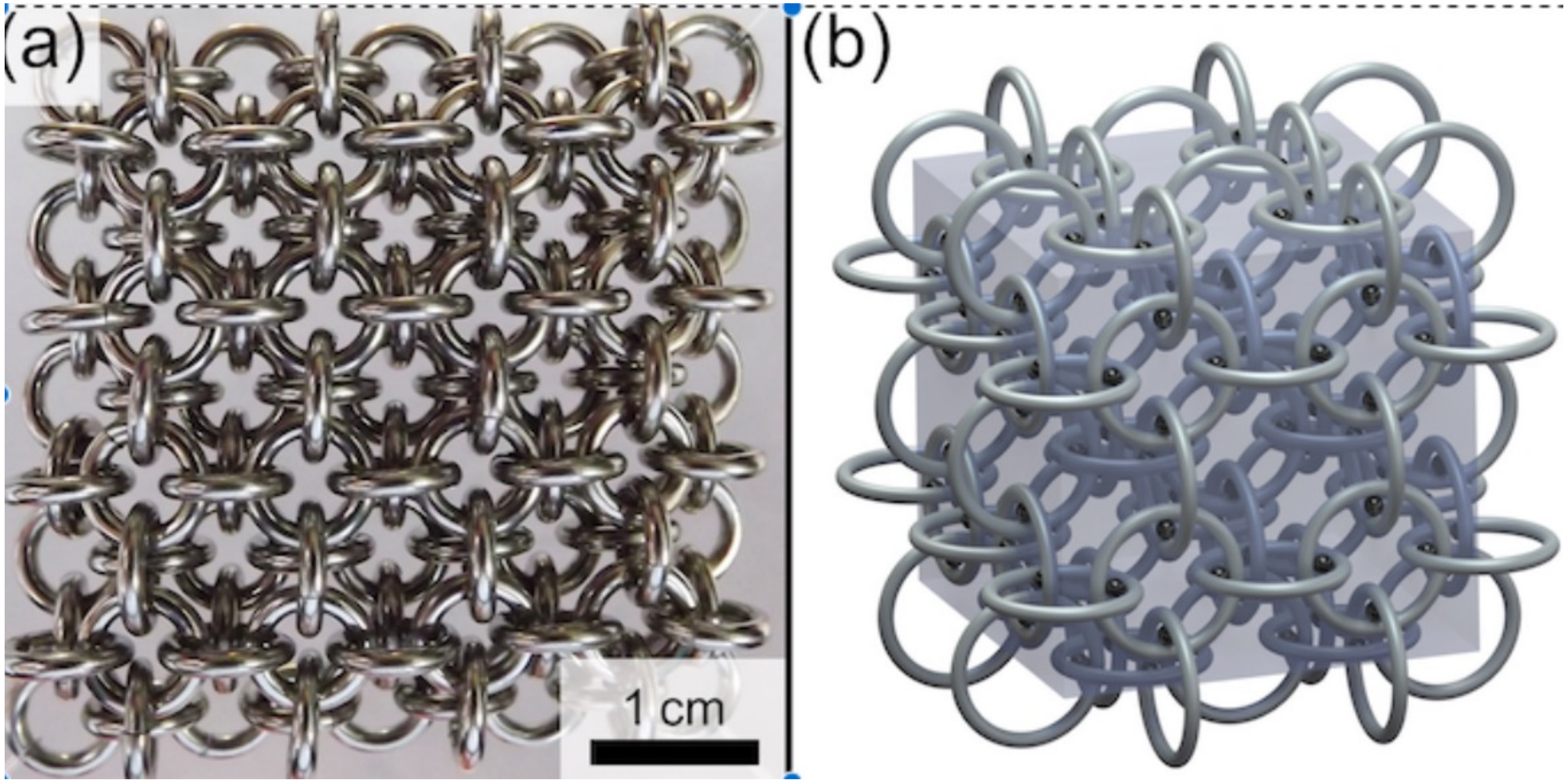
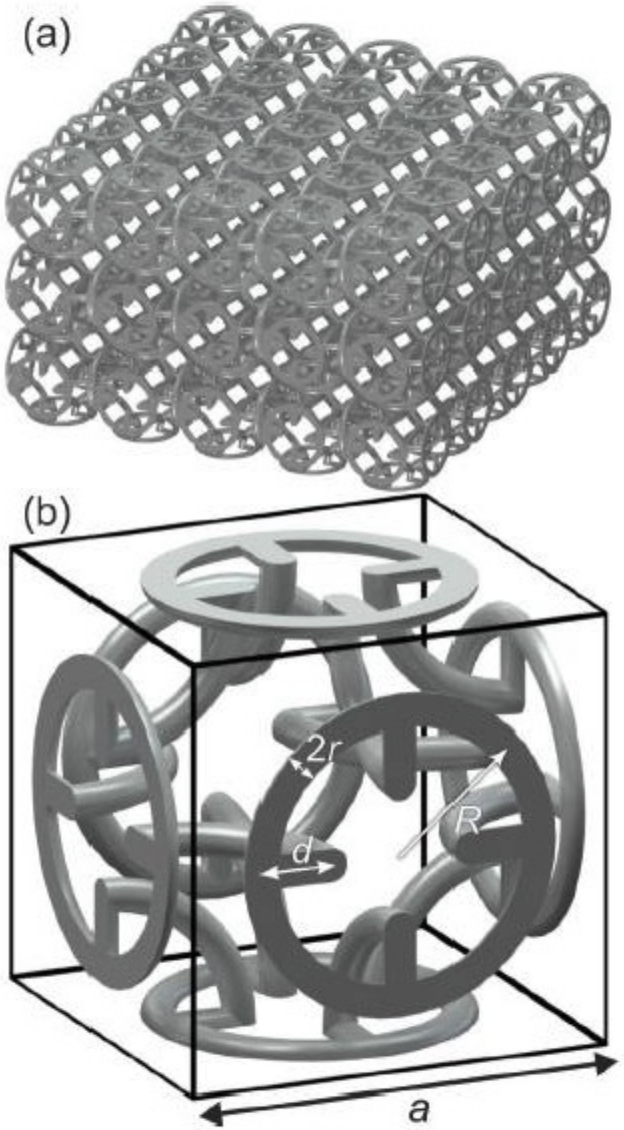


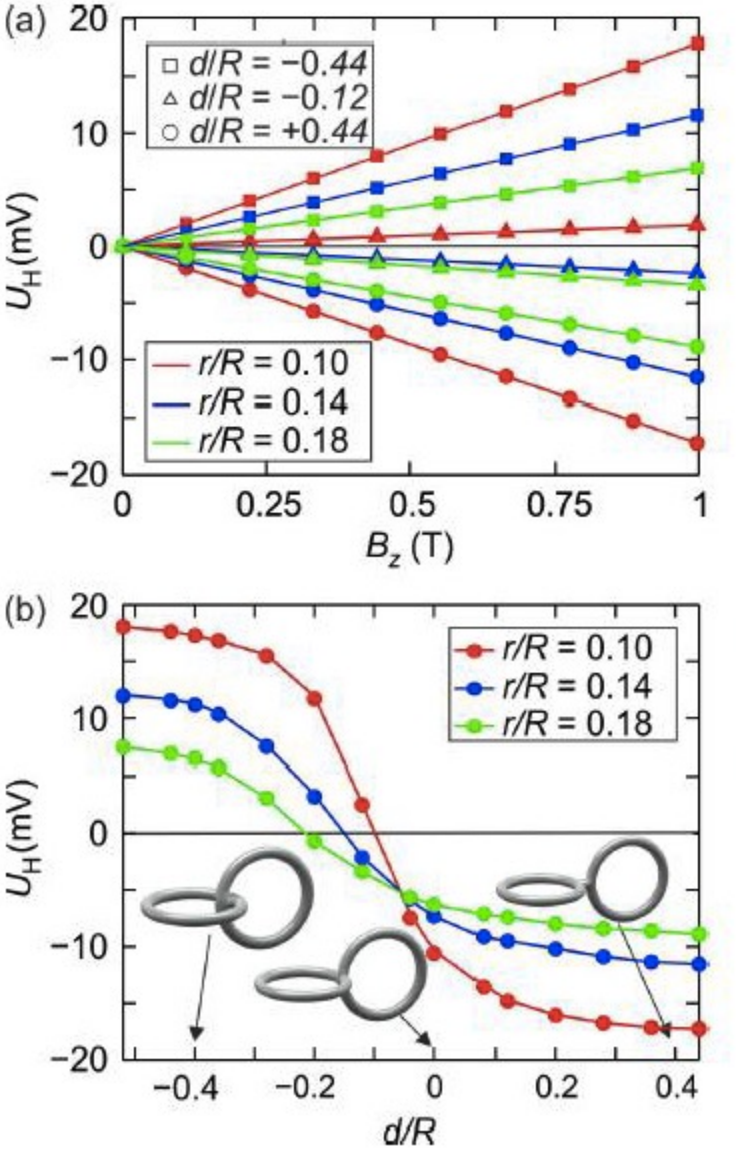
Image
Courtesy
Christian Kern

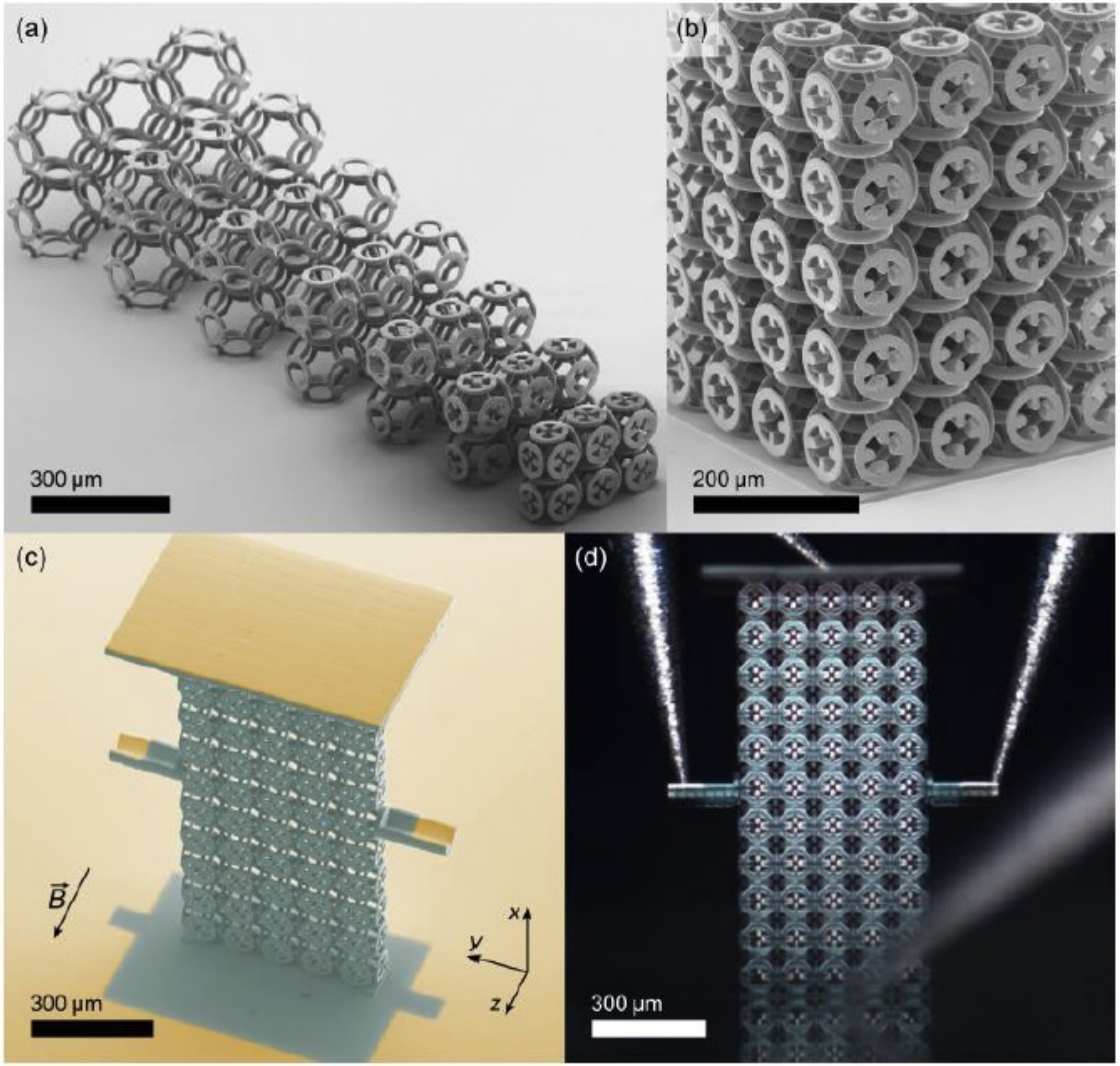
A material with cubic symmetry having a Hall coefficient opposite to that of the constituents.

M. Briane and G.W.M, Arch. Rat. Mech. Anal. 193, 715-738, (2009)

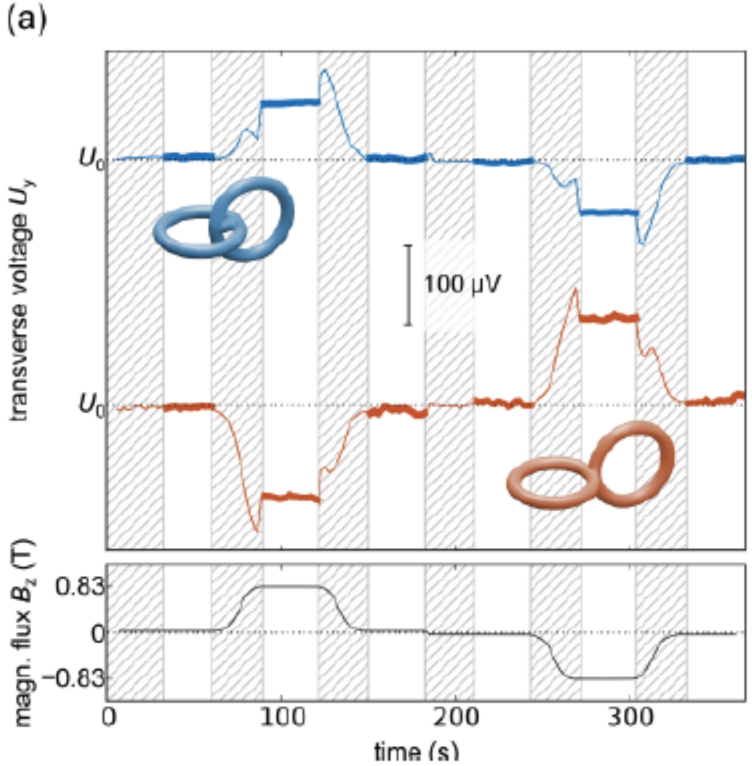
Simplification of Kadic et.al. (2015)

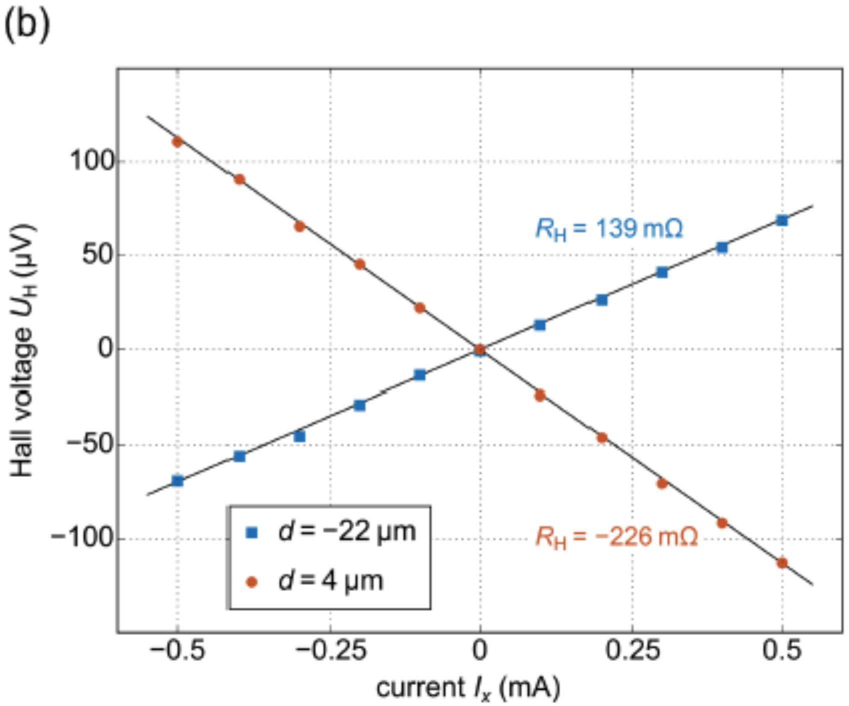


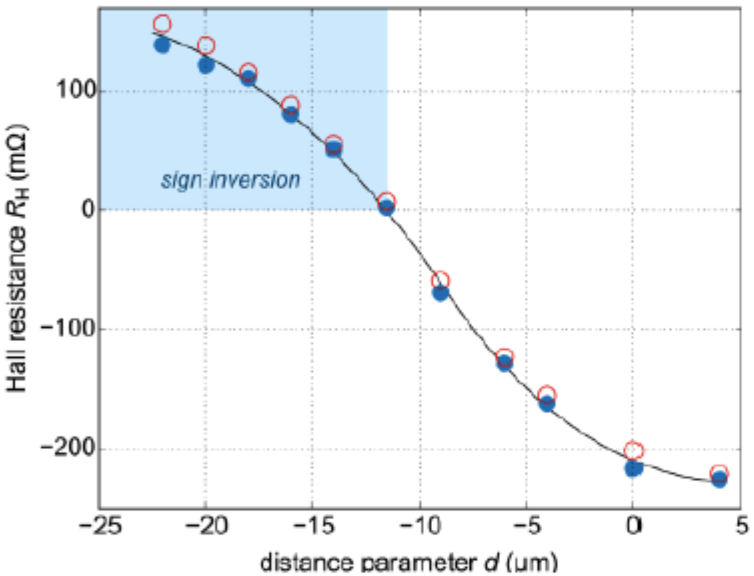




Experimental realization of Kern, Kadic, and Wegener



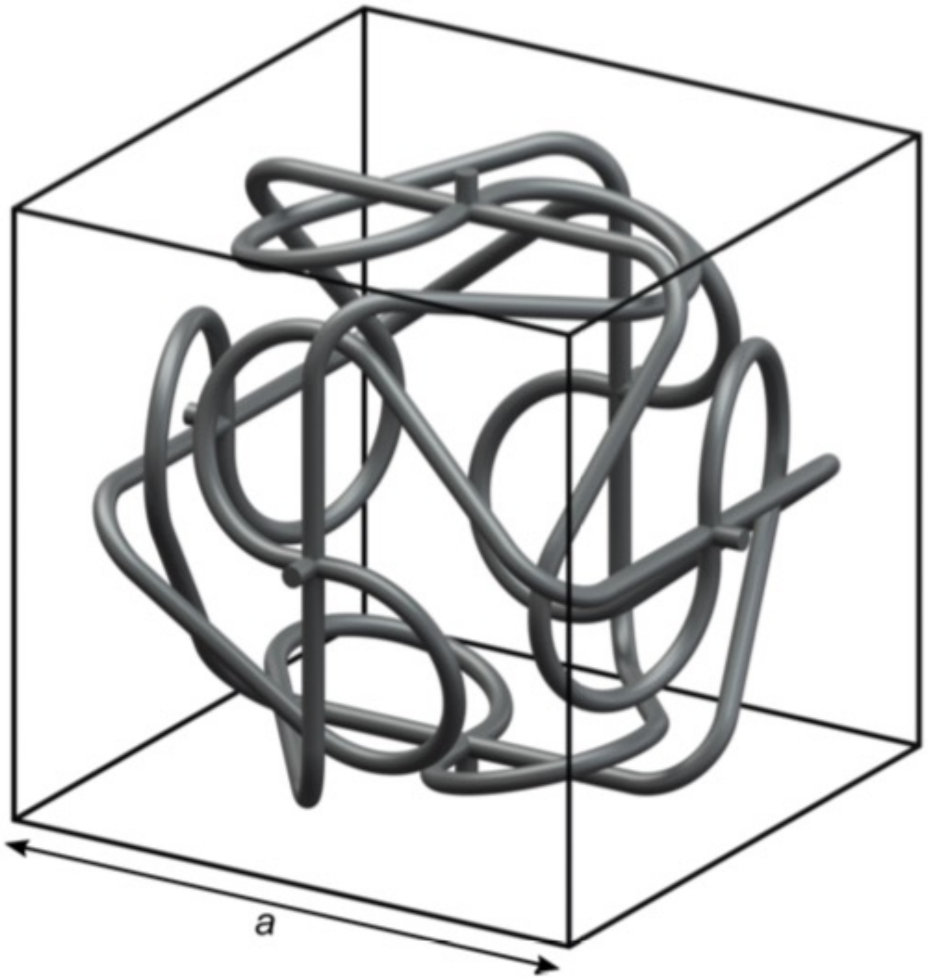




Their experimental results confirming Hall-effect reversal

Japanese 4 in cube / Jelly Cube available on ETSY

Alternate Structure of Christian Kern:



C. Kern, G.W.M., M. Kadic, and M. Wegener, New. J. Phys., 20, 083034, (2018)

The parallel Hall effect:

twisting the induced electric field in each unit cell

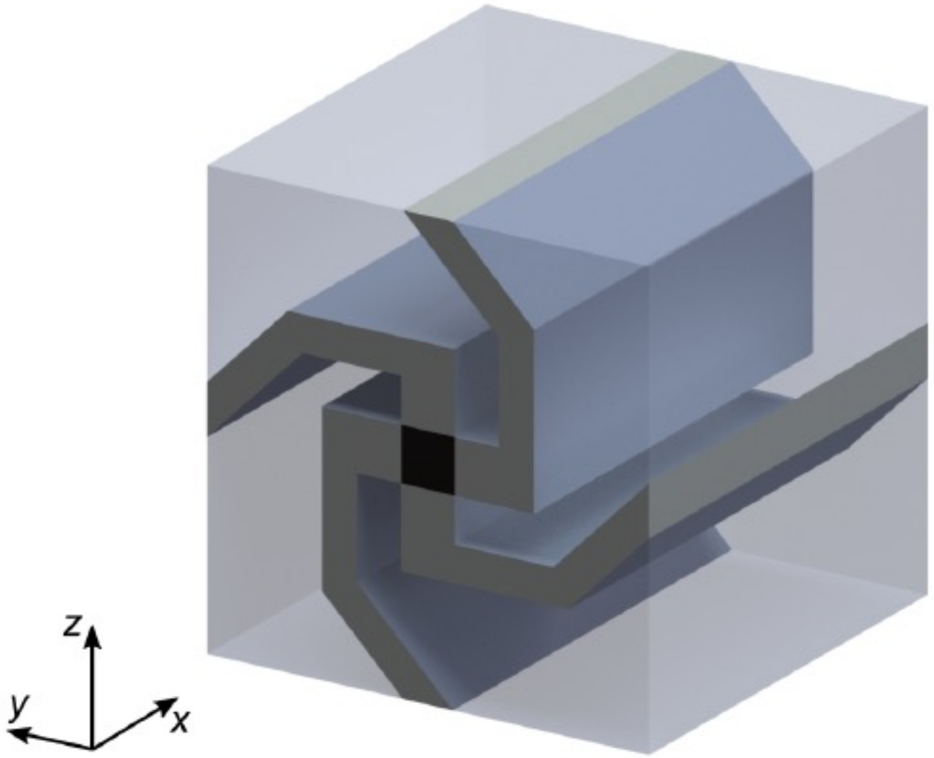


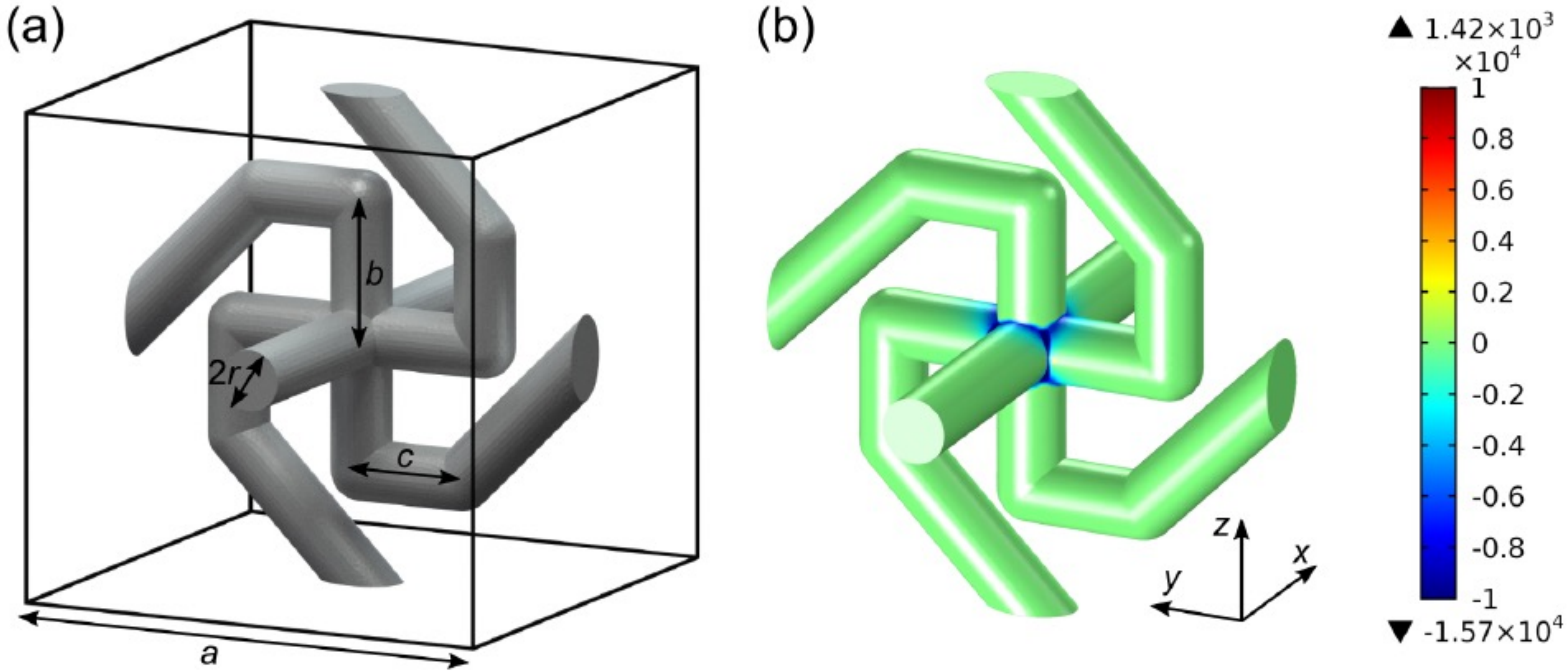
Image courtesy Christian Kern

$$\boldsymbol{A}_{\mathrm{H}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & A_{23} \\ 0 & -A_{23} & 0 \end{pmatrix}$$

$$\boldsymbol{e}_{\mathrm{H}} = -A_{23} j_x \left(b_y \hat{\boldsymbol{y}} + b_z \hat{\boldsymbol{z}} \right)$$

The Hall matrix becomes asymptotically an antisymmetric matrix.
(Milton and Briane, 2010)

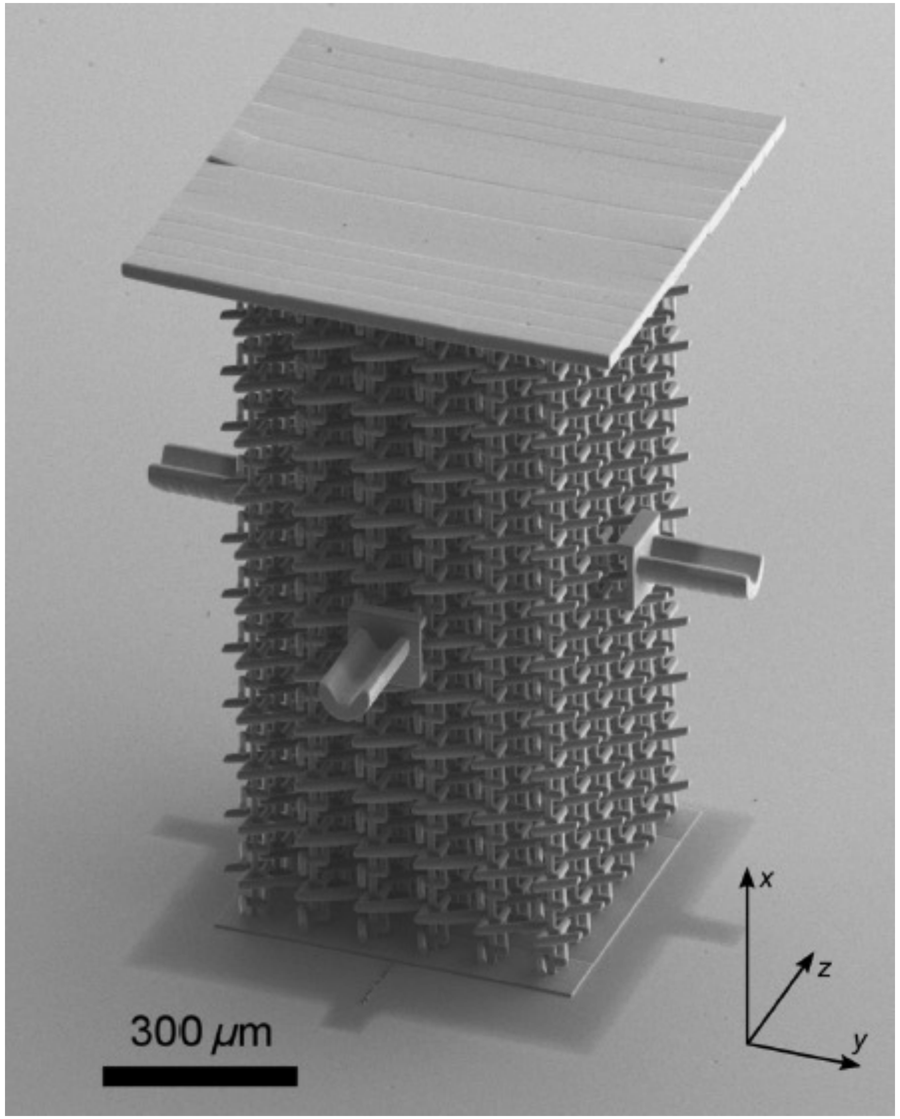
Simplified Design: (Kern, Kadic, Wegener 2015)

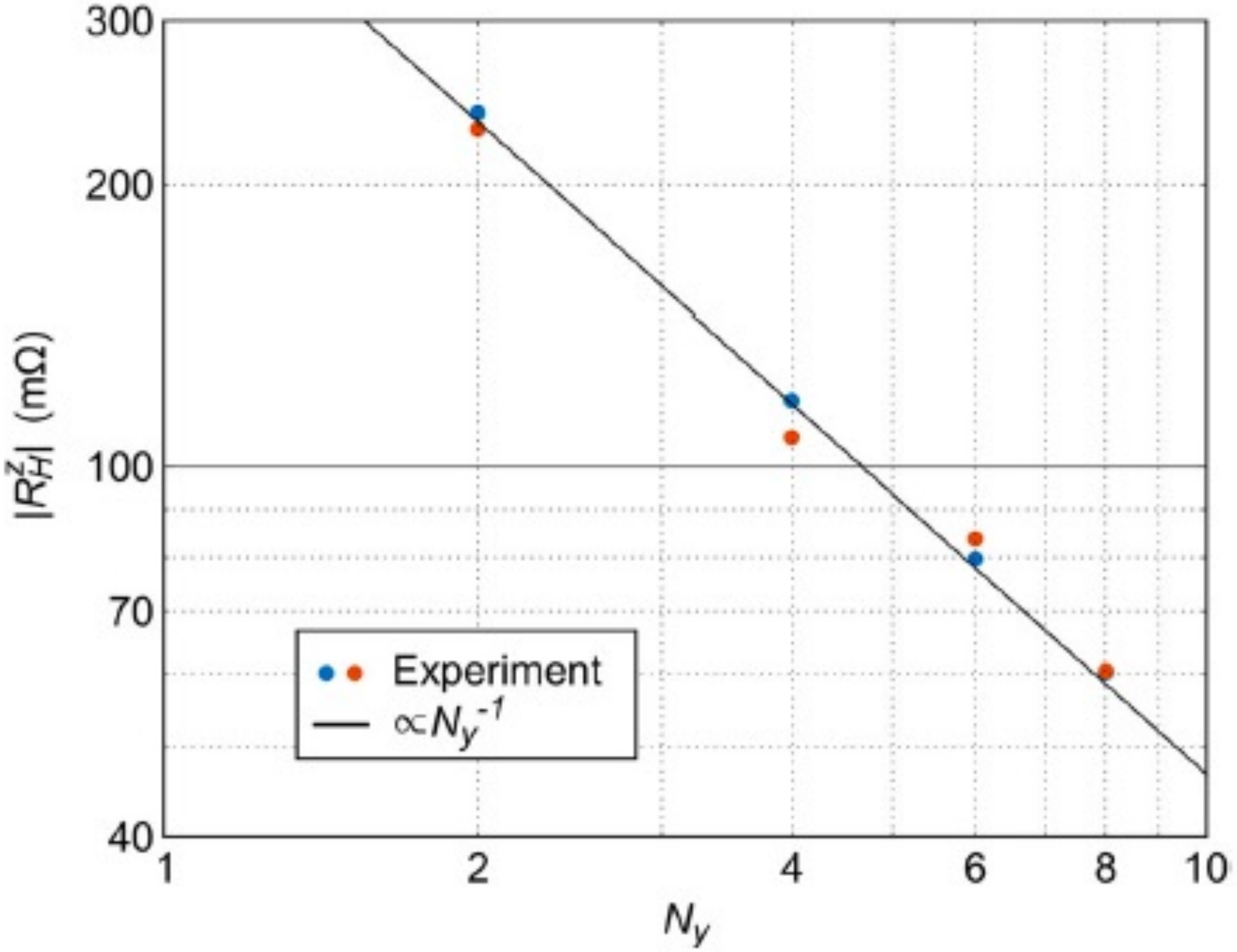


$$\boldsymbol{A}_{\mathrm{H}}^{*} = \begin{pmatrix} 6.85 & 0 & 0 \\ 0 & 0.04 & 6.84 \\ 0 & -6.84 & 0.04 \end{pmatrix} A_{\mathrm{H}}^{0}$$

Plot of the cofactor

Experiments: Kern, Schuster, Kadic, and Wegener (2017)





Transformation to self-adjoint form

Conductivity equation: $\nabla \cdot \boldsymbol{\sigma} \nabla V = -\nabla \cdot \mathbf{s}$,

$\boldsymbol{\sigma}$ not symmetric: $\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_s(\mathbf{x}) + \boldsymbol{\sigma}_a(\mathbf{x})$

$$\mathbf{j_0} = \boldsymbol{\sigma}\mathbf{e} - \mathbf{s}, \quad \nabla \cdot \mathbf{j_0} = 0, \quad \mathbf{e} = -\nabla V$$

Can be manipulated into the extended Cherkaev-Gibiansky form:

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{j_0} \end{pmatrix} = \mathbf{L} \begin{pmatrix} \mathbf{j_0} \\ \mathbf{e} \end{pmatrix} - \mathbf{s_0}, \quad \nabla \cdot \mathbf{j_0} = 0, \quad \mathbf{e} = -\nabla V$$

$$\mathbf{L} = \begin{pmatrix} \boldsymbol{\sigma}_s^{-1} & -\boldsymbol{\sigma}_s^{-1}\boldsymbol{\sigma}_a \\ \boldsymbol{\sigma}_a\boldsymbol{\sigma}_s^{-1} & \boldsymbol{\sigma}_s - \boldsymbol{\sigma}_a\boldsymbol{\sigma}_s^{-1}\boldsymbol{\sigma}_a \end{pmatrix}, \qquad \mathbf{s_0} = \begin{pmatrix} -\boldsymbol{\sigma}_s^{-1}\mathbf{s} \\ \mathbf{s} - \boldsymbol{\sigma}_a\boldsymbol{\sigma}_s^{-1}\mathbf{s} \end{pmatrix}$$

Field Patterns: A new type of Wave

With Ornella Mattei


설계 시
상담
32
8-0732
Space&Time
Architecture

Formulation of the problem

- Generic wave equation:

$$\frac{\partial}{\partial x}\left(\alpha(x,t)\frac{\partial u(x,t)}{\partial x}\right)-\frac{\partial}{\partial t}\left(\beta(x,t)\frac{\partial u(x,t)}{\partial t}\right)=0$$

The coefficients are time – dependent $\rightarrow$ DYNAMIC MATERIALS

- Boundary conditions: The medium is infinite in the x–direction

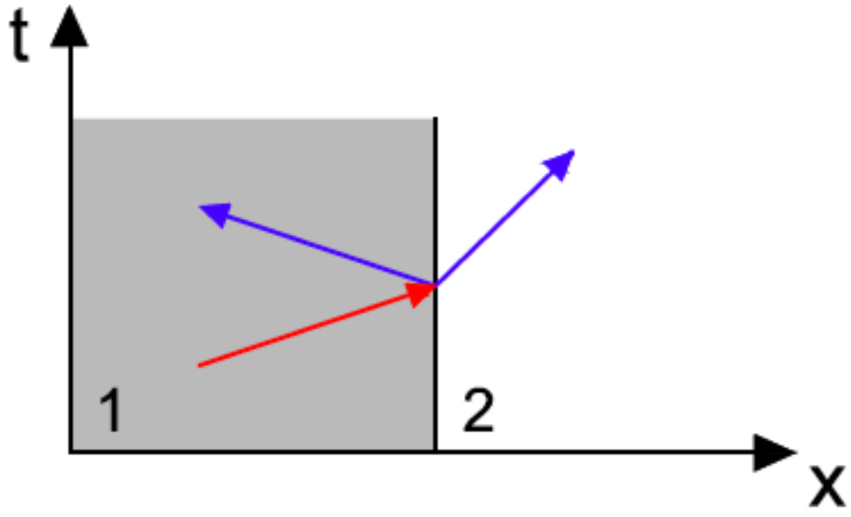
- Initial conditions:

$$u(x,0)=g(x)$$
$$\frac{\partial u(x,t)}{\partial t}|_{t=0}=f(x)$$

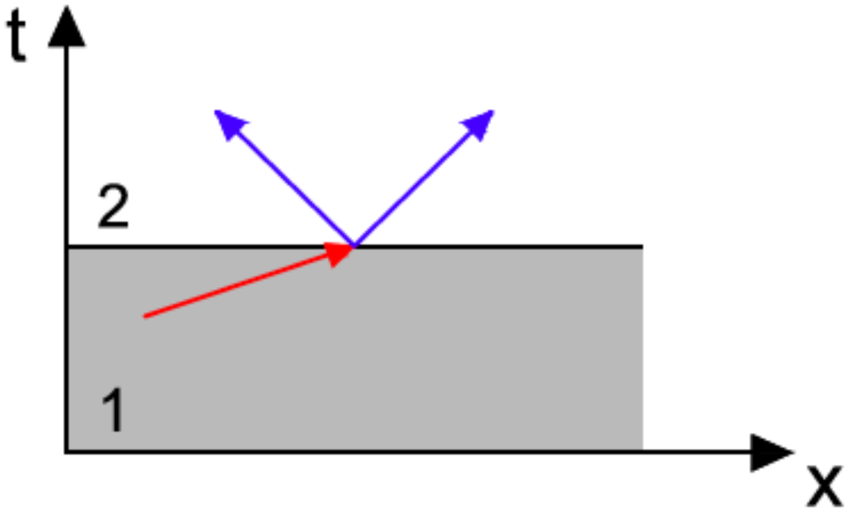
[see, e.g., Lurie, 2007]

Dynamic composites

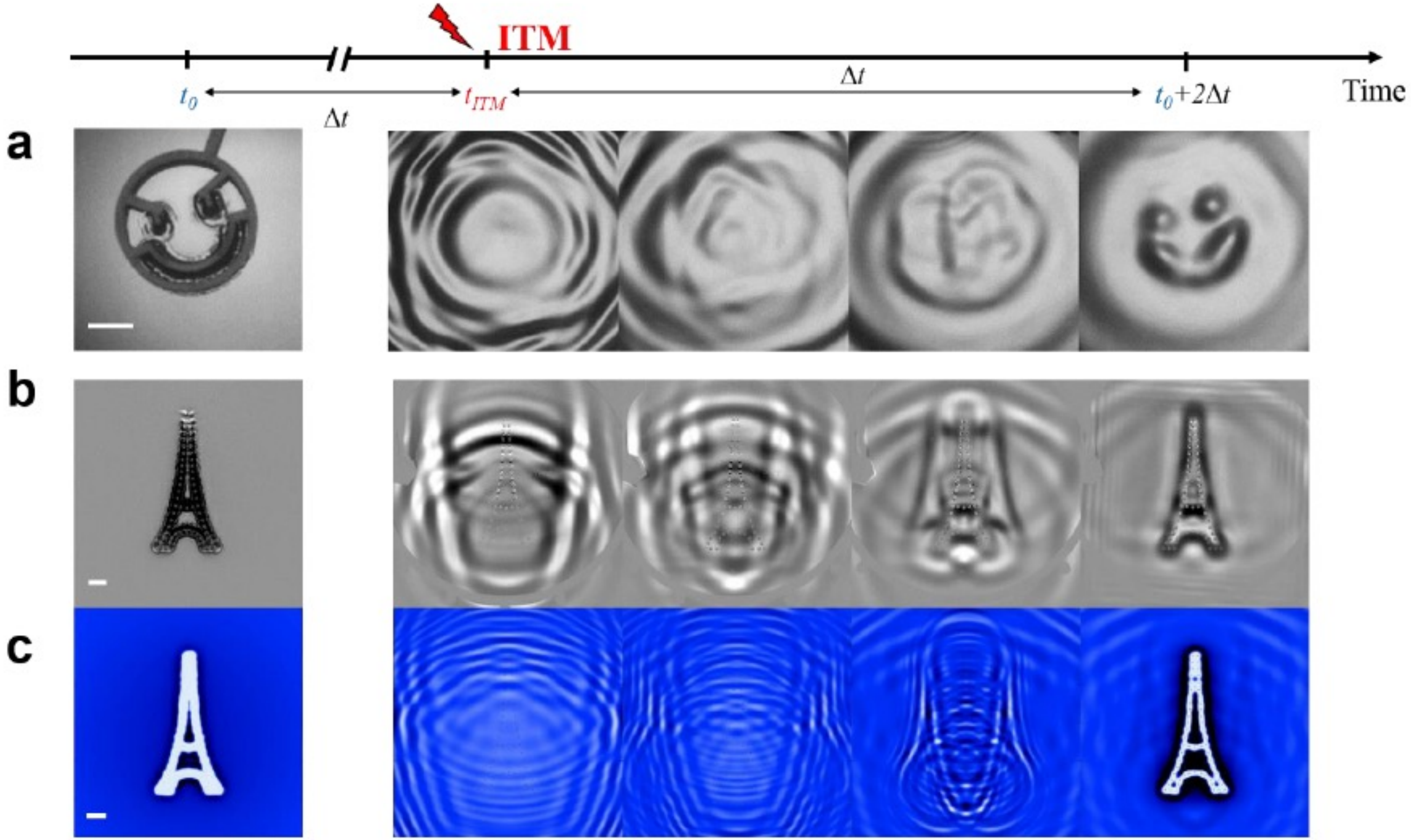
Pure space interface



Pure time interface



What happens at a time interface?



Bacot, Labousse, Eddi, Fink, and Fort, Nature 2016

Thinking of the wave equation as a conductivity problem

$$\mathbf{j}(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \text{where} \quad \nabla\cdot\mathbf{j} = 0, \quad \mathbf{e} = -\nabla V,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \begin{pmatrix} \alpha(\mathbf{x}) & 0 \\ 0 & -\beta(\mathbf{x}) \end{pmatrix}, \quad \begin{matrix} \text{material } 1 & \rightarrow & \alpha_1, \beta_1 \\ \text{material } 2 & \rightarrow & \alpha_2, \beta_2 \end{matrix}$$

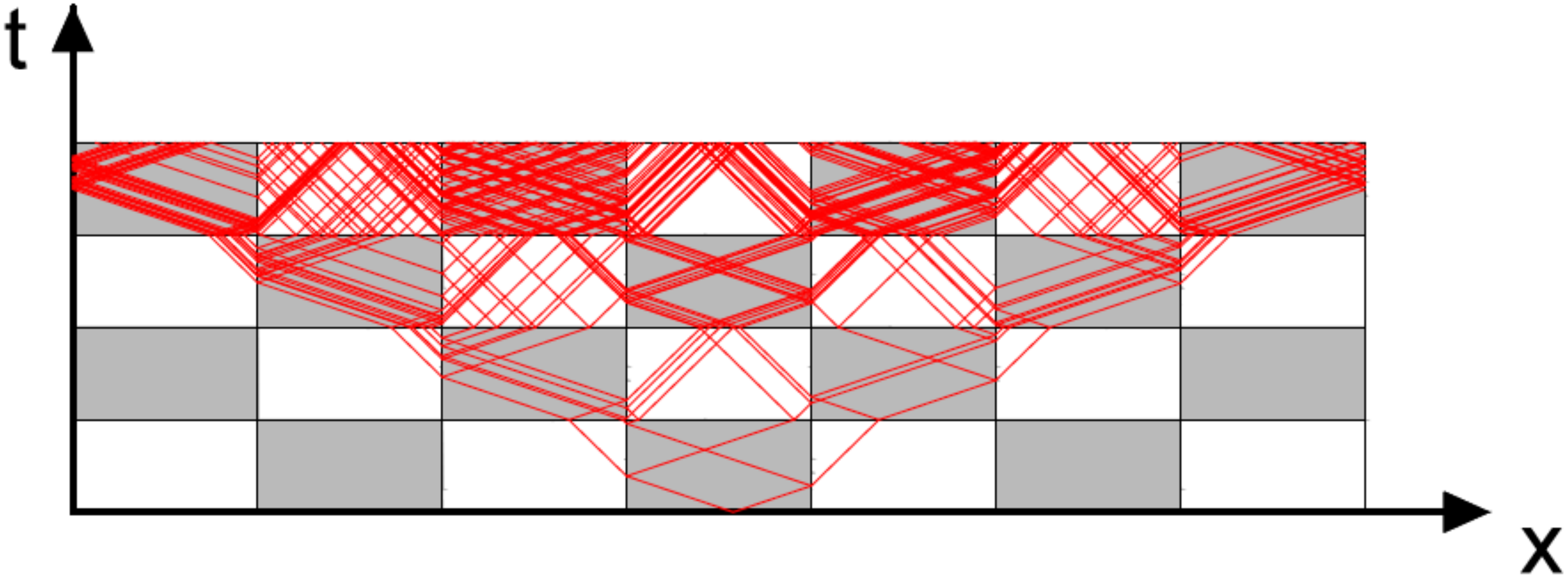
$$\frac{\partial}{\partial x_1}\left(\alpha(x_1,x_2)\frac{\partial V(x_1,x_2)}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\beta(x_1,x_2)\frac{\partial V(x_1,x_2)}{\partial x_2}\right) = 0$$

N.B. Hyperbolic materials!! [See, e.g. the review Poddubny, Iorsh, Belov, Kivshar, 2013]

$$\alpha_i\frac{\partial^2 V_i}{\partial x^2} - \beta_i\frac{\partial^2 V_i}{\partial t^2} = 0, \qquad i = 1,2$$

D'Alembert solution : $\quad V_i(x,t) = V_i^+(x - c_i t) + V_i^-(x + c_i t) \qquad c_i = \sqrt{\frac{\alpha_i}{\beta_i}}$

Evolution of a disturbance in a space-time checkerboard

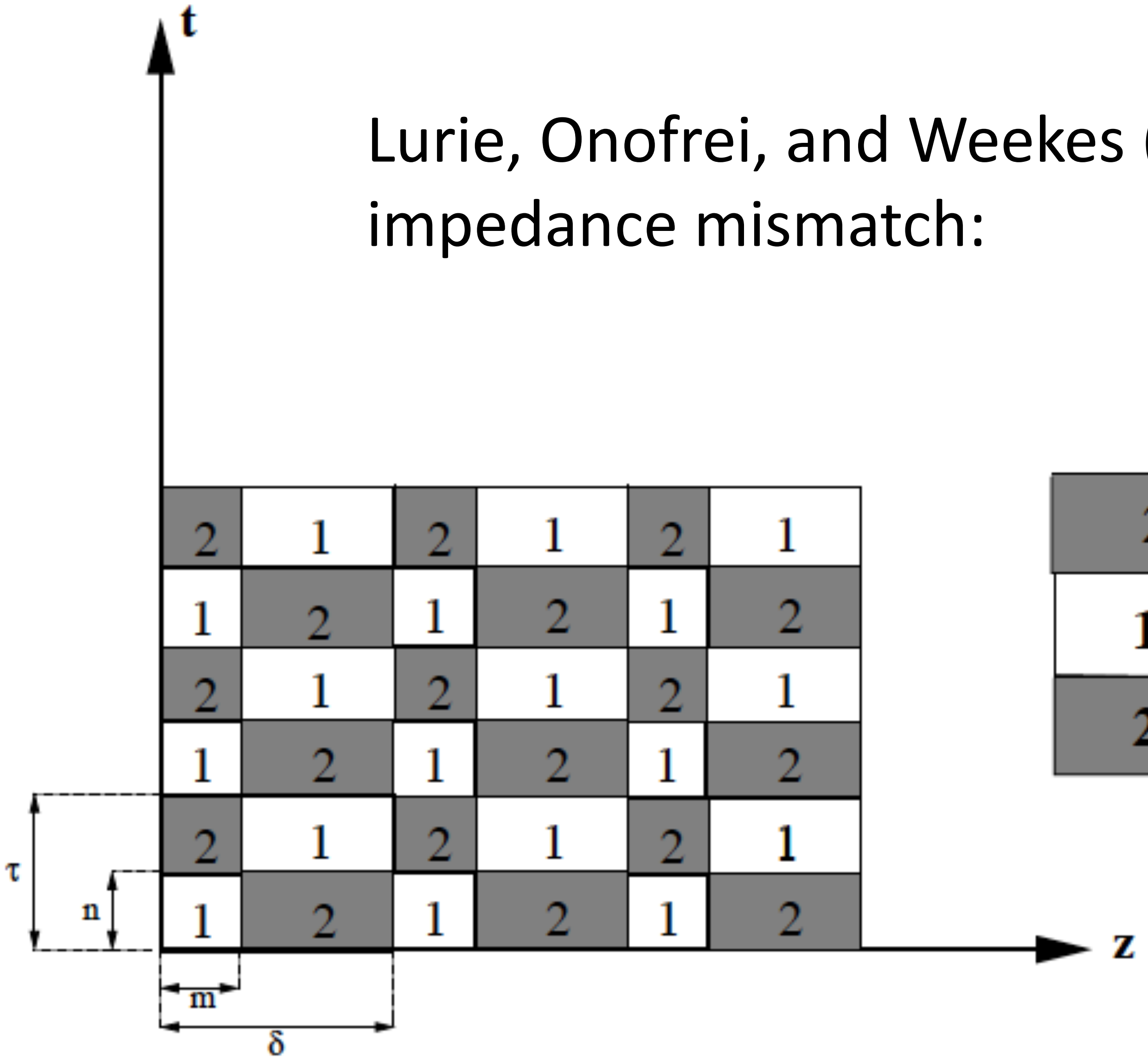


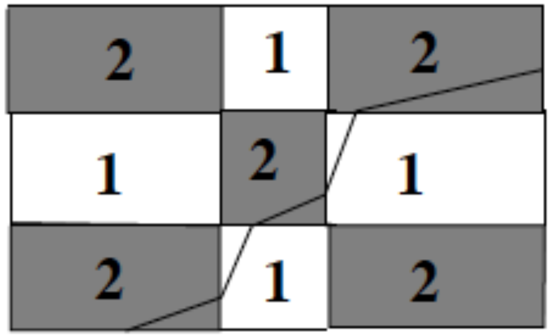
Transmission conditions:

$$\begin{cases} V_1 = V_2 \\ \mathbf{n} \cdot \sigma_1 \nabla V_1 = \mathbf{n} \cdot \sigma_2 \nabla V_2 \end{cases}$$

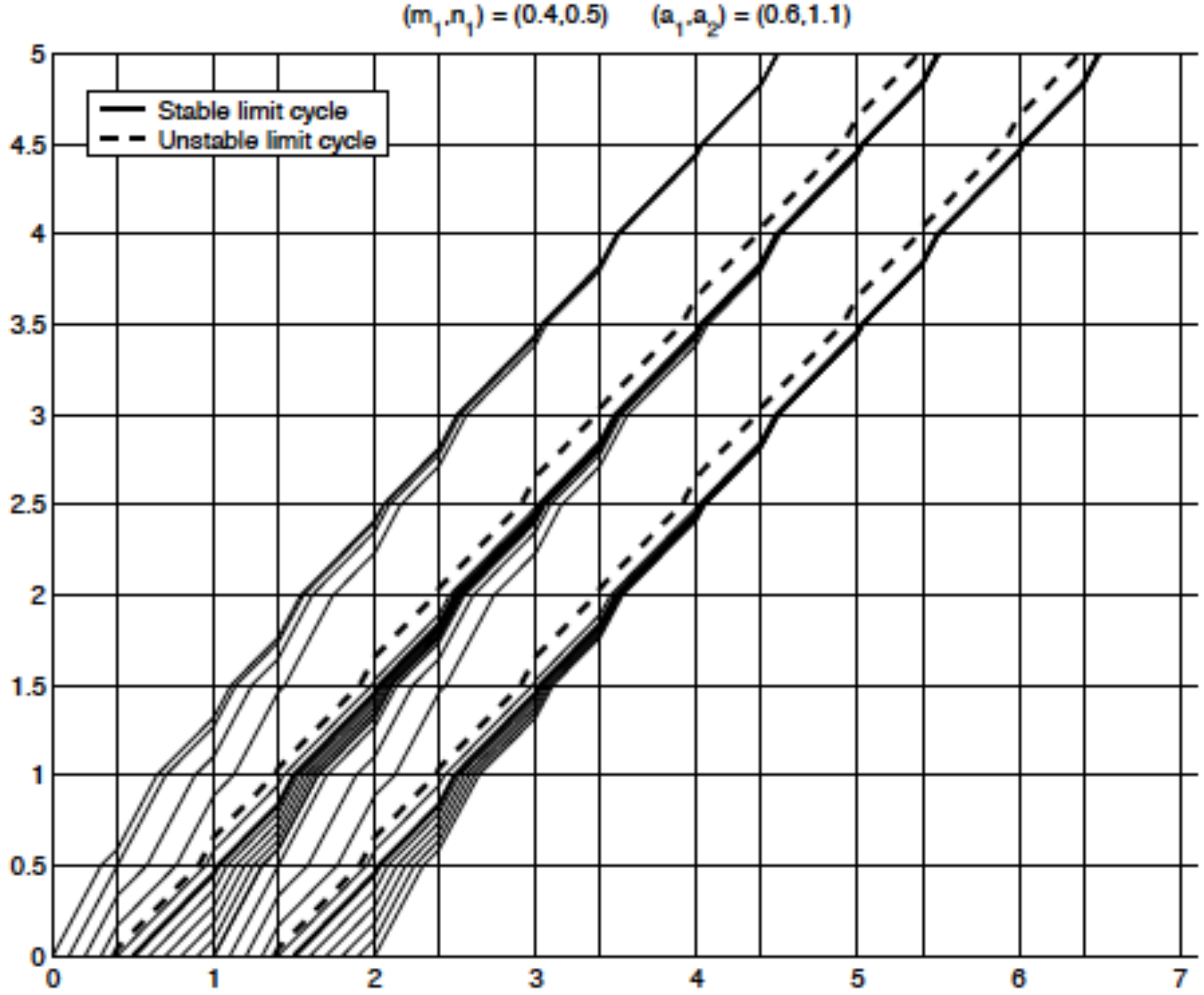
How to avoid this complicated cascade?

Lurie, Onofrei, and Weekes (2009) suggested having a zero impedance mismatch:





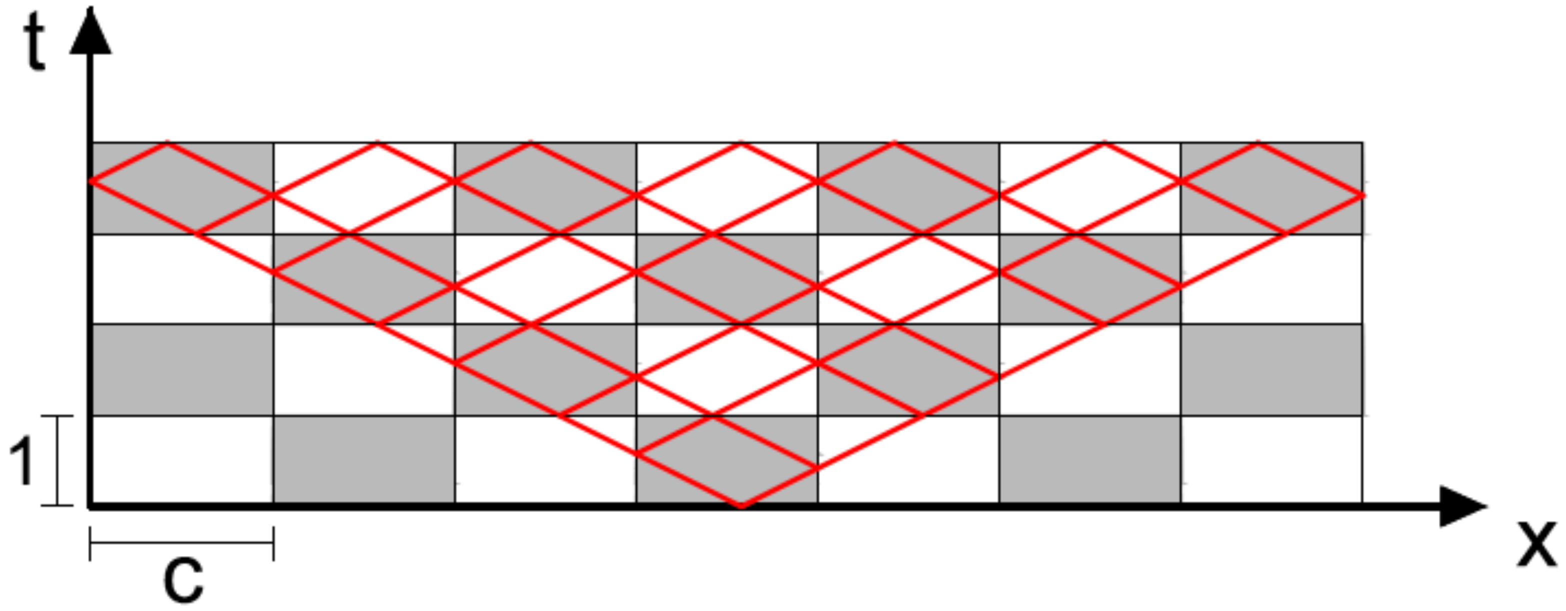
Curiously they found accumulations of the characteristic lines:



A bit like a shock but in a linear medium!

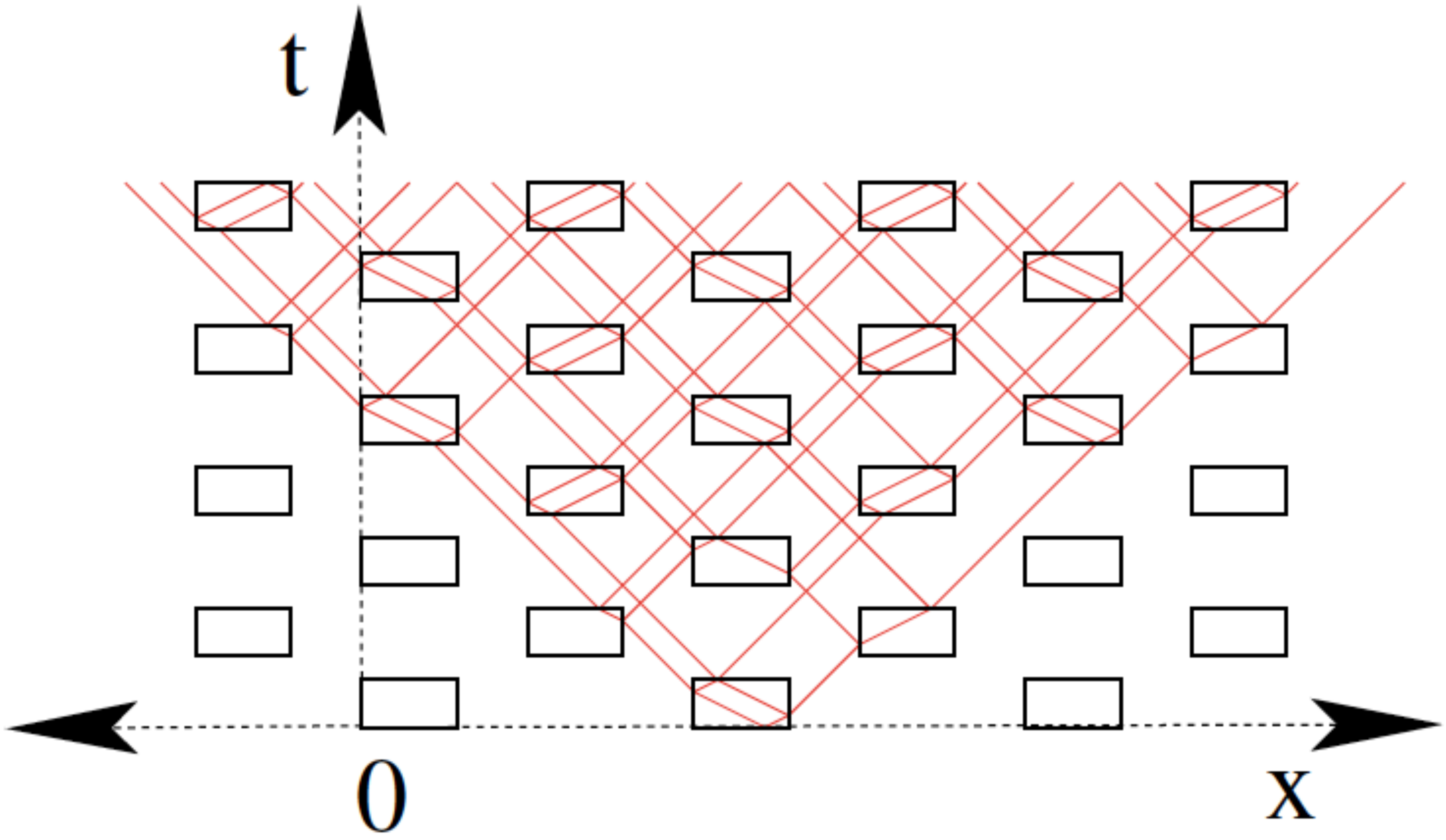
Field patterns in a space-time checkerboard

$$c_1 = c_2 = c \quad \Rightarrow \quad \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$$

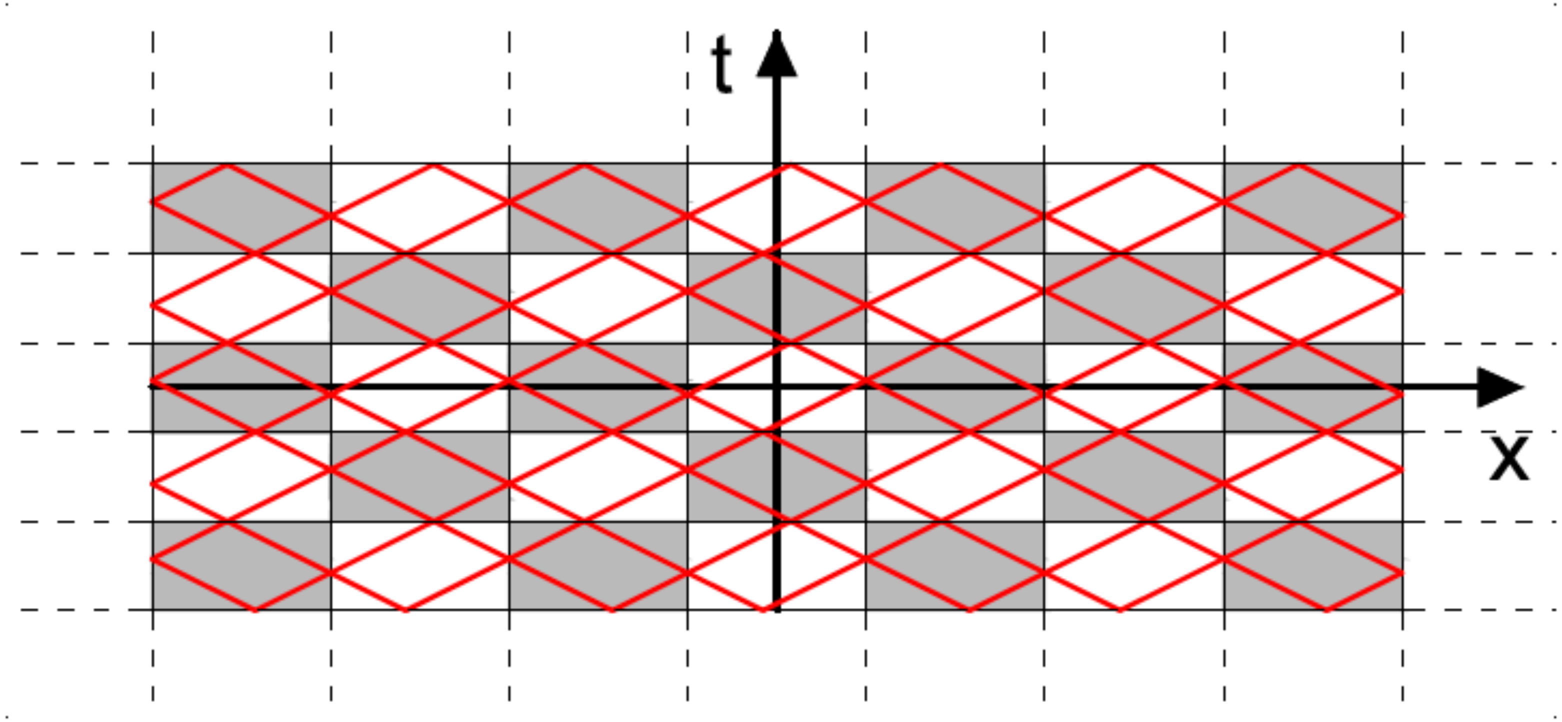


Field patterns are a new type of wave propagating along orderly patterns of characteristic lines!!!

Alternatively one can have staggered inclusions:



PT–symmetry of field patterns



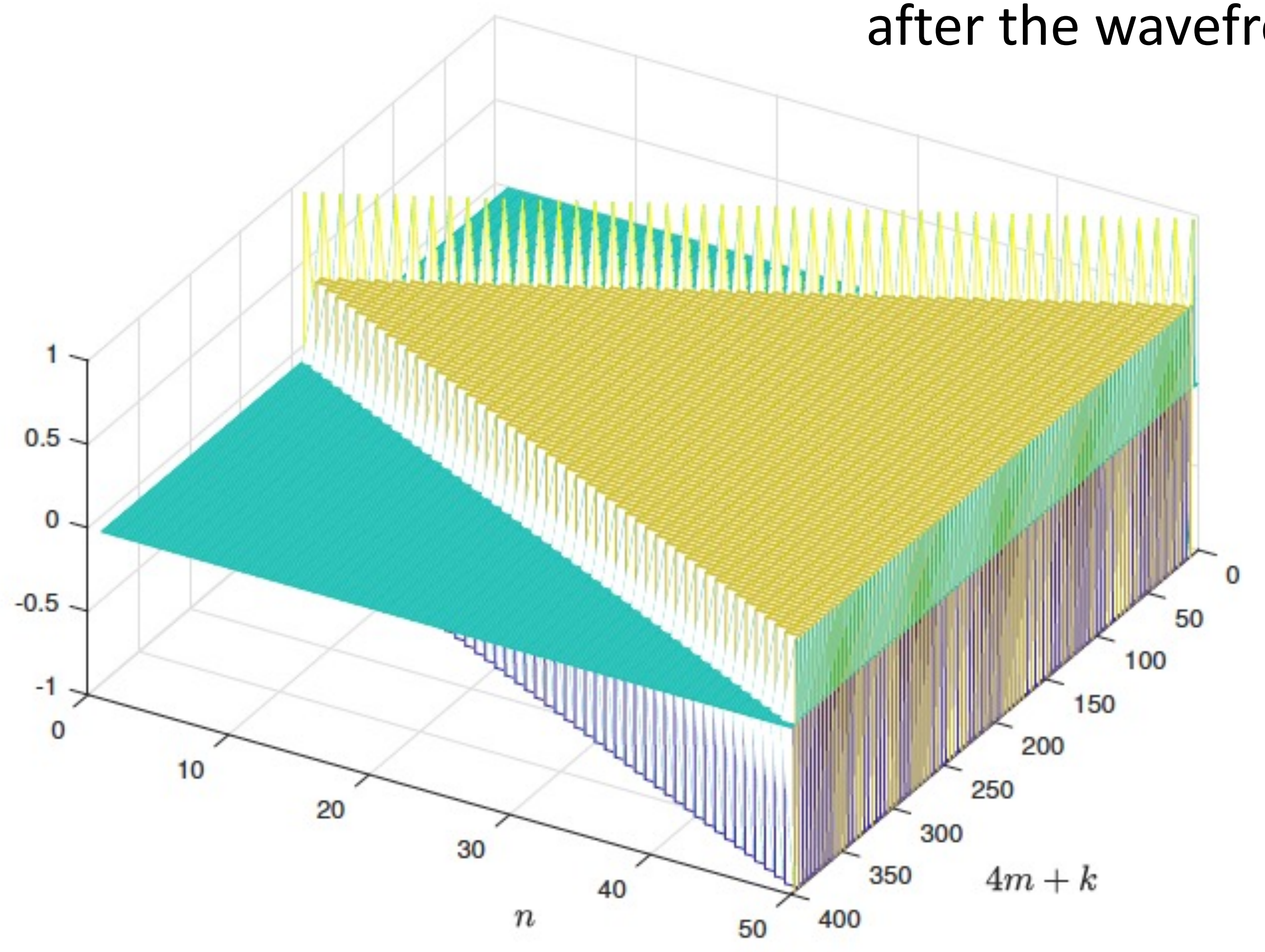
[Quantum physics, e.g., Bender and Boettcher, 1998,
Optics, e.g., Zyablovsky et al., 2014]

Unbroken PT–symmetry → real eigenvalues

Broken PT–symmetry → complex conjugate eigenvalues

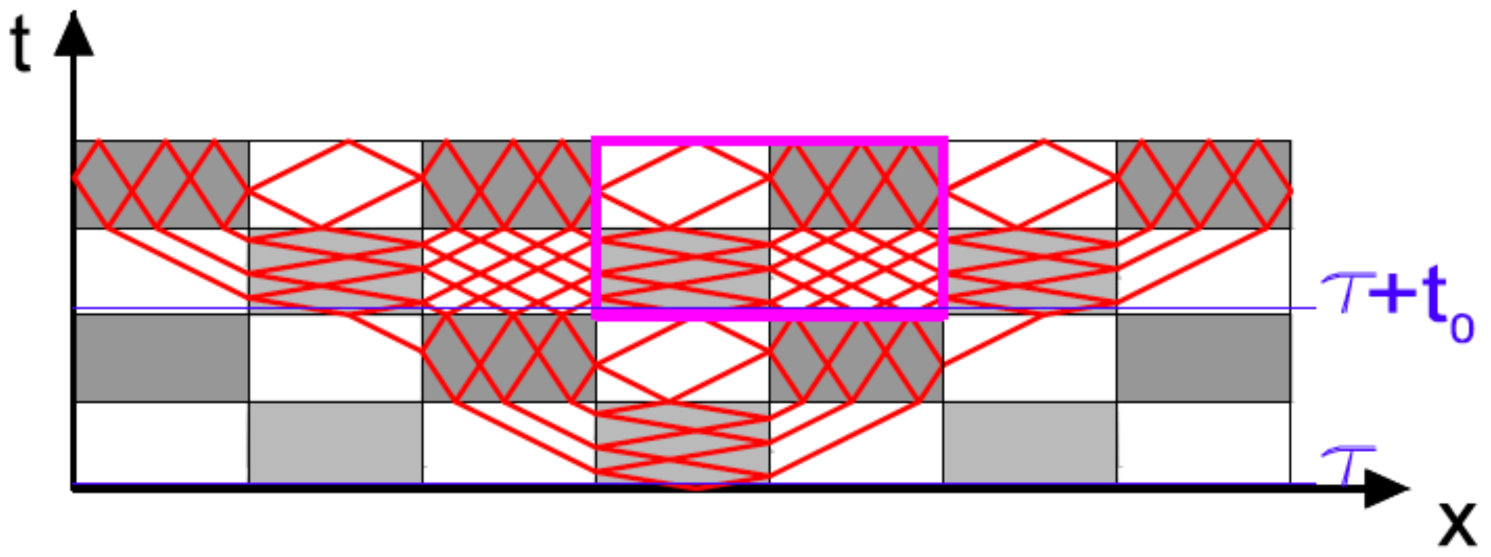
Evolution in time of the current distribution

Note: oscillations continue after the wavefront!



Three–phase space–time checkerboard

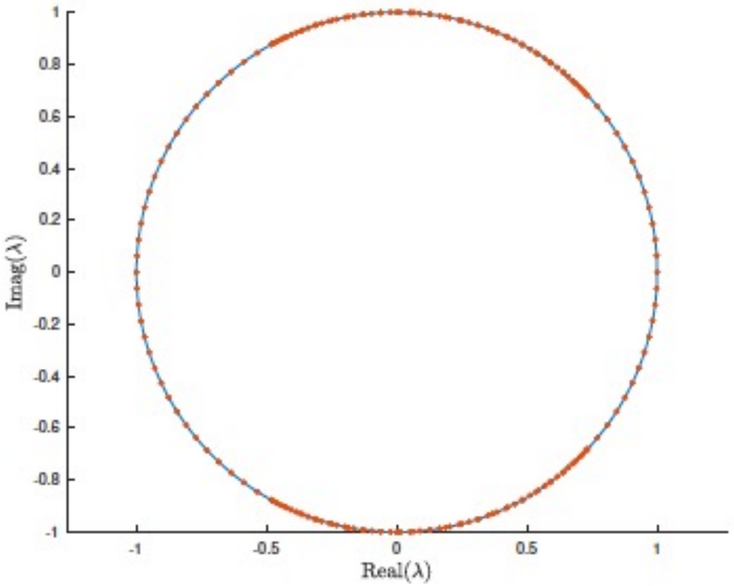
$$c_2/c_1 = c_1/c_3 = 3$$

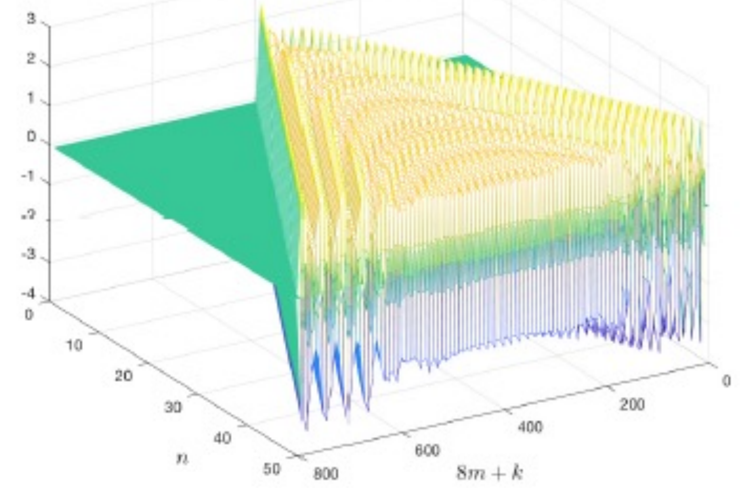


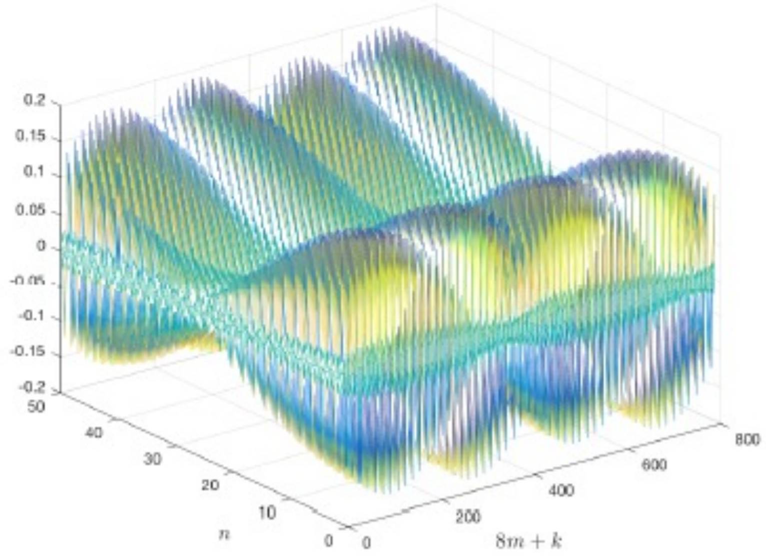
For some combinations of γ_1, γ_2, γ_3 : UNBROKEN PT–symmetry

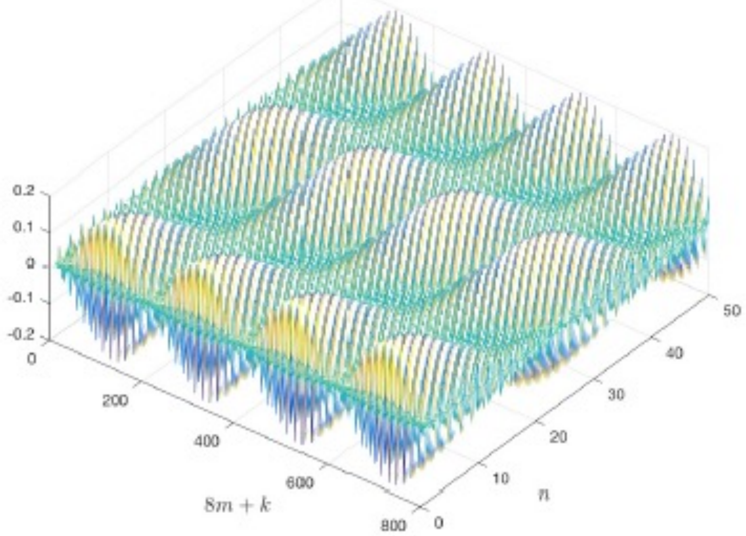
For other combinations of γ_1, γ_2, γ_3 : BROKEN PT–symmetry

Unbroken PT–symmetry for the three–phase checkerboard

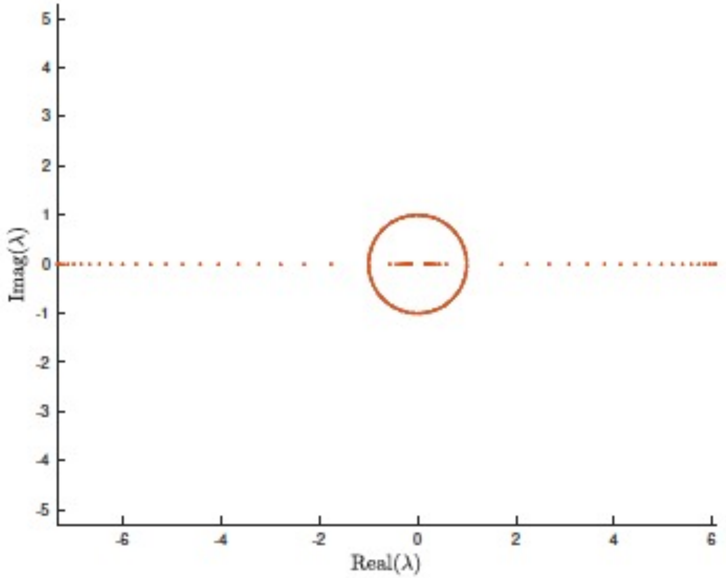


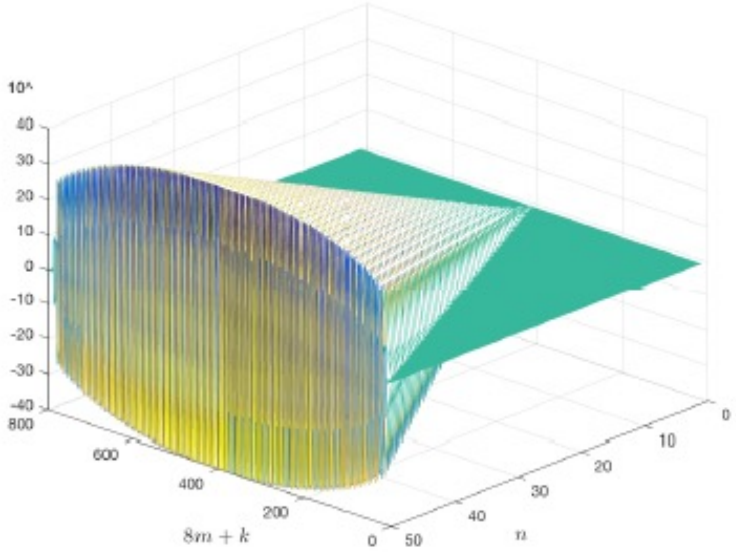




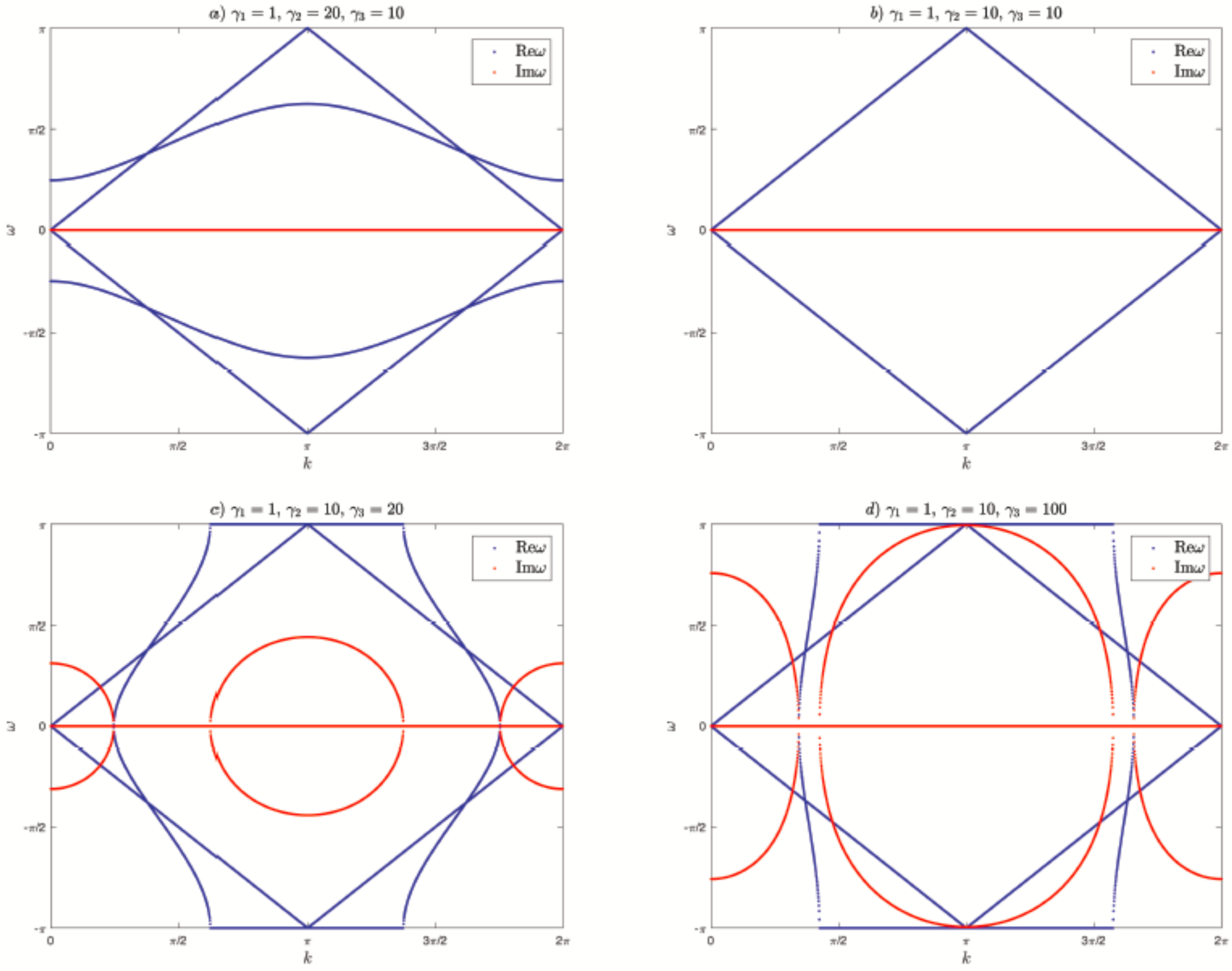


Broken PT–symmetry for the three–phase checkerboard





Dispersion diagrams for the three–phase checkerboard



Bloch Waves are:
Infinitely Degenerate!

Thank you!

Thank you!

Thank you!

Thank you!

Thank you!