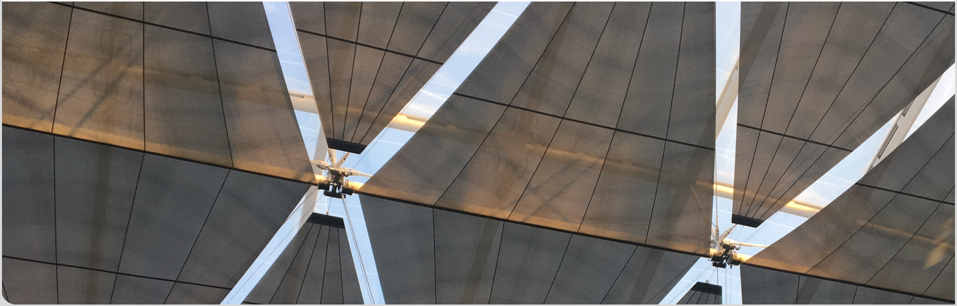
Karlsruher Institut für Technologie

The method of multipliers in spectral theory

Lucrezia Cossetti | July 11, 2022
Joint works with L. Fanelli and D. Krejčiřík
BIRS Workshop: *Mathematical aspects of the Physics with non-self-Adjoint Operators*, Banff, Alberta



The problem

Perturbed setting

$$H_V = H_0 + V$$

V possibly complex-valued (non-self-adjoint)

Question

$$\sigma(H_0) \text{ known } \implies \sigma(H_V)?$$

Goal

$$\sigma_p(H_0) = \varnothing \overset{\text{s.c.}}{\Longrightarrow} \sigma_p(H_V) = \sigma_p(H_0) = \varnothing$$ (absence of bound states)

- *repulsive V*
- V (*attractive*) small if compared to H_0

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The method of multipliers: the origine

Toy model: Linear Schrödinger equation

$$i\partial_t u = -\Delta u \qquad (*)$$

Using $(*)$

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 = \frac{d^2}{dt^2}\langle u, |x|^2 u\rangle = \frac{d}{dt}\Big(-i\langle u, [-\Delta, |x|^2]u\rangle\Big) \qquad (\bullet)$$
$$= -\langle u, [-\Delta, [-\Delta, |x|^2]]u\rangle$$

Since $[-\Delta, [-\Delta, |x|^2]] = 8\Delta$, then

$$\frac{d^2}{dt^2}\int |x|^2|u|^2 = 8\int |\nabla u|^2 = 16E$$

$$\implies \int |x|^2|u|^2 \to \infty \quad \text{for } t \to \pm\infty \quad \text{dispersion}(Morawetz'70)$$

- Identity $(\bullet)$ alternatively multiplying $(*)$ by $[-\Delta, |x|^2]$ and taking $\Re$

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Absence of bound states I

Self adjoint Schrödinger operator

$$H_V = -\Delta + V, \quad \text{in} \quad L^2(\mathbb{R}^d;\mathbb{C}), \qquad V\colon \mathbb{R}^d \to \mathbb{R}, \qquad d \geq 3$$

- By contradiction: $-\Delta u + Vu = \lambda u, \qquad \lambda \in \mathbb{R}$
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- If $x \cdot \nabla V \leq 0 \implies \int |\nabla u|^2 \leq 0 \implies \sigma_p(-\Delta + V) = \varnothing$

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Mourre Theory

$$H_V = -\Delta + V(x), \qquad V\colon \mathbb{R}^d \to \mathbb{R}$$

$$H_V u = \lambda u \xrightarrow{H_V \text{ symmetric}} \langle u, i[H_V, A]u\rangle = 0$$

$$A := -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x) \quad \text{(dilation operator)} \implies i[H_V, A] = -2\Delta - x\cdot\nabla V$$

$$\langle u, -2\Delta u\rangle = \langle u, x\cdot\nabla V u\rangle$$

$$\text{if } x\cdot\nabla V \leq 0 \implies \sigma_p(H_V) = \emptyset$$

How does the method of multipliers meet the Mourre theory?

- $A := -\frac{i}{2}(x\cdot\nabla + \nabla\cdot x)$ (dilation operator) $\implies [\Delta, |x|^2] = 4iA$
- $\sigma_p(H_V) = \emptyset$ under same condition on V

Mourre theory does not fit well with the *non self-adjoint* framework

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Absence of bound states II

Non self-adjoint Schrödinger operator (Fanelli, Krejčiřík, Vega '18)

$$H_V = -\Delta + V, \quad \text{in} \quad L^2(\mathbb{R}^d;\mathbb{C}), \qquad V\colon \mathbb{R}^d \to \mathbb{C}, \quad d \geq 3$$

$\Longrightarrow$ the spectrum is no more necessarily real

- By contradiction: $H_V u = (\lambda + i\varepsilon)u$ (Case $\lambda \geq |\varepsilon| > 0$)
- Multiply (in L^2) by $[-\Delta, |x|^2]u$ and take the real parts

$$\Re\langle H_V, [-\Delta, |x|^2]u\rangle = -\varepsilon\Im\langle u, [-\Delta, |x|^2]u\rangle$$

$\Longrightarrow$ one identity is not enough to get the absence of eigenvalues!

- Note $a, b \geq 0 \implies -2ab \leq 0$, but $a^2 - 2ab + b^2 = (a-b)^2 \geq 0$
- Multiply (in L^2) by φu and take the real parts and the imaginary parts
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$$\int|\nabla u|^2+\lambda\int|u|^2-2\lambda^{1/2}\operatorname{sgn}\varepsilon\,\Im\int\frac{x}{|x|}\bar u\nabla u$$
$$+\frac{|\varepsilon|}{\lambda^{1/2}}\Big[\int|x||\nabla u|^2+\lambda\int|x||u|^2-2\lambda^{1/2}\operatorname{sgn}\varepsilon\,\Im\int|x|\frac{x}{|x|}\nabla u\bar u\Big]$$
$$+\int\Re V|u|^2+2\Re\int xVu\nabla\bar u-2\Im\lambda^{1/2}\operatorname{sgn}\varepsilon\int x\frac{x}{|x|}V|u|^2=0$$

$u^-(x) := e^{-i\lambda^{1/2}\operatorname{sgn}\varepsilon|x|}u(x)$ (Eidus '62, Ikebe-Saito '72)

$$|\nabla u^-|^2=|\nabla u|^2+\lambda|u|^2-2\lambda^{1/2}\operatorname{sgn}\varepsilon\frac{x}{|x|}\Im(\bar u\nabla u)$$

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- Integrating by parts

$$\int |\nabla u^-|^2 + \frac{|\varepsilon|}{\lambda^{1/2}} \int |x||\nabla u^-|^2 - \int \partial_r(|x|\Re V)|u^-|^2 - 2\Im \int x \Im V u^- \overline{\nabla u^-} = 0$$

- Let a, b be suitable constants such that

$$\int [\partial_r(|x|\Re V)]_+ |u|^2 \leq a^2 \int |\nabla u|^2$$
$$\int |x|^2 |\Im V|^2 |u|^2 \leq b^2 \int |\nabla u|^2$$

$$(1 - a^2 - 2b) \int |\nabla u^-|^2 \leq 0 \xrightarrow{a^2+2b<1} \sigma_p(-\Delta + V) = \varnothing$$

Absence of bound states III

Non self-adjoint Schrödinger operator on the half-space (C., Krejčiřík '19)

$$\begin{cases} -\Delta u = (\lambda + i\varepsilon)u & \Omega = \mathbb{R}^{d-1} \times (0,\infty) \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \partial\Omega = \{x_d = 0\} \end{cases}$$

with $\alpha \colon \partial\Omega \to \mathbb{C}$, $\alpha \in L^\infty(\partial\Omega)$

The analogous identity reads

$$\int_\Omega |\nabla u^-|^2 + \frac{|\varepsilon|}{\lambda^{1/2}} \int_\Omega |x||\nabla u^-|^2 + \int_{\partial\Omega} \Re\alpha |u|^2 \, d\sigma + 2\Re \int_{\partial\Omega} x'\alpha u^- \overline{\nabla u^-} \, d\sigma = 0$$

- $\Re\alpha \geq 0$ (repulsivity)
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Absence of bound states III

Non self-adjoint Schrödinger operator on the half-space (C., Krejčiřík '19)

$$\begin{cases} -\Delta u = (\lambda + i\varepsilon)u & \Omega = \mathbb{R}^{d-1} \times (0,\infty) \\ \frac{\partial u}{\partial \nu} + \alpha u = 0 & \partial\Omega = \{x_d = 0\} \end{cases}$$

with $\alpha \colon \partial\Omega \to \mathbb{C}$, $\alpha \in L^\infty(\partial\Omega)$

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Generalizations to other models:

- Electromagnetic Schrödinger (Fanelli, Krejčiřík, Vega '18 $d \geq 2$)

$$H_{A,V} := -(\nabla + iA)^2 + V \qquad A\colon \mathbb{R}^d \to \mathbb{R}^d, \quad V\colon \mathbb{R}^d \to \mathbb{C}$$

New term depending on $B := (\nabla A) - (\nabla A)^T$:

$$\Im \int x \cdot B \cdot \overline{\nabla_A u} u \, dx$$

if $\int |x|^2 |B|^2 |u|^2 \leq b^2 \int |\nabla_A u|^2 \xrightarrow{b^2 \text{ small}} \sigma_p(H_{A,V}) = \varnothing$.

[Koch,Tataru 2006] Gauge dependent conditions.

- Lamé operators (C. '17)

$$H_V := -\mu\Delta - (\lambda + \mu)\nabla \operatorname{div} + V(x) \qquad V\colon \mathbb{R}^d \to \mathbb{C}^{d\times d}, \quad d \geq 3$$

- Matrix-valued electromagnetic Schrödinger (C., Fanelli, Krejčiřík '20)

$$\Longrightarrow \quad H_{A,V} = -(\nabla + iA)^2 I_{\mathbb{C}^n} + V(x) \qquad V\colon \mathbb{R}^d \to \mathbb{C}^{n\times n}, \quad d \geq 1$$

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Generalizations to other models:

- Pauli operators (C., Fanelli, Krejčiřík '20)

$$H_P(A,V) := -(\nabla + iA)^2 I_{\mathbb{C}^2} - \sigma \cdot B + V \qquad \sigma = (\sigma_1, \sigma_2, \sigma_3)$$

$$\sigma_i, V(x) \in \mathbb{C}^{2\times 2}$$

- Dirac operators (C., Fanelli, Krejčiřík '20)

$$H_D(A) := -i\alpha \cdot (\nabla + iA(x)) + \frac{1}{2}\alpha_4 \qquad \alpha = (\alpha_1, \alpha_2, \alpha_3)$$

$$\alpha_\mu \in \mathbb{C}^{4\times 4}, \quad \mu = 1,2,3,4$$

$$H_D(A)^2 = \begin{pmatrix} H_P(A) + \frac{1}{4} I_{\mathbb{C}^2} & 0 \\ 0 & H_P(A) + \frac{1}{4} I_{\mathbb{C}^2} \end{pmatrix} \quad \text{(supersymmetry)}$$

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Main theorem (C., Fanelli, Krejčiřík '20)

Let $d \geq 3, n \geq 1$ and let $A \in W^{1,d}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, $V = V^{(1)} + V^{(2)}$, with $V^{(2)} = vI_{\mathbb{C}^n}$ and $\Re v \in W^{1,d/2}_{\text{loc}}$. If, $\forall \psi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} r^2 \Big(|B|^2 + |V^{(1)}|^2 + |\Re v_-|^2 + |\Im v|^2 + [\partial_r(r \Re v)]_+ \Big) |u|^2 \leq c_d \int_{\mathbb{R}^d} |\nabla_A u|^2,$$

then $\sigma_{\text{p}}(H(A, V)) = \varnothing$.