

SCHRÖDINGER

OPERATORS

with complex potentials

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Setting

$$-\Delta + V(x) \quad \text{in } L^2(\mathbb{R}^d)$$

$$\uparrow \\ \mathbb{E}$$

V decaying : $V \in L^q(\mathbb{R}^d)$

for some $q < \infty$

Aim: Quantitative bounds
on **eigenvalues**



Many results by many authors :

Aslangan - Abramov - Davies, Frank - Laptev -
Lieb - Seiringer, Demuth - Hausmann - Katriel,
Laptev - Sazonov, Frank - Simon, Fanelli -
Krejcirik - Vega, Mizutani, Lee - Seo,
Cossetti, Bögli - Stampach, ...

For real potentials,

$$\sum_j |E_j|^\delta \lesssim \int_{\mathbb{R}^d} V_-(x)^{\frac{d}{2} + \delta} dx$$

$\forall \delta > 0$ if $d \geq 2$.

For complex potentials, Frank proved

$$|E_j|^\delta \lesssim \int_{\mathbb{R}^d} |V(x)|^{\frac{d}{2} + \delta} dx$$

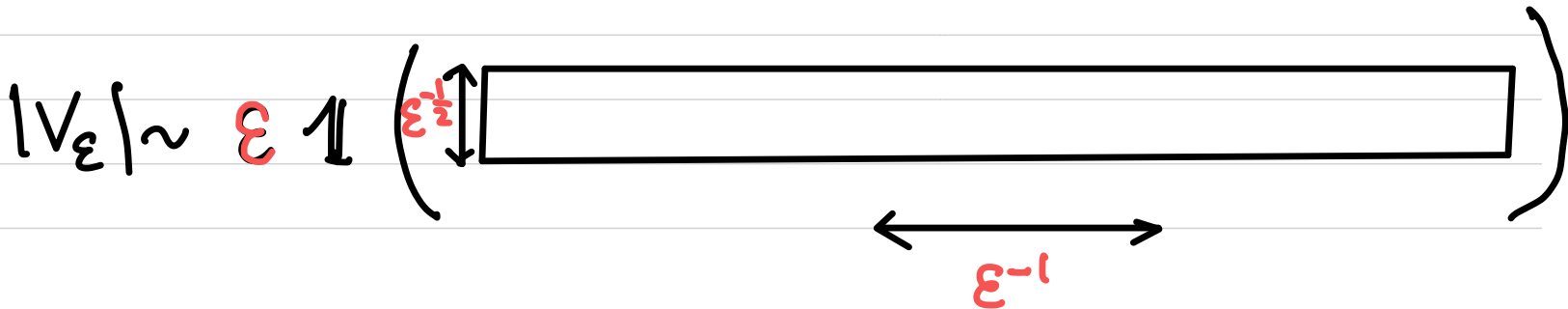
$\forall 0 < \delta \leq \frac{1}{2}$ if $d \geq 2$.

Conjecture (Laptev - Safonov 2009)

$$|E_j|^\delta \lesssim \int_{\mathbb{R}^d} |V(x)|^{\frac{d}{2} + \delta} dx$$

$\forall \frac{1}{2} \leq \delta \leq \frac{d}{2}$ if $d \geq 2$.

Counterexample (Bögli - C. 2021)



Optimality

Counterexample also shows optimality of :

$$\left. \begin{aligned} * |E|^{q-\frac{d}{2}} &\lesssim \int |V|^q \quad \forall q \leq \frac{d+1}{2} \\ * d(E, \mathbb{R}_+)^{q-\frac{d+1}{2}} |E|^{\frac{1}{2}} &\lesssim \int |V|^q \quad \forall q > \frac{d+1}{2} \end{aligned} \right\} \text{(Frank)}$$
$$* |E|^{\frac{1}{2}} \lesssim \sup_y \int \exp(-\ln E |x-y|) |V(x)|^{\frac{d+1}{2}} dx \quad (C.)$$

Q: Under what **structural** aspts. on V do we have **improvements**?

Improved bounds

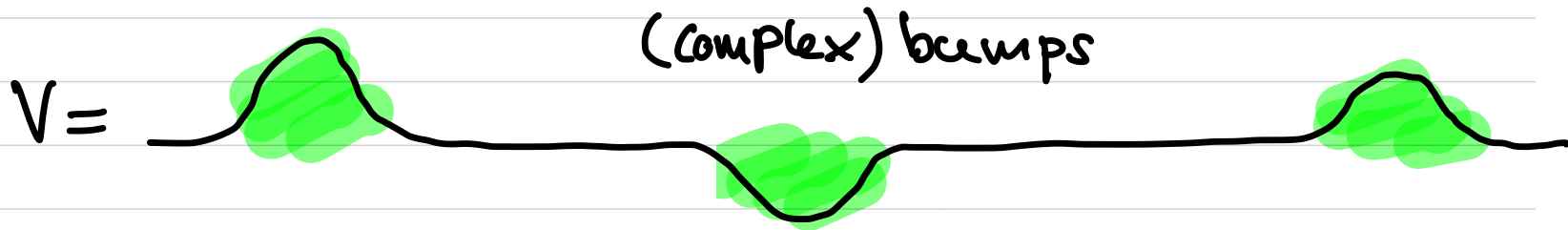
* V radial \Rightarrow L-S conj. tree (Frank-Simon)

* V sparse

* V random

* ...

Sparse Potentials



Bögli constructed examples $\forall q > d \forall \varepsilon > 0$ s.t.

$\|V\|_q + \|V\|_\infty < \varepsilon$ and $-\Delta + V$ has eigenvalues

accumulating to every point in \mathbb{R}_+ .



Almost Orthogonality

Exp. decay of Green's fct. + large separation \rightarrow

$$-\Delta + V \sim \bigoplus_j (-\Delta + V_j) + \text{small error}$$

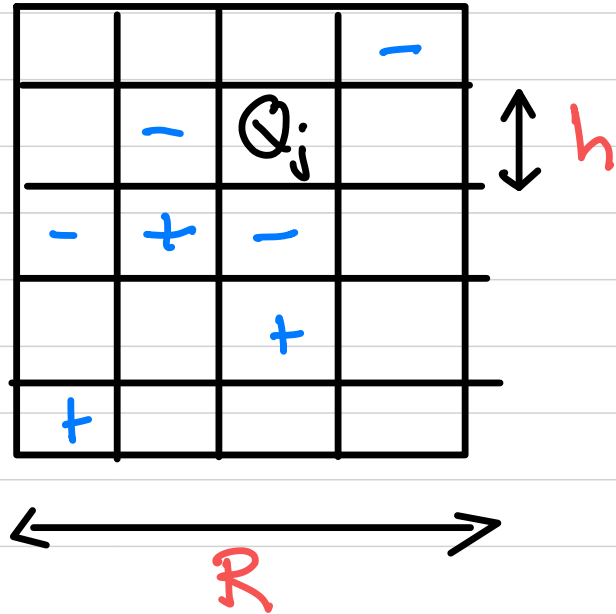
\Rightarrow can replace $\int |V|^q$ by $\sup_j \int |V_j|^q$
in Frank's bounds.

Random Potentials

$$V_\omega(x) = \omega_j v_j \quad \forall x \in Q_j$$

$$P(\omega_j = \pm 1) = \frac{1}{2}$$

$$\sum_j |v_j|^q < \infty$$



Thm (C.-Merz 122)

For $q \leq d+1$, every eigenvalue z of $-\Delta + V_w$ satisfies

$$\frac{|z|^{1 - \frac{d}{2q}}}{\langle |z|^{\frac{1}{2}} \rangle [\log \langle |z|^{\frac{1}{2}} \rangle]^{7/2}} \lesssim M \left(\sum_j |v_j|^q \right)^{\frac{1}{q}}$$

except for w in a set of measure $\leq \exp(-cM^2)$.

If we sacrifice the **endpoint**, we can remove the compact support assumption.

In particular: $\forall q < d+1$,

$$\sup_{\mathbb{Z}} \frac{|z|^{q-\frac{d}{2}}}{\left(\sum_j |v_j|^q\right)^{\frac{1}{q}}} < \infty \quad \text{a.s.}$$

e.v. of $-\Delta + V_w$

Proof Sketch

$$\textcircled{1} (-\Delta + V_\omega - z) = (-\Delta - z) \left(1 + \underbrace{(-\Delta - z)^{-1} V_\omega}_{R_0(z)} \right)$$

Born series:

$$(1 + R_0 V)^{-1} = \sum_{n=0}^{\infty} (-1)^n [R_0 V]^n R_0$$



$$R_0 V R_0^{\frac{1}{2}} |R_0|^{\frac{1}{2}} V R_0^{\frac{1}{2}} |R_0|^{\frac{1}{2}} V \dots$$

$|\mathbb{R}_0|^{\frac{1}{2}}$ is the Fourier multiplier

$$|\xi^2 - z|^{-\frac{1}{2}}$$

Localization in x -space ($\text{supp } V \subset B_R$)

$$\Rightarrow (|\xi^2 - z| + 1/R)^{-\frac{1}{2}}$$

(Smoothing in ξ -space)

② By spectral thm.

$$|R_0(z)| = \int |\lambda - z|^{-1} dE(\lambda)$$

$$\frac{dE(\lambda)}{d\lambda} = c \lambda^{\text{power}} E(\lambda) E(\lambda)^*$$

$$E(\lambda): L^2(\lambda S^{d-1}, d\sigma_\lambda) \rightarrow L^\infty(\mathbb{R}^d)$$
$$g \mapsto (g d\sigma_\lambda)^\vee$$

$$\int_{1-c}^{1+c} (|\lambda - 1| + \frac{1}{R})^{-1} \ll \log \frac{1}{R}$$

Up to factors of $\log \frac{1}{R}$ it suffices
to control the operators

$$\mathcal{E}(\lambda)^* V_w \mathcal{E}(\lambda) : L^2(\lambda^{\alpha+1} d\sigma_\lambda) \rightarrow$$

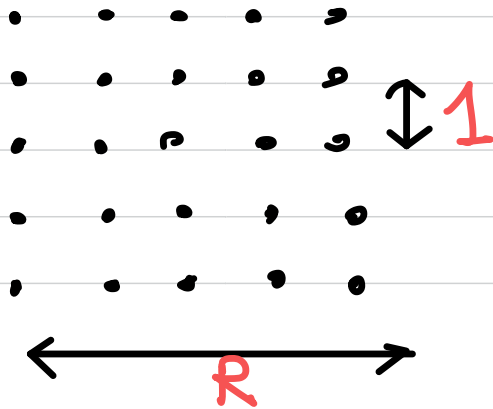
③ Reductions :

- Suff. to consider $z = 1 + i\epsilon$
- Localization in Ξ -space (S^{d-1} compact)

\Rightarrow smoothing of V on unit scale

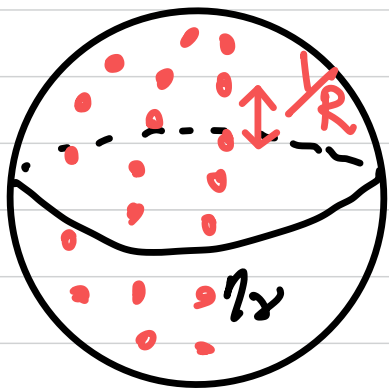
\Rightarrow discretize

(# lattice pts. $\sim R^d$)



• Localization in x -space ($\text{supp } \psi \subset \mathbb{B}_R$)

\Rightarrow discretize S^{d-1} :



$\{\eta_\nu\}$ $\frac{1}{R}$ -net

$$\mathcal{H} := \ell^2(\mathbb{1}, \dots, R^{d-1} \mathbb{1})$$

Discrete extension operator:

$$(\mathbb{S}g)(x) := \sum_{\nu} e^{2\pi i x \cdot \eta_\nu} g(\eta_\nu)$$

Reduced to control

$$S^* V_w S : \mathcal{H} \rightarrow \mathcal{H}$$

- By large deviation estimates suff. to control

$$\mathbb{E} \|S^* V_w S\| = \sup_{\|f\| = \|g\| = 1} |\langle V_w S f, S g \rangle|$$

- Replace sup by a sup over a finite set, pay an entropy cost.

Entropy bound

Consider a linear operator $S : \mathcal{H} \rightarrow \ell_m^\infty$, where

- ▶ \mathcal{H} is a finite-dimensional Hilbert space and
- ▶ $\ell_m^\infty = \ell^\infty(\{1, \dots, m\})$.

For $\epsilon > 0$ let $\mathcal{N}(\epsilon)$ be the minimal number of balls in ℓ_m^∞ of radius ϵ needed to cover the set $\{Sx : x \in \mathcal{H}, \|x\|_{\mathcal{H}} \leq 1\}$.

Using an entropy bound known as the “dual Sudakov inequality”, [Pajor and Tomczak-Jaegermann 1986], Bourgain shows that

$$\log \mathcal{N}(\epsilon) \lesssim (\log m) \epsilon^{-2} \|S\|_{\mathcal{H} \rightarrow \ell_m^\infty}^2.$$

Crucial:

- ▶ Independent of $\dim \mathcal{H}$,
- ▶ Only log loss in m .

We apply the entropy bound to a discrete version of the Fourier extension operator ($\log m = \log R$):

$$S : \mathcal{H} \rightarrow \ell^\infty(\Lambda_1 \cap B_R), \quad \{a_\nu\} \mapsto \left\{ \sum_{\nu \in \Lambda_R^*} a_\nu e(\nu \cdot x) \right\}_x$$

where $\mathcal{H} := \ell^2(\Lambda_R^*)$ with norm $\|a\|_{\mathcal{H}} := R^{-\frac{d-1}{2}} (\sum_\nu |a_\nu|^2)^{1/2}$.

Discrete Stein–Tomas inequality

For each $R \geq 2$, each collection Λ_R^* consisting of $1/R$ -separated points on S^{d-1} , each sequence $a_\nu \subset \mathbb{C}$, each ball B_R and each collection Λ_1 of 1-separated points in \mathbb{R}^d :

$$\|S\|_{\mathcal{H} \rightarrow \ell^{p'}(\Lambda_1 \cap B_R)} \lesssim 1 \quad \text{for} \quad p' \geq 2(d+1)/(d-1).$$

In particular, we have the trivial bound ($p' = \infty$)

$$\|S\|_{\mathcal{H} \rightarrow \ell^\infty(\Lambda_1 \cap B_R)} \lesssim 1.$$

Thus the entropy number satisfies the bound

$$\log \mathcal{N}(\epsilon) \lesssim (\log R) \epsilon^{-2}.$$

Consequence:

Chaining

Let $p' \geq 2(d+1)/(d-1)$. For every $k \in \mathbb{Z}_+$, there exist sets $\mathcal{F}_k \subset \ell^\infty(\Lambda_1 \cap B_R)$ with the following properties.

- (a) $\log |\mathcal{F}_k| \lesssim \log(R)4^k$ (here $|\cdot|$ denotes counting measure).
- (b) For $\xi \in \mathcal{F}_k$,

$$\|\xi\|_{\ell^\infty(\Lambda_1)} \lesssim 2^{-k}, \quad \|\xi\|_{\ell^{p'}(\Lambda_1)} \lesssim 1.$$

- (c) For each $a \in \mathcal{H}$ with $\|a\|_{\mathcal{H}} \leq 1$ there is a representation

$$Sa = \sum_{k \in \mathbb{Z}_+} \xi^{(k)} \quad \text{for some } \xi^{(k)} \in \mathcal{F}_k.$$

Thank you for your
attention !