Fokas diagonalization

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Motivating problems: full line heat

$$
\begin{aligned}
\partial_t q(x,t) - \partial_{xx} q(x,t) &= 0 & (x,t) &\in \mathbb{R}\times(0,T), & \text{(1.PDE)}\\
q(x,0) &= Q(x) & x &\in \mathbb{R}, & \text{(1.IC)}\\
q(\,\cdot\,,t) &\in \mathcal{S}(\mathbb{R}) & t &\in [0,T], & \text{(1.BC)}
\end{aligned}
$$

Apply exponential Fourier transform $\mathrm{F}_{\mathbb{R}}\left[\phi\right](\lambda) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda x}\phi(x)\,\mathrm{d}x$ to get

$$
\begin{aligned}
\partial_t \mathrm{F}_{\mathbb{R}}\left[q\right](\lambda;t) &= \mathrm{F}_{\mathbb{R}}\left[\partial_{xx} q\right](\lambda;t)\\
&= -\lambda^2 \mathrm{F}_{\mathbb{R}}\left[q\right](\lambda;t), \qquad \text{by IBP.}
\end{aligned}
$$

Solve temporal ODE for $\mathrm{F}_{\mathbb{R}}\left[q\right](\lambda;t) = \mathrm{e}^{-\lambda^2 t}\mathrm{F}_{\mathbb{R}}\left[Q\right](\lambda)$. Apply inverse Fourier transform:

$$
q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x - \lambda^2 t}\mathrm{F}_{\mathbb{R}}\left[Q\right](\lambda)\,\mathrm{d}\lambda.
$$

Motivating problems: full line heat

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$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\lambda x-\lambda^2 t}\mathrm{F}_{\mathbb{R}}[Q](\lambda)\,\mathrm{d}\lambda.$$

Argument uses 2 properties of the exponential Fourier transform:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

Motivating problems: half line Dirichlet heat

$$\begin{aligned} \partial_t q(x,t) - \partial_{xx} q(x,t) &= 0 && (x,t) \in (0,\infty)\times(0,T), && \text{(2.PDE)} \\ q(x,0) &= Q(x) && x \in [0,\infty), && \text{(2.IC)} \\ q(0,t) = 0, \qquad q(\,\cdot\,,t) &\in \mathcal{S}[0,\infty) && t \in [0,T], && \text{(2.BC)} \end{aligned}$$

Apply Fourier sine transform $\mathrm{F_s}[\phi](\lambda) = \int_0^\infty \sin(\lambda x)\phi(x)\,\mathrm{d}x$ to get

$$\begin{aligned} \partial_t \mathrm{F_s}[q](\lambda;t) &= \mathrm{F_s}[\partial_{xx} q](\lambda;t) \\ &= -\lambda^2 \mathrm{F_s}[q](\lambda;t), && \text{by IBP.} \end{aligned}$$

Solve temporal ODE for $\mathrm{F_s}[q](\lambda;t) = \mathrm{e}^{-\lambda^2 t}\mathrm{F_s}[Q](\lambda)$. Apply inverse Fourier sine transform:

$$q(x,t) = \mathrm{F_s^{-1}}\left[\mathrm{e}^{-\lambda^2 t}\mathrm{F_s}[Q](\lambda)\right](x).$$

Motivating problems: half line Dirichlet heat

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q(x,0) &= Q(x) & x &\in [0,\infty), & \text{(2.IC)}\\
q(0,t) = 0, \qquad q(\,\cdot\,,t) &\in \mathcal{S}[0,\infty) & t &\in [0,T], & \text{(2.BC)}
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Solve temporal ODE for $\mathrm{F_s}\left[q\right](\lambda;t) = \mathrm{e}^{-\lambda^2 t}\mathrm{F_s}\left[Q\right](\lambda)$. Apply inverse Fourier sine transform:

$$q(x,t) = \mathrm{F_s^{-1}}\left[\mathrm{e}^{-\lambda^2 t}\mathrm{F_s}\left[Q\right](\lambda)\right](x).$$

Argument uses 2 properties of the Fourier sine transform:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

Motivating problems: finite interval Dirichlet heat

$$
\begin{aligned}
\partial_t q(x,t) - \partial_{xx} q(x,t) &= 0 & (x,t) &\in (0,1)\times(0,T), && \text{(3.PDE)}\\
q(x,0) &= Q(x) & x &\in [0,1], && \text{(3.IC)}\\
q(0,t) = 0, \qquad q(1,t) &= 0 & t &\in [0,T], && \text{(3.BC)}
\end{aligned}
$$

Apply finite interval Fourier sine transform:

$$
\mathrm{F}_{\mathrm{s\ ser}}\left[\phi\right](j) = 2\int_0^1 \phi(x)\sin(j\pi x)\,\mathrm{d}x, \qquad \mathrm{F}_{\mathrm{s\ ser}}^{-1}\left[(C_j)_{j\in\mathbb{N}}\right](x) = \sum_{j=1}^{\infty} C_j \sin(j\pi x).
$$

to get,

$$
\begin{aligned}
\partial_t \mathrm{F}_{\mathrm{s\ ser}}\left[q\right](j;t) &= \mathrm{F}_{\mathrm{s\ ser}}\left[\partial_{xx} q\right](j;t), && j\in\mathbb{N}\\
&= -(j\pi)^2 \mathrm{F}_{\mathrm{s\ ser}}\left[q\right](j;t), && \text{by IBP.}
\end{aligned}
$$

Solve temporal ODE for $\mathrm{F}_{\mathrm{s\ ser}}\left[q\right](j;t) = \mathrm{e}^{-(j\pi)^2 t}\mathrm{F}_{\mathrm{s\ ser}}\left[Q\right](j;t)$. Apply inverse finite interval Fourier sine transform:

$$
q(x,t) = \mathrm{F}_{\mathrm{s\ ser}}^{-1}\left[\mathrm{e}^{-(j\pi)^2 t}\mathrm{F}_{\mathrm{s}}\left[Q\right](j)\right](x).
$$

Motivating problems: observations

You need the right transform for the job in the sense that:

- The transform diagonalizes of the spatial differential operator.
- The transform is invertible.

But how do you find the right transform?

Classical generalization

Use separation of variables to derive the right transform pair.

Difficulty: relies on eigenfunctions forming a basis of the space of initial data.

Modern generalization

Use Fokas transform method. See works of Fokas, Pelloni, Mantzavinos, Deconinck, Trogdon, Vasan, Sheils...

Inverse transform is a contour integral. Forward transform accepts a continuous (complex contour) parameter.

Avoids difficulty of incompleteness of eigenfunctions.

Plan

1. Describe more efficient derivation of the Fokas transform pair.
2. Describe (in what sense) the Fokas transform pair diagonalizes the spatial differential operator.
3. Solve IBVP with the Fokas transform pair.
4. Generalize to interface problems.

3. Solve IBVP with Fokas transform pair: best scenario

Differential operator & boundary conditions:

$$L\phi = \omega(-i\partial_x)\phi, \qquad 0 = B_k\phi := \sum_{j=1}^{n} b_{k\,j}\phi^{(j-1)}(0) + \beta_{k\,j}\phi^{(j-1)}(1)$$

where ω monic of degree n and $k = 1, \dots, n$. Study problem

$$\begin{aligned} \partial_t q(x,t) + aLq(\,\cdot\,,t) &= 0 & (x,t) &\in (0,1)\times(0,T), & &\text{(4.PDE)} \\ q(x,0) &= Q(x) & x &\in [0,1], & &\text{(4.IC)} \\ B_k q(\,\cdot\,,t) &= 0 & t &\in [0,T],\ k = 1,\dots,n. & &\text{(4.BC)} \end{aligned}$$

Suppose we know a transform pair that obeys $\mathbf{F}^{-1}\left[\mathbf{F}\left[\phi\right]\right](x) = \phi(x)$ and $\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n\mathbf{F}\left[\phi\right](\lambda)$. Then (4.PDE):

$$\partial_t\mathbf{F}\left[q\right](\lambda;t) = -a\lambda^n\mathbf{F}\left[q\right](\lambda;t).$$

Solve temporal ODE for $\mathbf{F}\left[q\right](\lambda;t) = \mathrm{e}^{-a\lambda^n t}\mathbf{F}\left[Q\right](\lambda)$ and apply inverse transform:

$$q(x,t) = \mathbf{F}^{-1}\left[\mathrm{e}^{-a\lambda^n t}\mathbf{F}\left[Q\right](\lambda)\right](x).$$

3. Solve IBVP with Fokas transform pair: real scenario

Suppose $\mathbf{F}^{-1}\left[\mathbf{F}\left[\phi\right]\right](x) = \phi(x)$ holds, but $\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda)$ is replaced by

$$\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda) + \mathbf{R}\left[\phi\right](\lambda),$$

where, for appropriate functions $q(\,\cdot\,, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1}\left[\int_0^t \mathrm{e}^{a\lambda^n(s-t)}\mathbf{R}\left[q\right](\lambda; s)\,\mathrm{d}s\right](x) = 0.$$

Then (4.PDE) implies

$$\partial_t \mathbf{F}\left[q\right](\lambda; t) = -a\lambda^n \mathbf{F}\left[q\right](\lambda; t) - a\,\mathbf{R}\left[q\right](\lambda; t).$$

Solve temporal ODE for

$$\mathbf{F}\left[q\right](\lambda; t) = \mathrm{e}^{-a\lambda^n t}\mathbf{F}\left[Q\right](\lambda) - a\,\mathrm{e}^{-a\lambda^n t}\int_0^t \mathrm{e}^{a\lambda^n s}\mathbf{R}\left[q\right](\lambda; s)\,\mathrm{d}s$$

and apply inverse transform for

$$q(x,t) = \mathbf{F}^{-1}\left[\mathrm{e}^{-a\lambda^n t}\mathbf{F}\left[Q\right](\lambda)\right](x) - a\,\mathbf{F}^{-1}\left[\int_0^t \mathrm{e}^{a\lambda^n(s-t)}\mathbf{R}\left[q\right](\lambda; s)\,\mathrm{d}s\right](x).$$

3. Solve IBVP with Fokas transform pair: observation

We do need invertability: $\mathbf{F}^{-1}\left[\mathbf{F}\left[\phi\right]\right](x) = \phi(x)$.

But true diagonalization, $\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda)$, is not required.

It is enough to have

$$\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda) + \mathbf{R}\left[\phi\right](\lambda),$$

where, for appropriate functions $q(\,\cdot\,, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1}\left[\int_0^t \mathrm{e}^{a\lambda^n(s-t)} \mathbf{R}\left[q\right](\lambda; s)\,\mathrm{d}s\right](x) = 0,$$

for the transform method to work.

2. Diagonalization how?

Theorem

There exists a remainder transform $\mathbf{R}$ *such that, for all* ϕ *satisfying the boundary conditions,*

$$\mathbf{F}[L\phi](\lambda) = \lambda^n \mathbf{F}[\phi](\lambda) + \mathbf{R}[\phi](\lambda) \tag{5a}$$

and, if $q : [0,1] \times [0,T] \to \mathbb{C}$ *is such that, for all* $t \in [0,T]$, $q(\,\cdot\,,t)$ *satisfying the boundary conditions and, for all* $j \in \{0,1,\ldots,n-1\}$*, uniformly for all* $x \in [0,1]$, $\partial_x^j q(x,\,\cdot\,)$ *is a function of bounded variation, then, for all* $t \in [0,T]$ *and all* $x \in (0,1)$,

$$\mathbf{F}^{-1}\left[\int_0^t \mathrm{e}^{a\lambda^n(s-t)} \mathbf{R}[q](\lambda;s)\,\mathrm{d}s\right](x) = 0. \tag{5b}$$

Proof sketch.

Precisely define $\mathbf{R}[\phi](\lambda)$. Show $\mathbf{R}[\phi](\lambda)$ is analytic and $\mathcal{O}\left(|\lambda|^{n-1}\right)$ as $\lambda \to \infty$, uniformly in $\arg(\lambda)$. To show (5b): integrate by parts to get $1/a\lambda^n$ factor so $\mathcal{O}\left(|\lambda|^{-1}\right)$ overall. Use Jordan's lemma & Cauchy's theorem. $\square$

1. Derivation of transform: biholomorphic map

Proposition

For sufficiently large $\mathscr{R} > 0$, there is a function ν which satisfies

$$\omega(\nu(\lambda)) = \lambda^n, \qquad \lim_{\lambda\to\infty} \nu(\lambda)/\lambda = 1,$$

is biholomorphic outside the disc centered at zero with radius $\mathscr{R}$, and whose derivative approaches 1 as $\lambda \to \infty$, uniformly in the argument of λ.

Not a new theorem, but we give an elementary proof.

Allows us to think of the eigenvalues as λ^n instead of $\omega(\lambda)$, simplifying some arguments.

1. Derivation of transform: formal transform pair

"Formal" because it works with the formal differential operator; it ignores the boundary conditions.

$$\mathscr{F}[\phi](\lambda) := \left\langle \phi, \mathrm{e}^{\mathrm{i}\overline{\nu(\lambda)\cdot}} \right\rangle = \int_0^1 \phi(x)\mathrm{e}^{-\mathrm{i}\nu(\lambda)x}\,\mathrm{d}x, \qquad |\lambda| > \mathscr{R}, \tag{6a}$$

$$\mathscr{F}^{-1}[F](x) := \frac{1}{2\pi}\,\mathrm{p.v.}\int_\gamma \mathrm{e}^{\mathrm{i}\nu(\lambda)x}\nu'(\lambda)F(\lambda)\,\mathrm{d}\lambda, \tag{6b}$$

γ a perturbation of $\mathbb{R}$ to avoid the disc on which ν is undefined.

Theorem

Suppose that $\phi \in \mathrm{C}^1[0,1]$. Then, for all $x \in (0,1)$, $\mathscr{F}^{-1}[\mathscr{F}[\phi]](x) = \phi(x)$.

Sketch proof.

Change of variables $k = \nu(\lambda)$ in usual exponential Fourier transform. □

1. Derivation of transform: forward transform

Let $\alpha = \exp(2\pi\mathrm{i}/n)$, so that $\omega(\nu(\alpha^j\lambda)) = (\alpha^j\lambda)^n = \lambda^n$, for $j = 1, 2, \dots, n$.

Build forward transform out of inner products with all the eigenfunctions of the adjoint formal differential operator.

$$\mathbf{F}[\phi](\lambda) := \text{certain linear combination of } \left\langle \phi, \mathrm{e}^{\mathrm{i}\overline{\nu(\alpha^j\lambda)\cdot}} \right\rangle$$

The particular linear combination is given explicitly in terms of Birkhoff's characteristic determinant for classical (Lagrange) adjoint differential operator $L^\star$ whose action is

$$L^\star\psi = \overline{\omega}(-\mathrm{i}\partial_x)\psi,$$

for $\overline{\omega}$ the Schwarz conjugate of ω, with adjoint boundary conditions defined so that, for all ϕ in the domain of L and all ψ in the domain of $L^\star$,

$$\langle L\phi, \psi\rangle = \langle \phi, L^\star\psi\rangle .$$

1. Derivation of transform: inverse transform

Inverse transform is defined by

$$\mathbf{F}^{-1}\left[F\right](x) := \mathrm{p.v.}\int_{\Gamma} \mathrm{e}^{\mathrm{i}\nu(\lambda)x} F(\lambda)\,\mathrm{d}\lambda,$$

where Γ is a perturbation of $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda^n) = 0\}$.

Theorem

Suppose $\phi \in \mathrm{C}^1[0,1]$ *and* **certain integrals evaluate to $\mathbf{0}$**. *Then, for all* $x \in (0,1)$, $\mathbf{F}^{-1}\left[\mathbf{F}\left[\phi\right]\right](x) = \phi(x)$.

Sketch proof.

Contours chosen so that, under the assumption, much of the contour integrals cancels. Forward transform chosen so that remaining parts simplify to $\mathscr{F}^{-1}\left[\mathscr{F}\left[\phi\right]\right](x)$. Apply validity theorem of formal transform pair. □

Assumption is not too restrictive. Requires eigenvalues aren't in the wrong places and certain ratios of analytic functions decay. Such assumptions *always* appear (or are proved) in Fokas transform method.

Related to wellposedness of IBVP.

1. Derivation of transform: observations

The problem of deriving the Fokas transform pair is reduced to construction of the classical adjoint.

And that can be done algorithmically; see Coddington & Levinson book 1955.

A student, Linda Linfan Xiao, implemented it in julia as undergraduate thesis.

Another student, Sultan Aitzhan, reimplemented it in ApproxFun.

4. Interface problems: statement of problem

Several finite intervals.

On each a different formal differential operator, all same degree.

Boundary conditions prescribe linear combinations of all boundary values from all intervals.

Defines L, a list differential operator. The initial interface value problem is

$$\begin{aligned} \partial_t q(x,t) + \mathbf{a} \circ Lq(x,t) &= \mathbf{0} & (x,t) &\in (0,1)\times(0,T), & \text{(7.PDE)} \\ q(x,0) &= Q(x) & x &\in [0,1], & \text{(7.IC)} \\ q(\,\cdot\,,t) \text{ obeys the boundary conditions} & & t &\in [0,T], & \text{(7.BC)} \end{aligned}$$

where $q(\,\cdot\,,t), Q$ are lists of functions, one on each copy of spacial interval $[0,1]$, $\mathbf{a}$ a list of temporal coefficients, $\circ$ is entrywise multiplication.

Note: this also includes piecewise constant coefficient PDE.

4. Interface problems: via usual Fokas method

See Sheils's work for how to do this, several in collaboration with Deconinck.

Arduous to derive D-to-N map, because you have to solve very large linear systems by hand.

4. Interface problems: via new approach

Sultan Aitzhan & Sambhav Bhandari constructed the adjoint of the interface differential operator.

Sultan implemented in julia & ApproxFun.

That gives you the definition of the transform pair.

Analogous diagonalization result holds.

So the same transform method can be used to solve the initial interface value problem.

Conclusion

Fokas diagonalization is **not** like

$$\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda).$$

Instead, Fokas diagonalization is like

$$\mathbf{F}\left[L\phi\right](\lambda) = \lambda^n \mathbf{F}\left[\phi\right](\lambda) + \mathbf{R}\left[\phi\right](\lambda),$$

where, for appropriate functions $q(\,\cdot\,, t)$ obeying boundary conditions,

$$\mathbf{F}^{-1}\left[\int_0^t \mathrm{e}^{a\lambda^n(s-t)}\mathbf{R}\left[q\right](\lambda; s)\,\mathrm{d}s\right](x) = 0.$$

But that is enough for the transform method to work, essentially unaltered.

We've got a quicker way to derive the transform, in terms of the characteristic determinant of the classical adjoint.

All this works for interface problems, too.

Thanks

Thank you