Linear growth of quantum complexity

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

NICOLE YUNGER HALPERN

INSTITUTE FOR PHYSICAL SCIENCE & TECHNOLOGY

NIST

JOINT CENTER FOR QUANTUM INFORMATION AND COMPUTER SCIENCE

physics University of Maryland

NSF

Institute for Robust Quantum Simulation

Quantum complexity 

Quantum complexity

- What it is: the difficulty of implementing a unitary on n qubits

Quantum complexity

- What it is: the difficulty of implementing a unitary on n qubits
- Ex.: complexity of $\mathbf{1} = 0$

Quantum complexity 

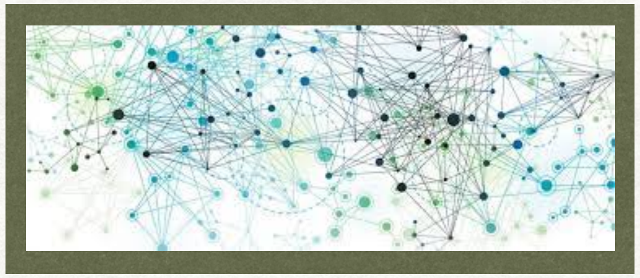
- What it is: the difficulty of implementing a unitary on n qubits
- Ex.: complexity of $\mathbf{1} = 0$
- Maximum: $\sim 4^n$
  - Counting argument: Susskind, arXiv:1810.11563 (2018).

Quantum complexity

- What it is: the difficulty of implementing a unitary on n qubits
- Ex.: complexity of $\mathbf{1} = 0$
- Maximum: $\sim 4^n$
    - Counting argument: Susskind, arXiv:1810.11563 (2018).
- Multiplicity of quantifications

Quantum complexity has been echoing across many-body physics.

Quantum complexity has been echoing across many-body physics.



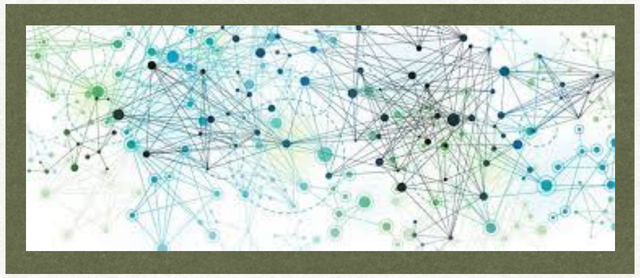
(1) Quantum computation

Quantum complexity has been echoing across many-body physics.

(1) Quantum computation → quantum advantage/supremacy

- Ex.: Arute *et al.*, Nature **574**, 505 (2019).

Quantum complexity has been echoing across many-body physics.



(1) Quantum computation → quantum advantage/supremacy

- Ex.: Arute *et al.*, Nature **574**, 505 (2019).
- Quantum computer achieves advantage upon preparing a sufficiently complex state

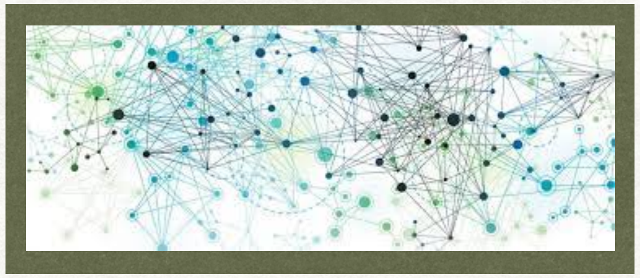
Quantum complexity has been echoing across many-body physics.

(1) Quantum computation ⟶ quantum advantage/supremacy

- Ex.: Arute *et al.*, Nature **574**, 505 (2019).
- Quantum computer achieves advantage upon preparing a sufficiently complex state

(2) Condensed matter

Quantum complexity has been echoing across many-body physics.



(1) Quantum computation ⟶ quantum advantage/supremacy

- Ex.: Arute *et al.*, Nature **574**, 505 (2019).
- Quantum computer achieves advantage upon preparing a sufficiently complex state

(2) Condensed matter

- Gapped Hamiltonian is in nontrivial topological phase only if the ground state has a complexity $>$ constant in the system size
- Chen, Gu, and Wen, Phys. Rev. B **83** (3) (2011).

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS

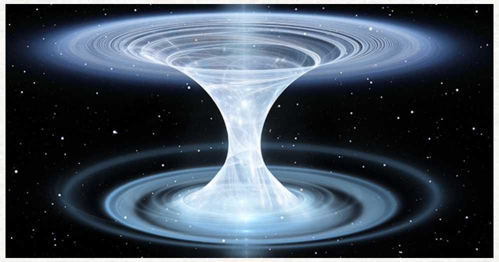
CFT

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



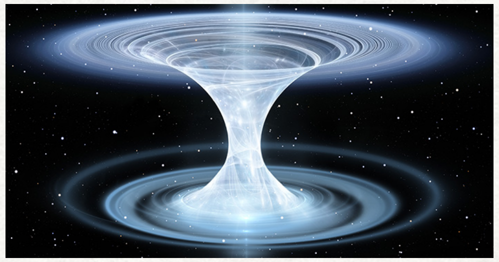
CFT

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



Volume grows linearly for exponentially long time

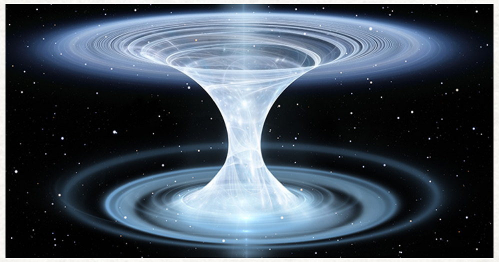
CFT

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



Volume grows linearly for exponentially long time

CFT

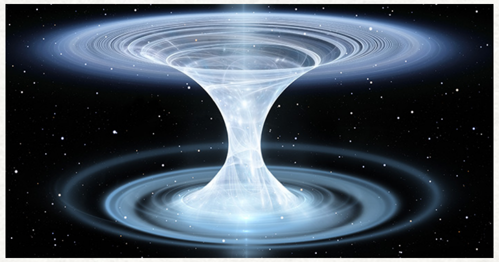
$$|\psi\rangle$$

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



Volume grows linearly for exponentially long time

CFT

$|\psi\rangle$

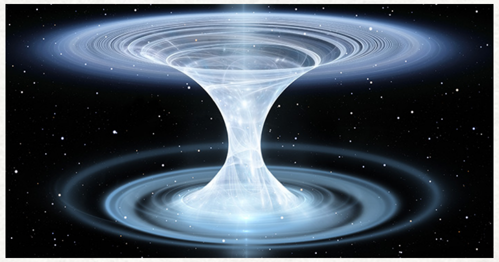
Local observables & out-of-time-ordered correlators equilibrate more quickly.

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



Volume grows linearly for exponentially long time

CFT

$$|\psi\rangle$$

Local observables & out-of-time-ordered correlators equilibrate more quickly.

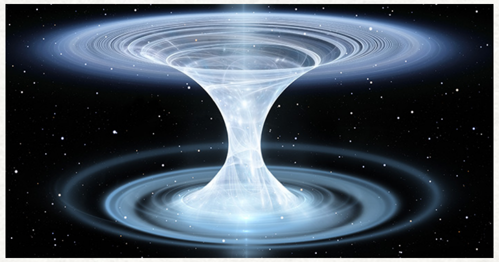
- Proposed resolution: complexity = volume
  - Susskind, arXiv:1402.5674 (2014).
  - Susskind, Fort. Phys. **64**, 24 (2016).
  - Stanford and Susskind, Phys. Rev. D **90**, 126007 (2014).

Quantum complexity has been echoing across many-body physics.

(3) Wormhole-growth paradox in AdS/CFT

- Hartman and Maldacena, JHEP **5**, 014 (2013).
- Susskind, Fort. Phys. **64**, 24 (2016).

AdS



Volume grows linearly for exponentially long time

CFT

$$|\psi\rangle$$

Local observables & out-of-time-ordered correlators equilibrate more quickly.

- Proposed resolution: complexity = volume
  - Susskind, arXiv:1402.5674 (2014).
  - Susskind, Fort. Phys. **64**, 24 (2016).
  - Stanford and Susskind, Phys. Rev. D **90**, 126007 (2014).
- Complexity = other stuff

A quantification of quantum complexity:

A quantification of quantum complexity:
exact circuit complexity

A quantification of quantum complexity: exact circuit complexity

- Useful in the context of quantum complexity

A quantification of quantum complexity:
exact circuit complexity

- Useful in the context of quantum complexity
- $\mathscr{C}(U)$ = least number of 2-qubit gates required to effect U

A quantification of quantum complexity: exact circuit complexity

- Useful in the context of quantum complexity
- $\mathscr{C}(U)$ = least number of $\underbrace{\text{2-qubit gates}}_{\text{Needn't be geometrically local}}$ required to effect U

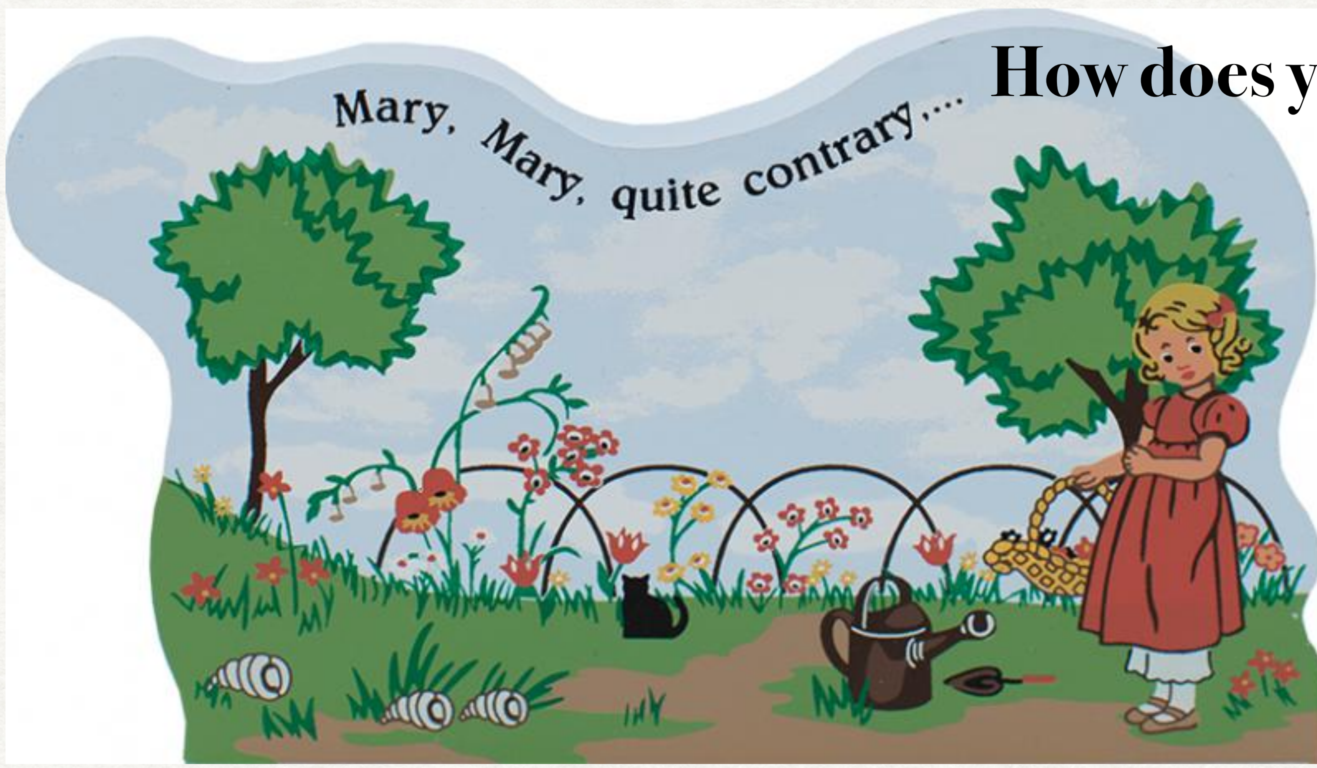
2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(1)

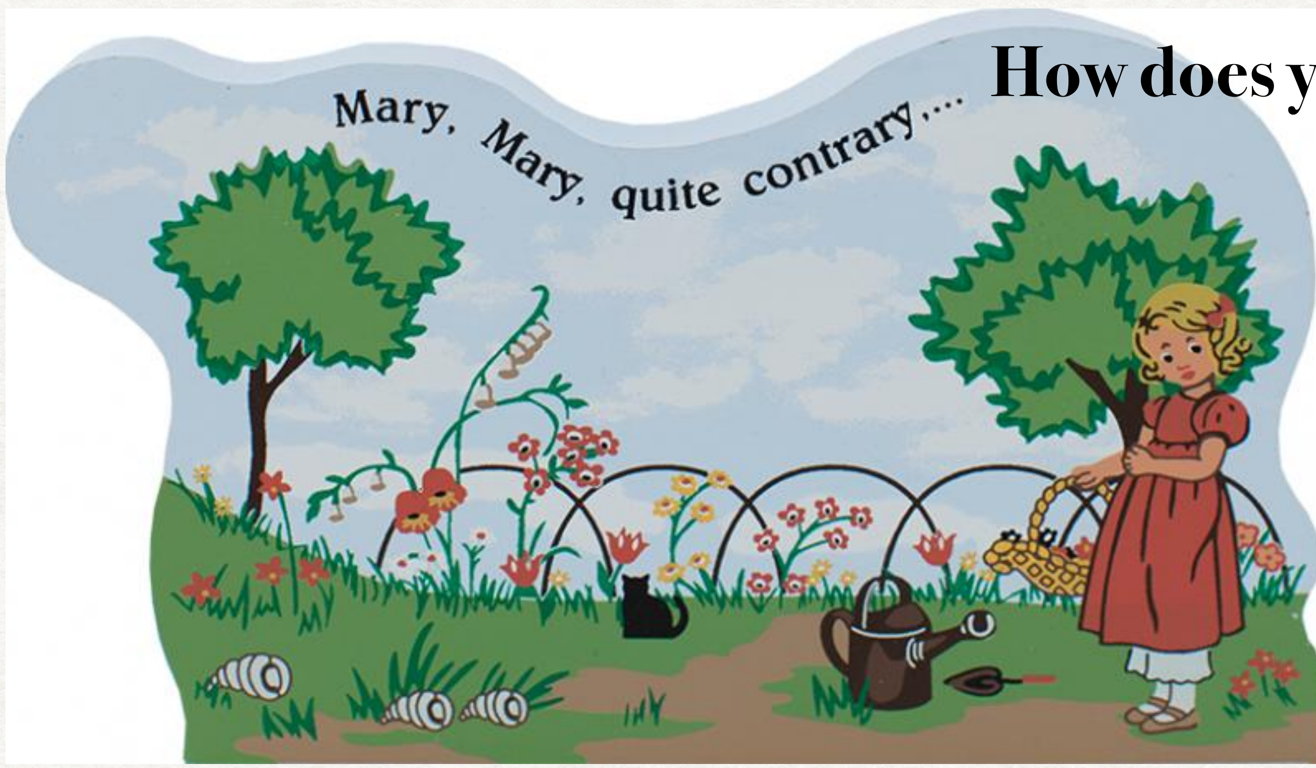


How does your complexity grow?

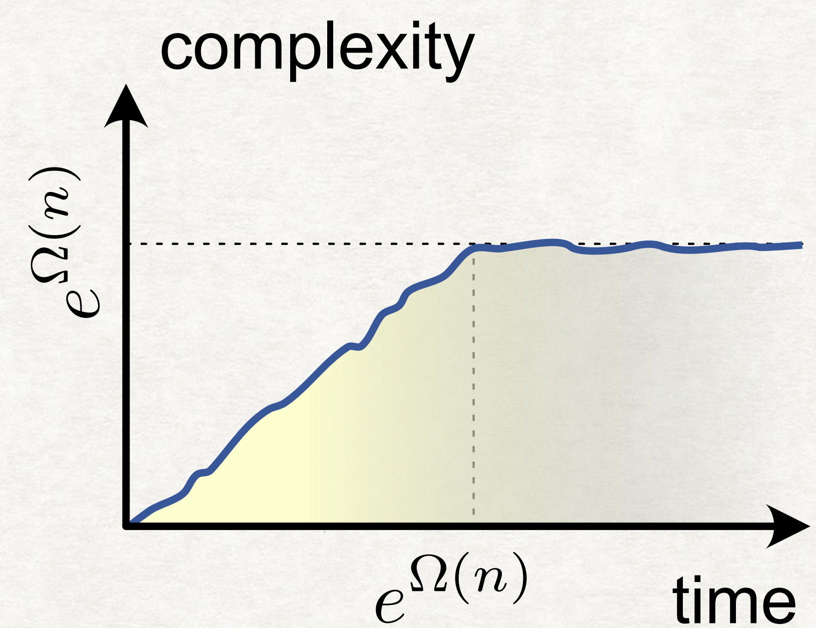
2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(1)



How does your complexity grow?



Quantum complexity generically grows linearly in time for a time exponential in the system size.

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is useful.

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is useful. $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is useful. $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation
  ~ clean scrap paper

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is $\underbrace{\text{useful}}_{\text{a resource}}$ $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation

  ~ clean scrap paper

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is $\underbrace{\text{useful}}_{\text{a resource}}$ $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation
  ~ clean scrap paper
- **Resource theory**: simple quantum-information-theoretic model for constrained operations
  - Review: Chitambar and Gour, Rev. Mod. Phys. **91**, 025001 (2019).

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for <u>uncomplexity</u> can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- <u>Uncomplexity is useful</u> $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation

  a resource

  ~ clean scrap paper

- **Resource theory**: simple quantum-information-theoretic model for constrained operations 
  - <u>Review</u>: Chitambar and Gour, Rev. Mod. Phys. **91**, 025001 (2019).

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is useful $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation

  a resource

  ~ clean scrap paper
- **Resource theory**: simple quantum-information-theoretic model for constrained operations

  - Review: Chitambar and Gour, Rev. Mod. Phys. **91**, 025001 (2019).
  - Uses: formalize, and calculate optimal efficiencies of, operational tasks

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is useful $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation
  a resource
  ~ clean scrap paper
- **Resource theory**: simple quantum-information-theoretic model for constrained operations 
  - Review: Chitambar and Gour, Rev. Mod. Phys. **91**, 025001 (2019).
  - Uses: formalize, and calculate optimal efficiencies of, operational tasks

Example: distill
high-quality entanglement

2 complexity conjectures by Brown and Susskind

Brown and Susskind, Phys. Rev. D **97**, 086015 (2018).

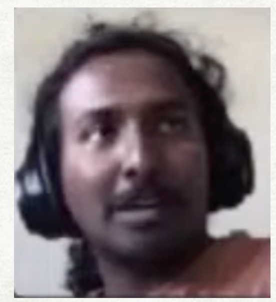
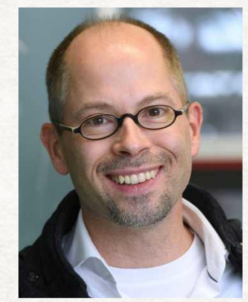
(2) A resource theory for uncomplexity can be defined.

- **State complexity**: difficulty of preparing a desired state from $|0\rangle^{\otimes n}$
- **Uncomplexity**: (maximal complexity) – (actual complexity)
- Uncomplexity is $\underbrace{\text{useful}}_{\text{a resource}}$ $\longrightarrow$ $|0\rangle^{\otimes n}$ as input to quantum computation
  ~ clean scrap paper
- **Resource theory**: simple quantum-information-theoretic model for constrained operations 
  - Review: Chitambar and Gour, Rev. Mod. Phys. **91**, 025001 (2019).
  - Uses: formalize, and calculate optimal efficiencies of, operational tasks
    Example: distill high-quality entanglement

Proofs of

2 complexity conjectures by Brown and Susskind



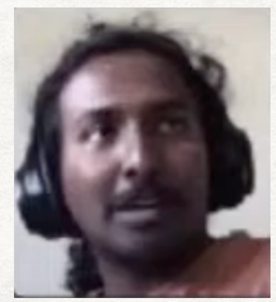
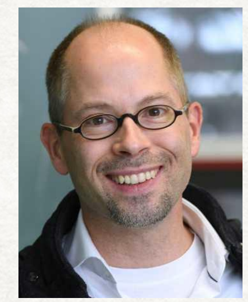

(1) Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

(2) NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Proofs of

2 complexity conjectures by Brown and Susskind




(1) Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

(2) NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Where we're headed

Where we're headed

- Why is the problem hard?

Where we're headed

- Why is the problem hard?
  - Setting the stage

Where we’re headed

- Why is the problem hard?
- Setting the stage
- Introduce terminology + mindset

Where we're headed

- Why is the problem hard?
- Setting the stage
- Introduce terminology + mindset
- Main theorem

Where we're headed

- Why is the problem hard?
- Setting the stage
- Introduce terminology + mindset
- Main theorem
- Proof sketch

Where we're headed

- Why is the problem hard?
- Setting the stage
- Introduce terminology + mindset
- Main theorem
- Proof sketch
- Opportunities

Lower-bounding quantum complexity
for an exponentially long time is difficult.

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …
- Workarounds, earlier approaches

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …
- <u>Workarounds, earlier approaches</u>
  - Focus on short times.

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …
- Workarounds, earlier approaches
  - Focus on short times.
  - Focus on high-dimensional subsystems.

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …
- <u>Workarounds, earlier approaches</u>
  - Focus on short times.
  - Focus on high-dimensional subsystems.
  - Use unitary t-designs.

Lower-bounding quantum complexity for an exponentially long time is difficult.

- Knill, arXiv:9508006 (1995).
  Nielsen, arXiv:0502070 (2005).
  Gosset *et al.*, Quant. Inf. Comp. **14**, 1277 (2014).
  Roberts and Yoshida, JHEP **121**, 121 (2017).
  Brandão *et al.*, PRX Quantum **2**, 030316 (2021).
  Brandão, Harrow, and Horodecki, Phys. Rev. Lett. **116**, 170502 (2016).
  Eisert, Phys. Rev. Lett. **127**, 020501 (2021).
  …
- Workarounds, earlier approaches
  - Focus on short times.
  - Focus on high-dimensional subsystems.
  - Use unitary t-designs.
  - Assume a lack of collisions.

Lower-bounding quantum complexity is difficult.

$\longrightarrow$ Why:

Lower-bounding quantum complexity is difficult.

$\longrightarrow$ Why:

- Later gates can cancel earlier gates

Lower-bounding quantum complexity is difficult.

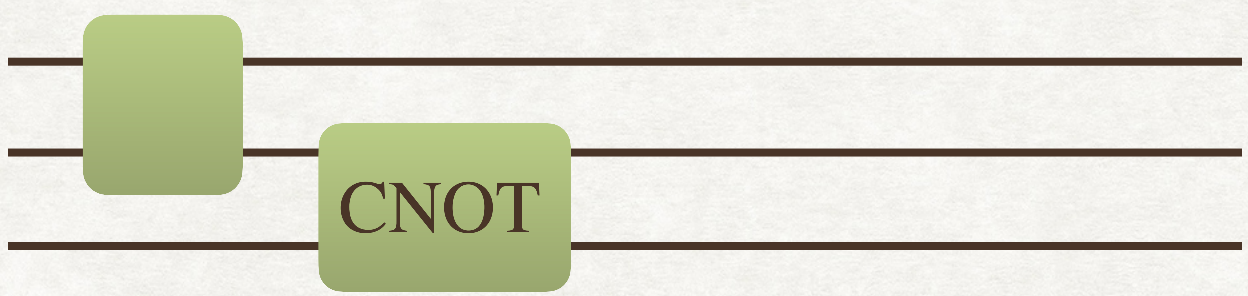
→ Why:

- Later gates can cancel earlier gates → complexity can conceivably decrease

Lower-bounding quantum complexity is difficult.

⟶ Why:

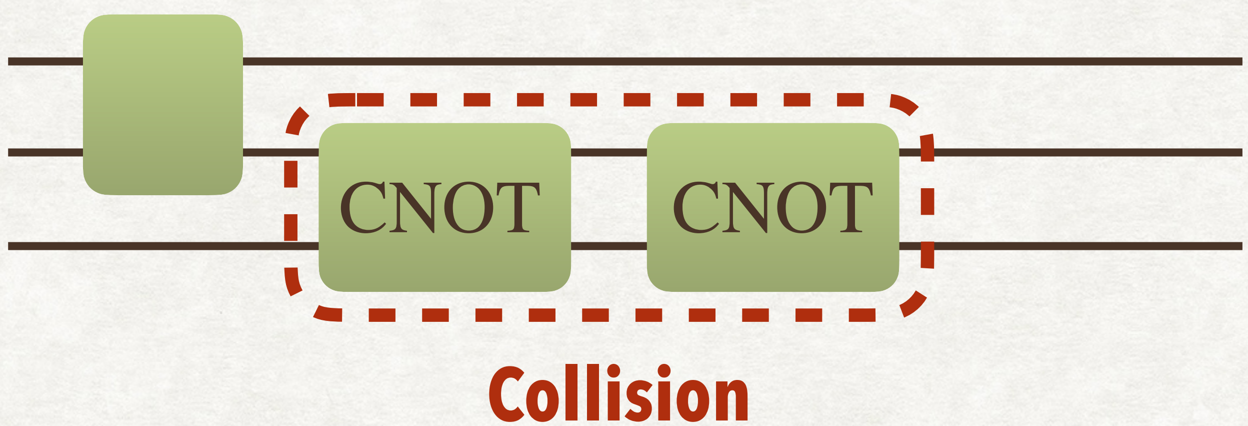
- Later gates can cancel earlier gates ⟶ complexity can conceivably decrease



Lower-bounding quantum complexity is difficult.

⟶ Why:

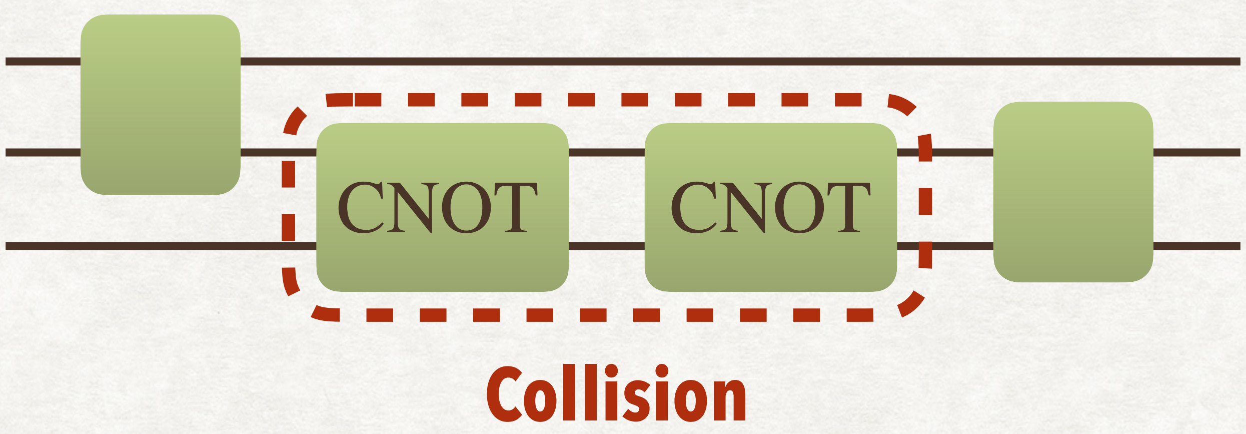
- Later gates can cancel earlier gates ⟶ complexity can conceivably decrease



Lower-bounding quantum complexity is difficult.

⟶ Why:

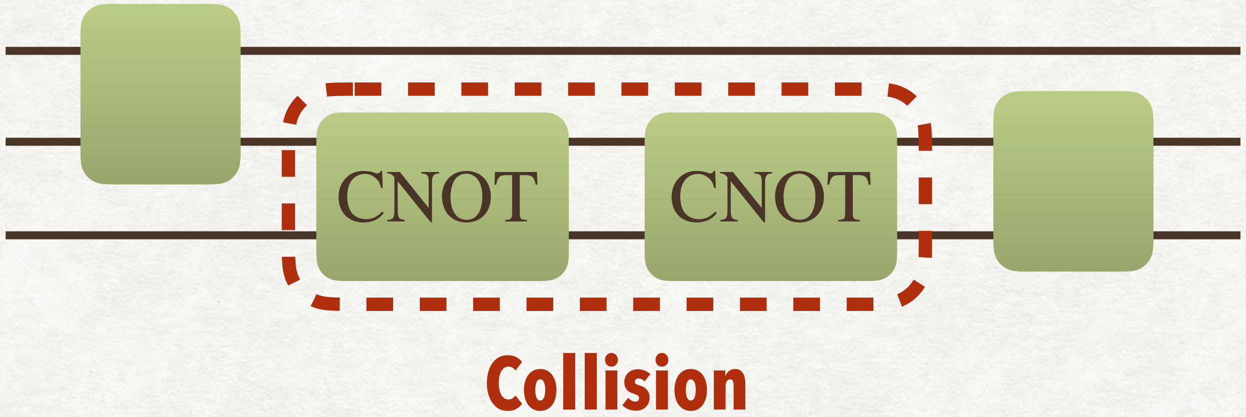
- Later gates can cancel earlier gates ⟶ complexity can conceivably decrease
- <u>Common assumption</u>: Collisions almost never happen.



Lower-bounding quantum complexity is difficult.

⟶ Why:

- Later gates can cancel earlier gates ⟶ complexity can conceivably decrease
- Common assumption: Collisions almost never happen.
  - Difficult to prove



Setting the stage

Setting the stage

- n qubits

Setting the stage

- n qubits

Assume even, for simplicity

Setting the stage

- n qubits

  Assume even, for simplicity

- Circuit of Haar-random 2-qubit unitary gates $\in \mathrm{SU}(4)$

Setting the stage

- n qubits

  Assume even, for simplicity

- Circuit of Haar-random 2-qubit unitary gates $\in \mathrm{SU}(4)$
  - Needn't be geometrically local

Setting the stage

- n qubits

  Assume even, for simplicity

- Circuit of Haar-random 2-qubit unitary gates $\in \mathrm{SU}(4)$
  - Needn't be geometrically local
  - Captures features of chaos

Setting the stage

- n qubits

  Assume even, for simplicity

- Circuit of Haar-random 2-qubit unitary gates $\in \mathrm{SU}(4)$
  - Needn't be geometrically local
  - Captures features of chaos
- $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Setting the stage

- n qubits

  Assume even, for simplicity

- Circuit of Haar-random 2-qubit unitary gates $\in \mathrm{SU}(4)$
  - Needn't be geometrically local
  - Captures features of chaos
- $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$
- $|\psi^k\rangle := |\psi\rangle^{\otimes k}$

Terminology + mindset

Terminology + mindset

- **Architecture** (A): arrangement of a fixed number of gates

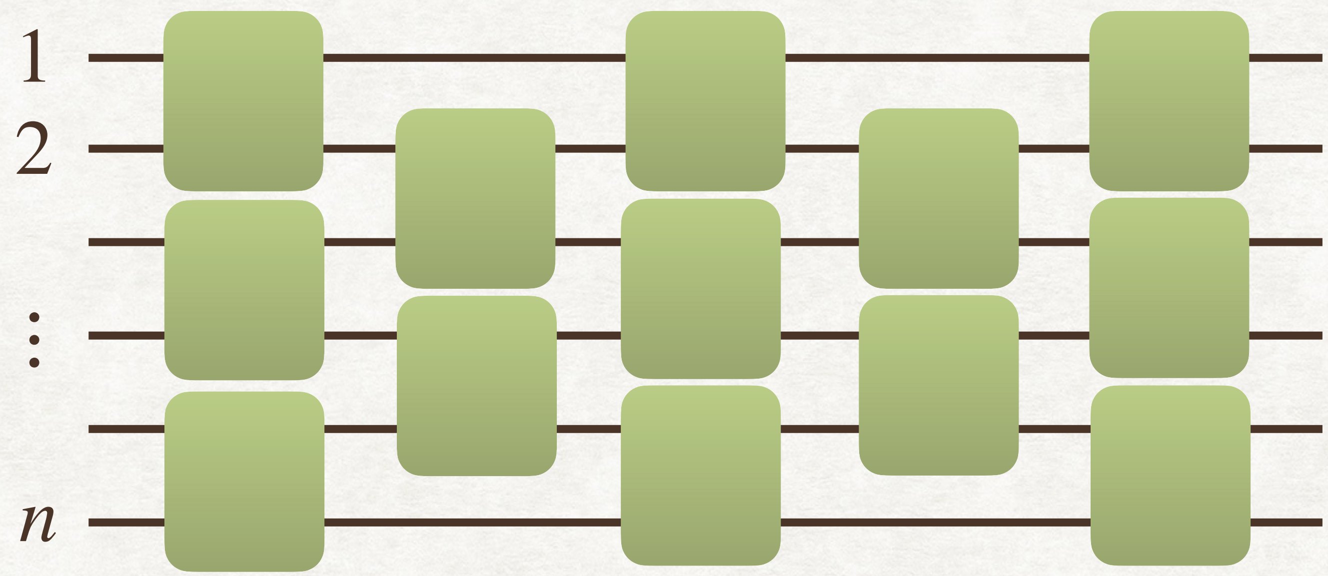
Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

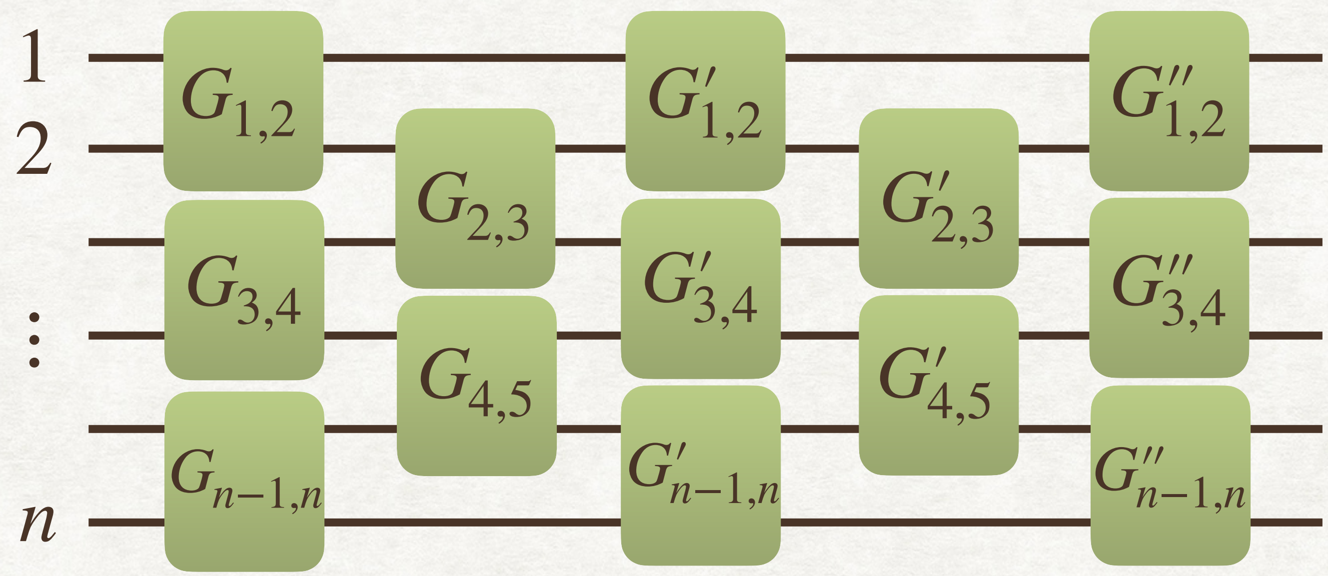
- Example: **brickwork architecture**



Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

- Example: **brickwork architecture**

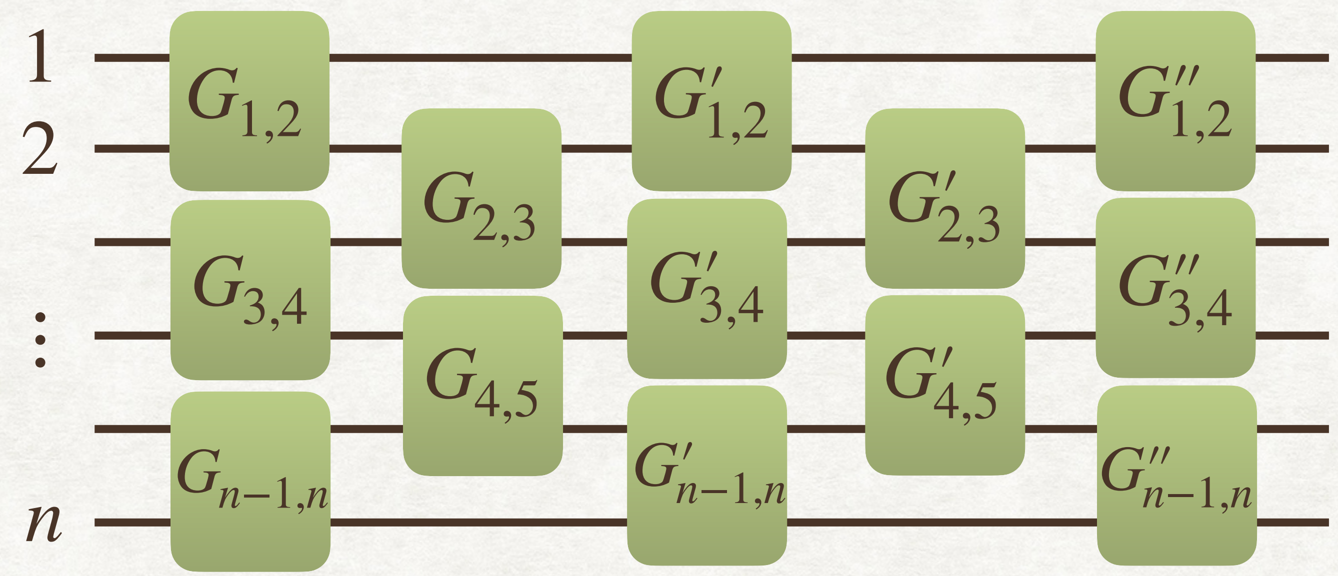


- Slot particular gates into architecture

Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

- Example: **brickwork architecture**

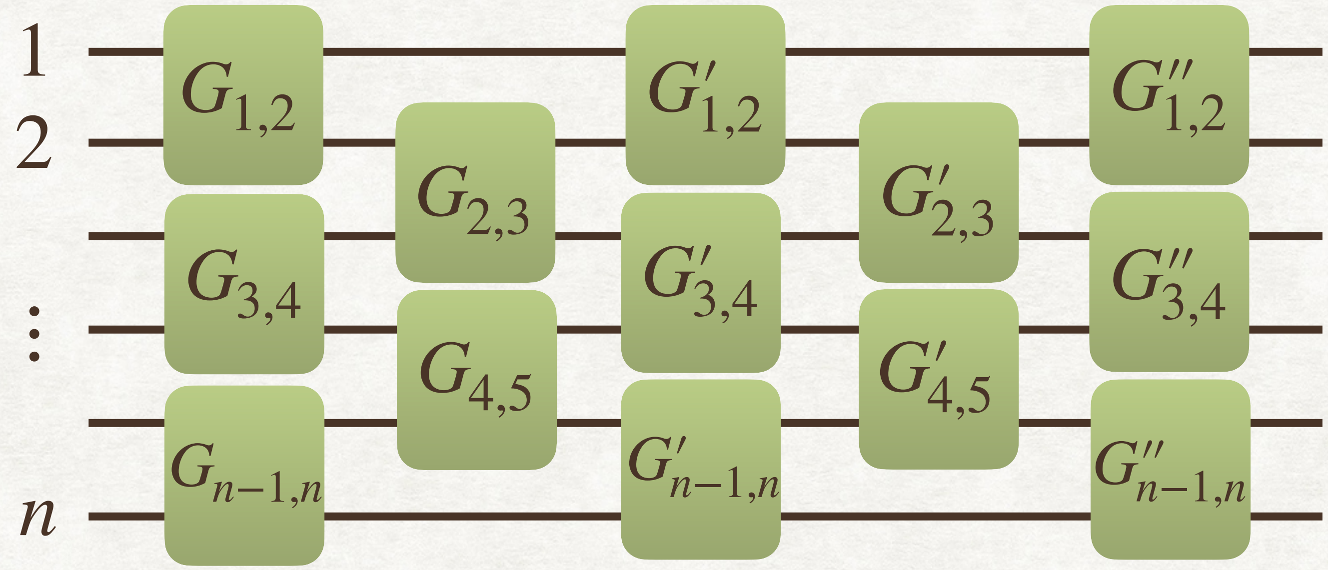


- Slot particular gates into architecture $\longrightarrow$ **circuit**

Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

- Example: **brickwork architecture**

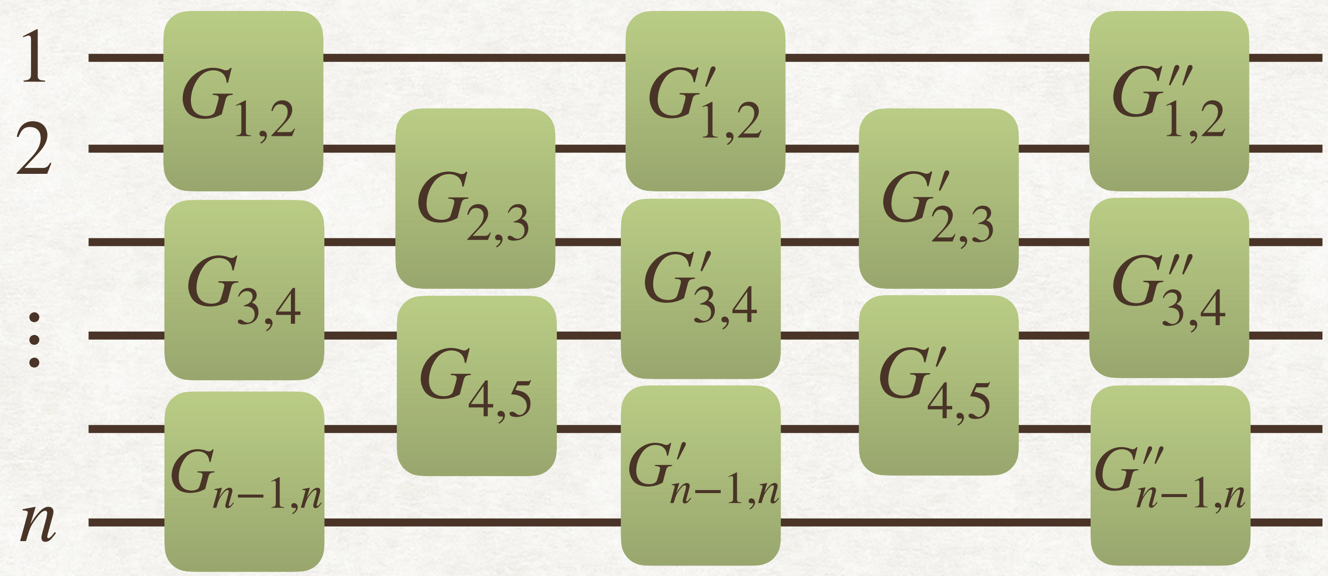


- Slot particular gates into architecture $\longrightarrow$ **circuit**
- Contract the gates in the circuit $\longrightarrow$ **unitary** $U \in \mathrm{SU}(2^n)$

Terminology + mindset

- **Architecture** (A): arrangement of $\underbrace{\text{a fixed number}}_{R}$ of gates

- Example: **brickwork architecture**

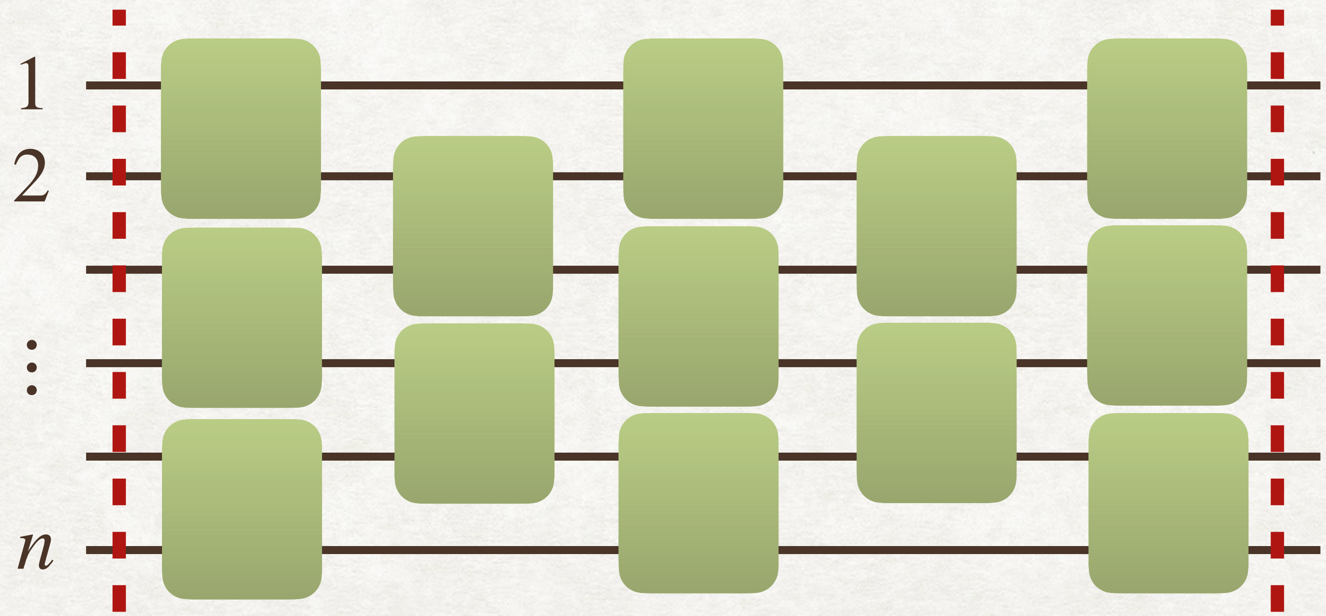


- Slot particular gates into architecture $\longrightarrow$ **circuit**

- $\underbrace{\text{Contract the gates in the circuit}}_{\textbf{Contraction map}\ F^A}$ $\longrightarrow$ **unitary** $U \in \mathrm{SU}(2^n)$

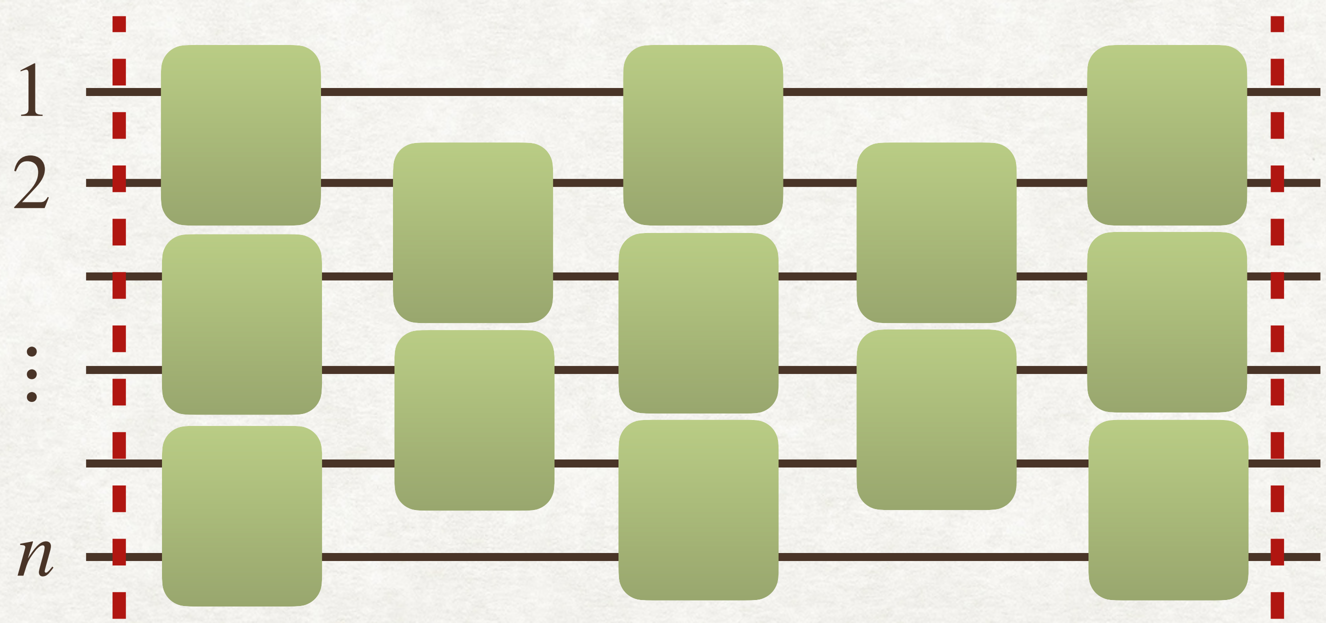
Terminology + mindset

- **Block**: the gates between 2 vertical cuts



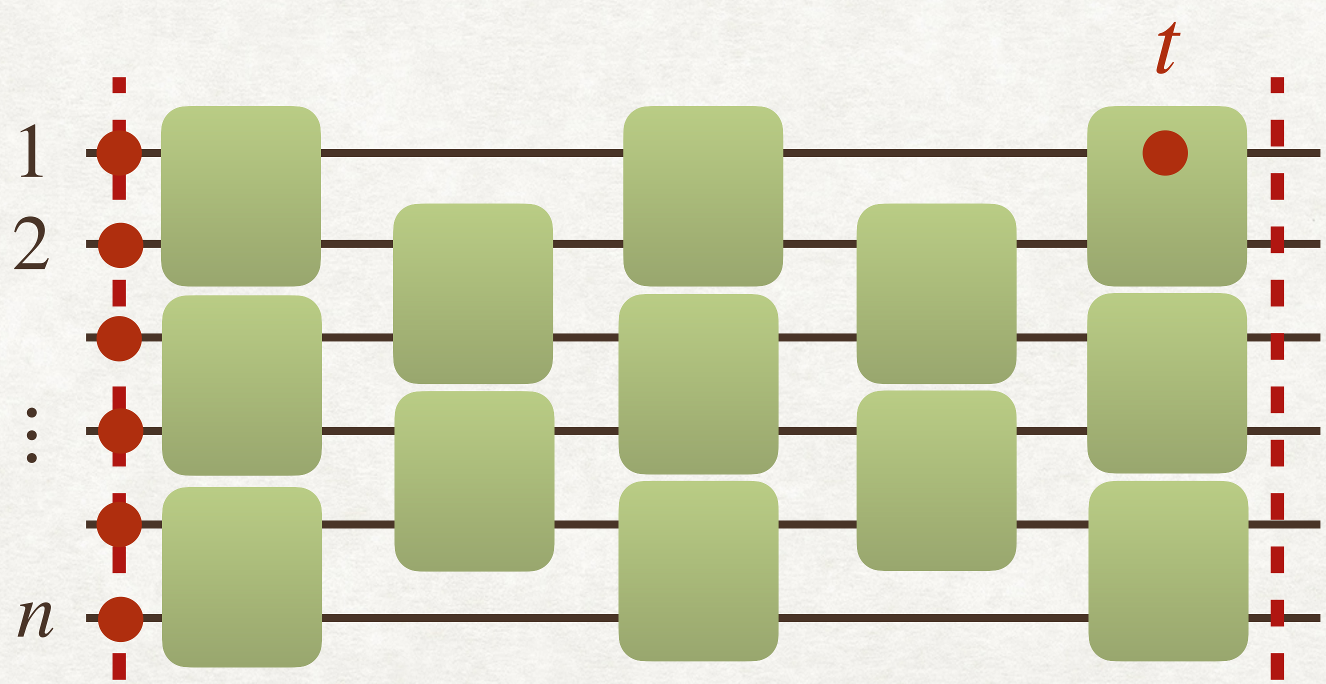
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**



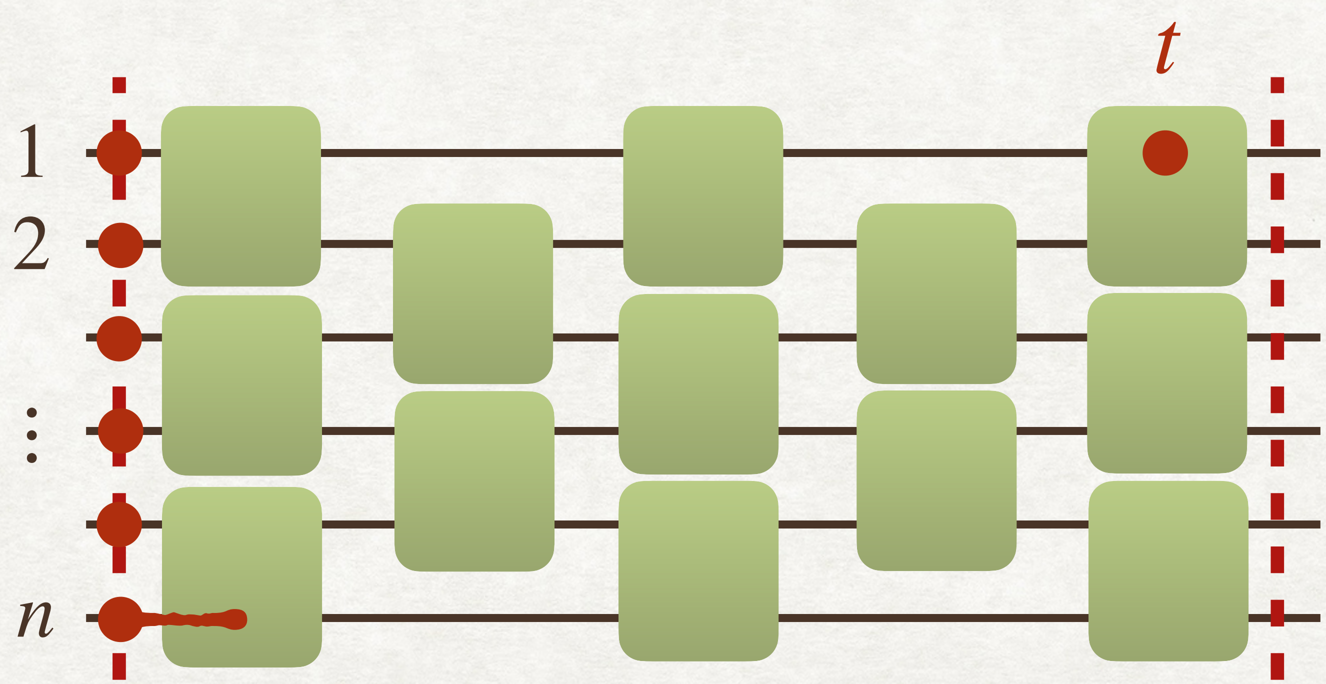
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



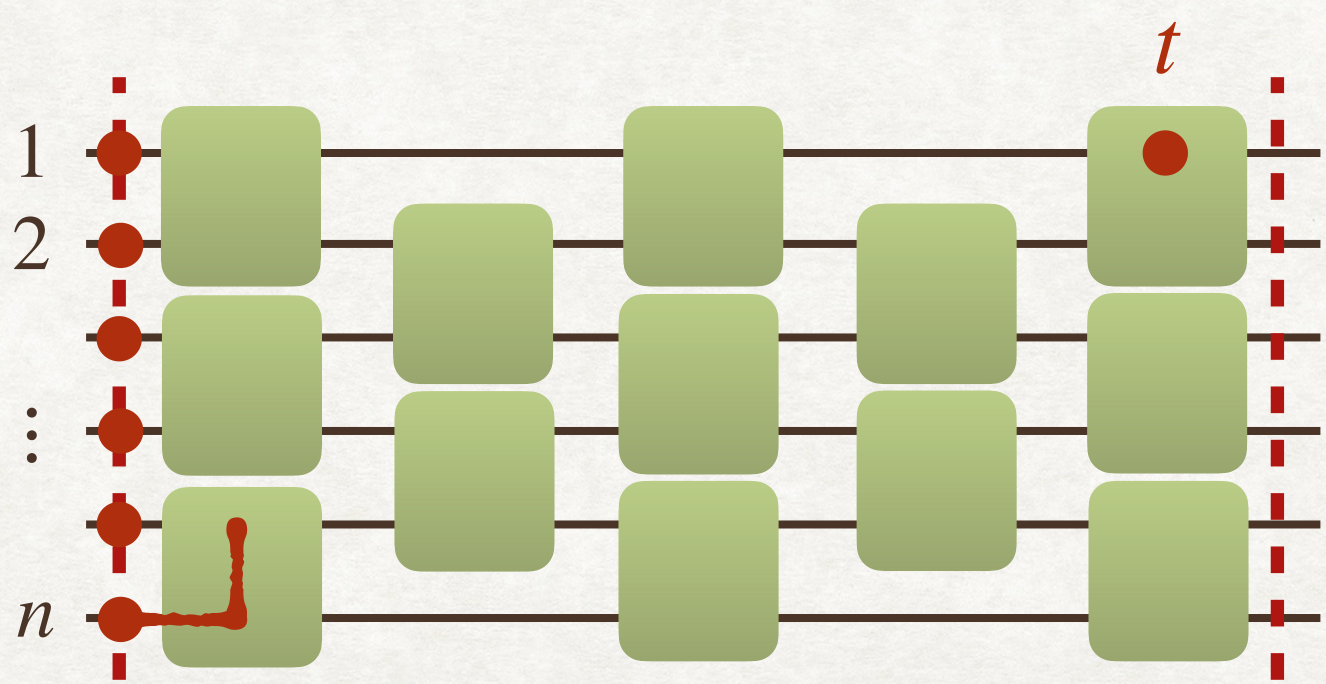
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



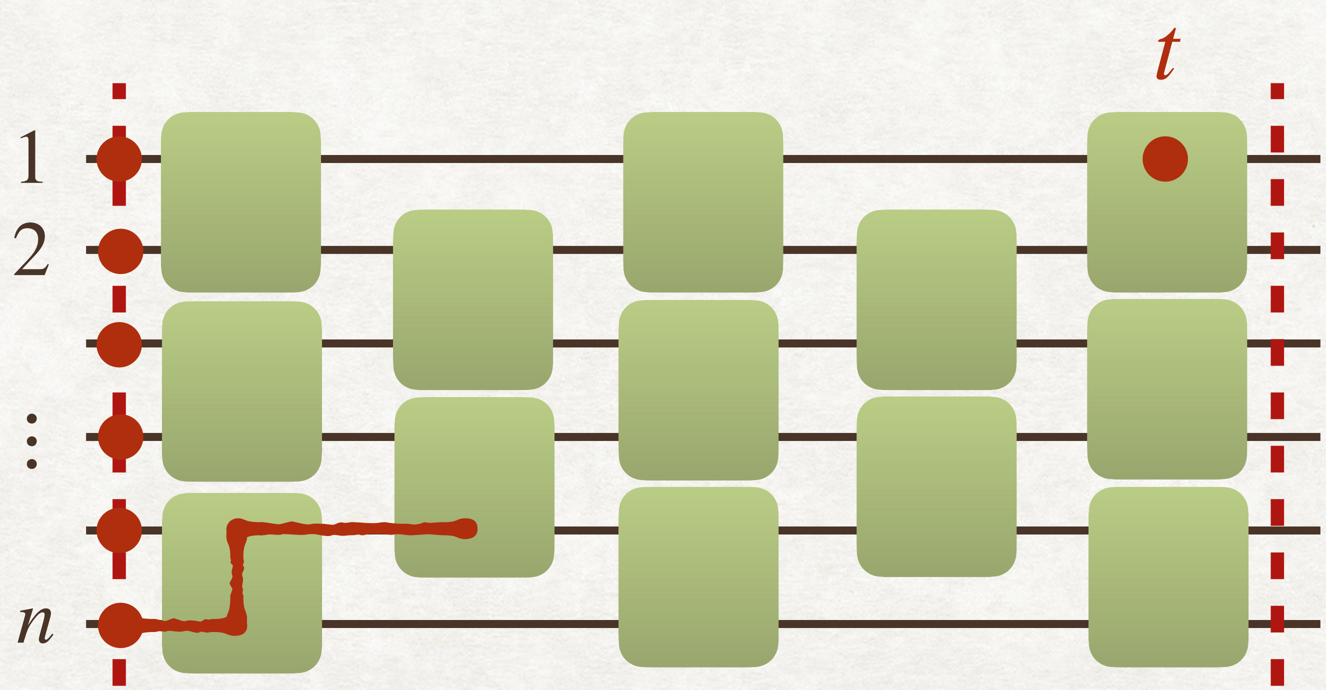
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



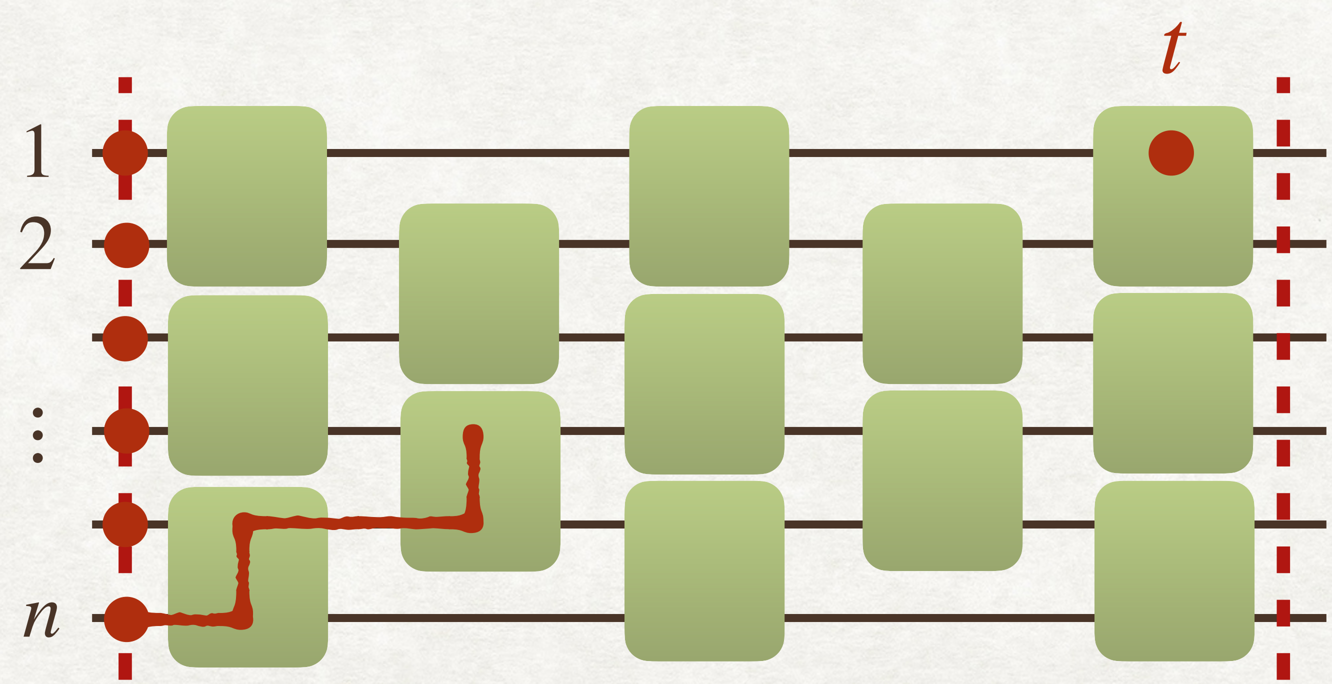
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
    - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



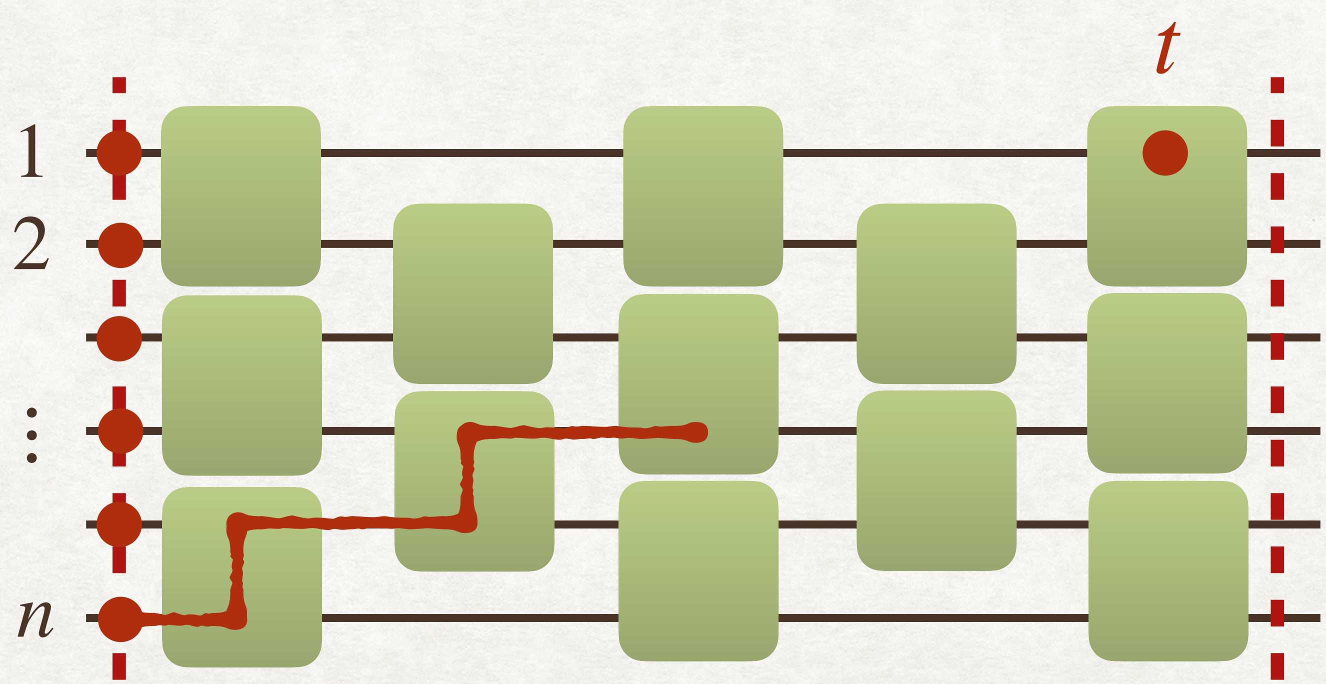
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



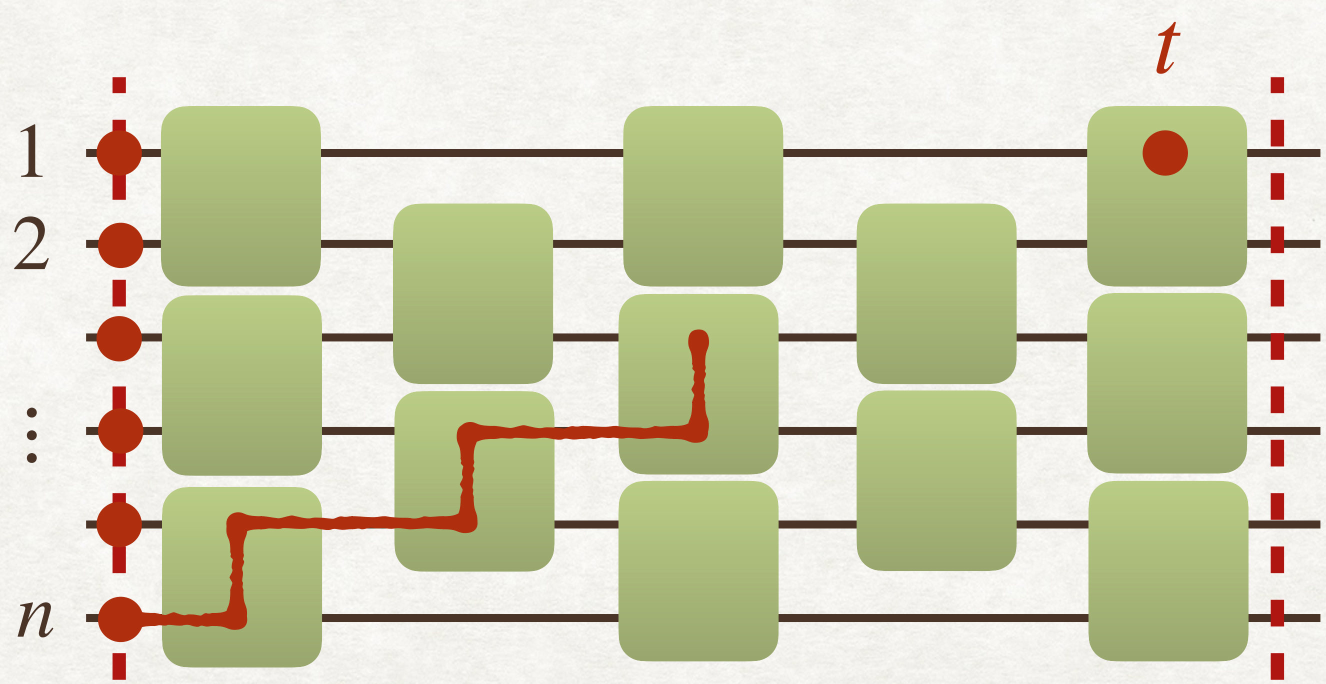
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



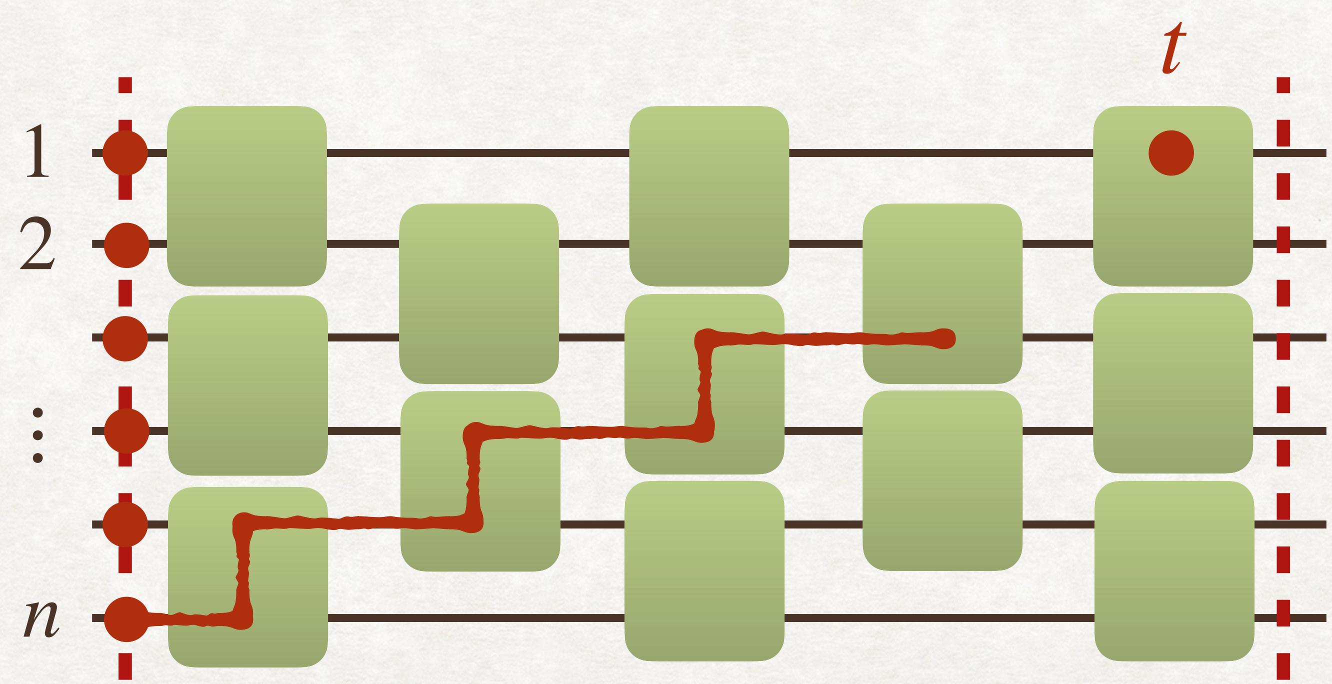
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



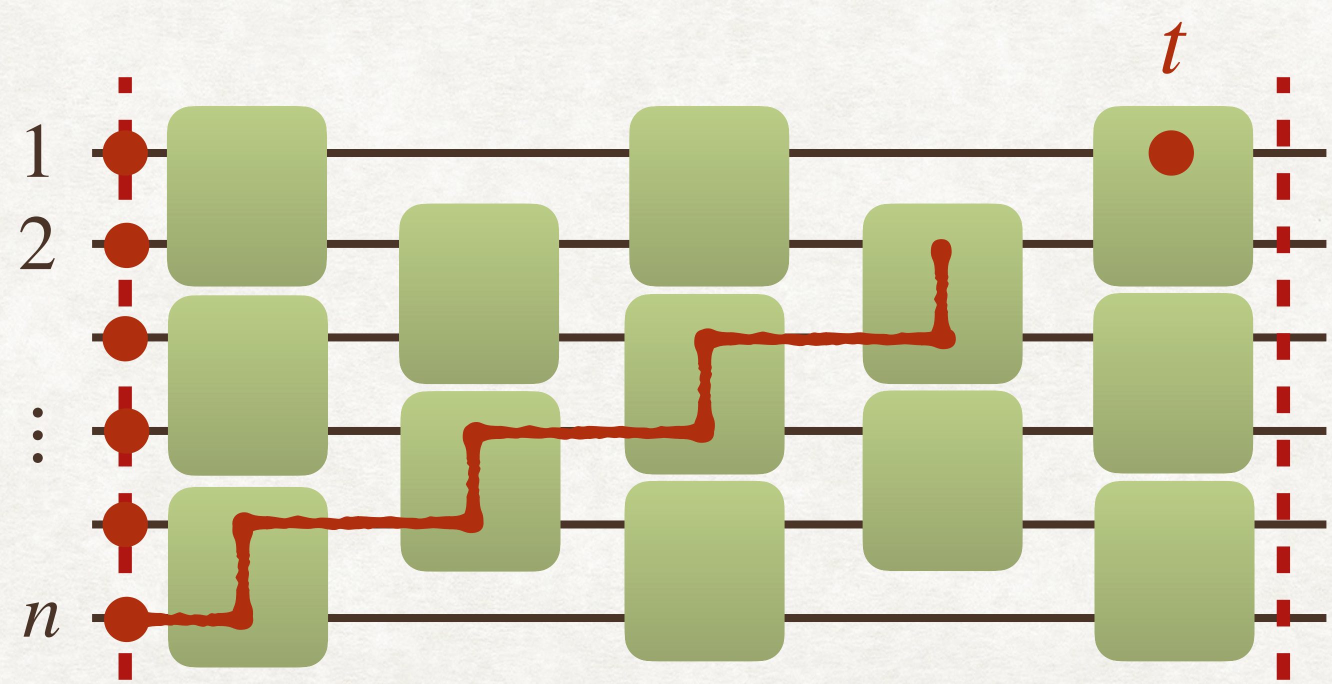
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



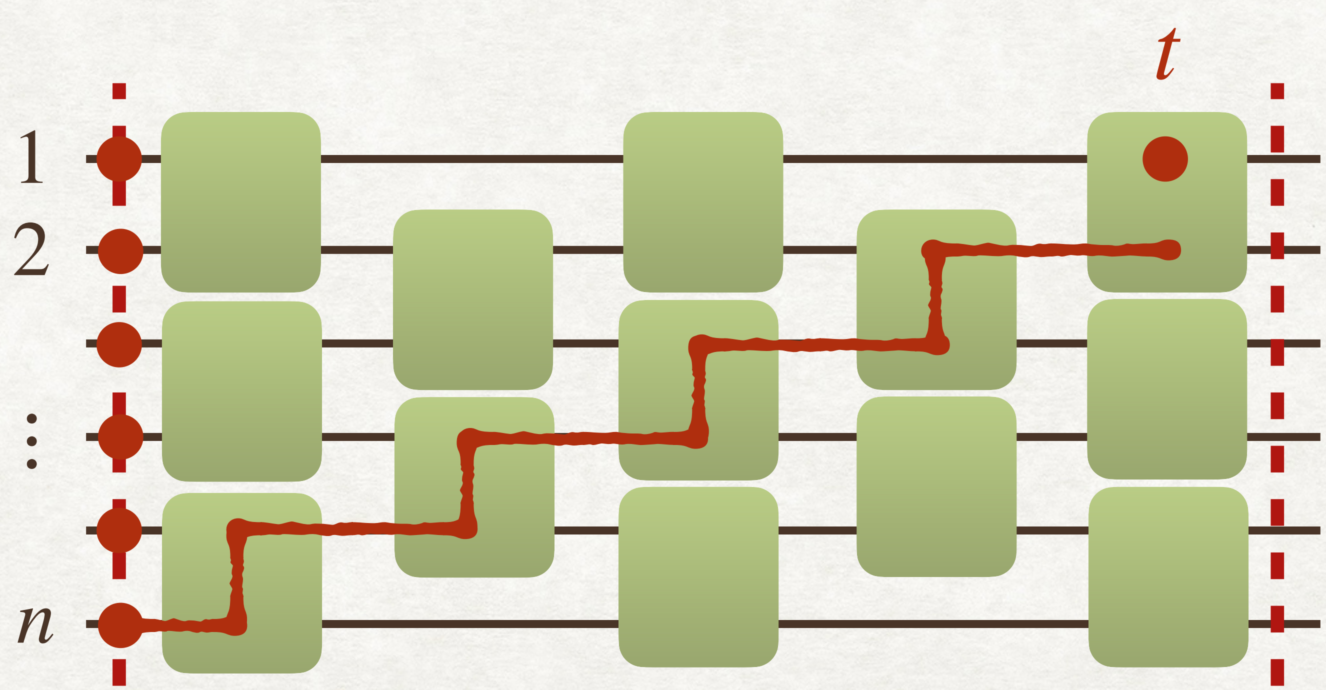
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



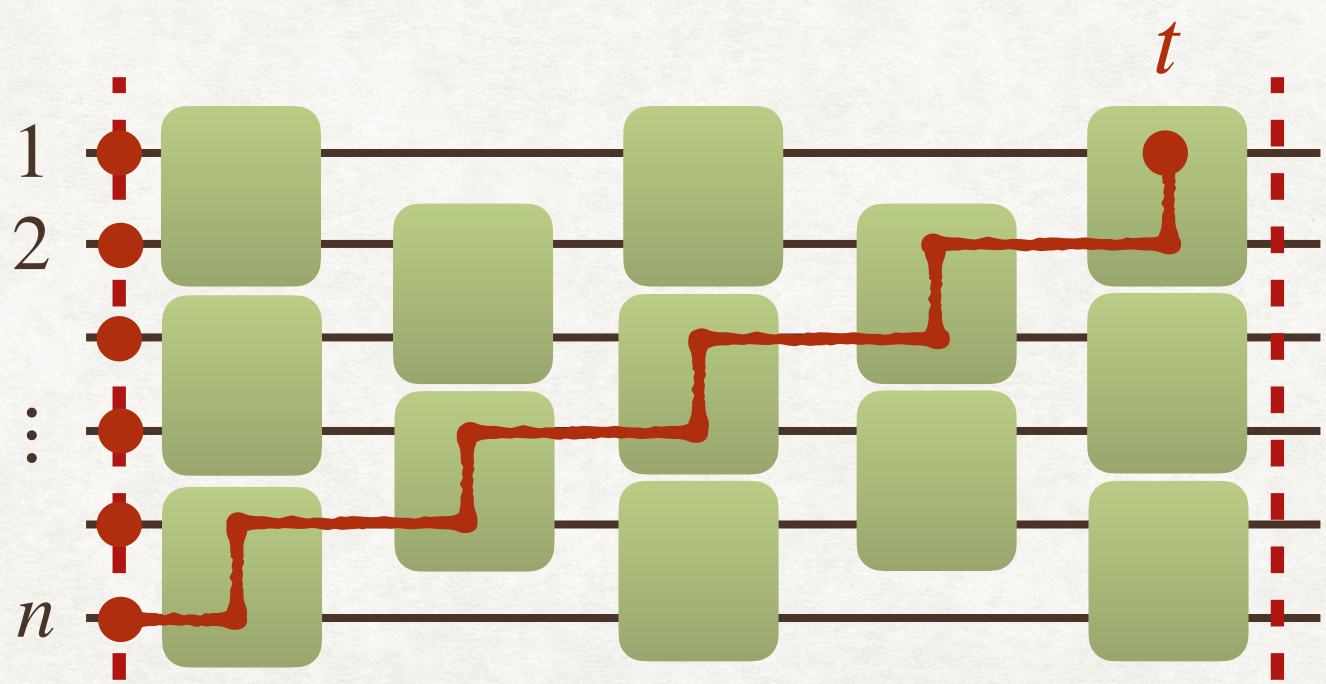
Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



Terminology + mindset

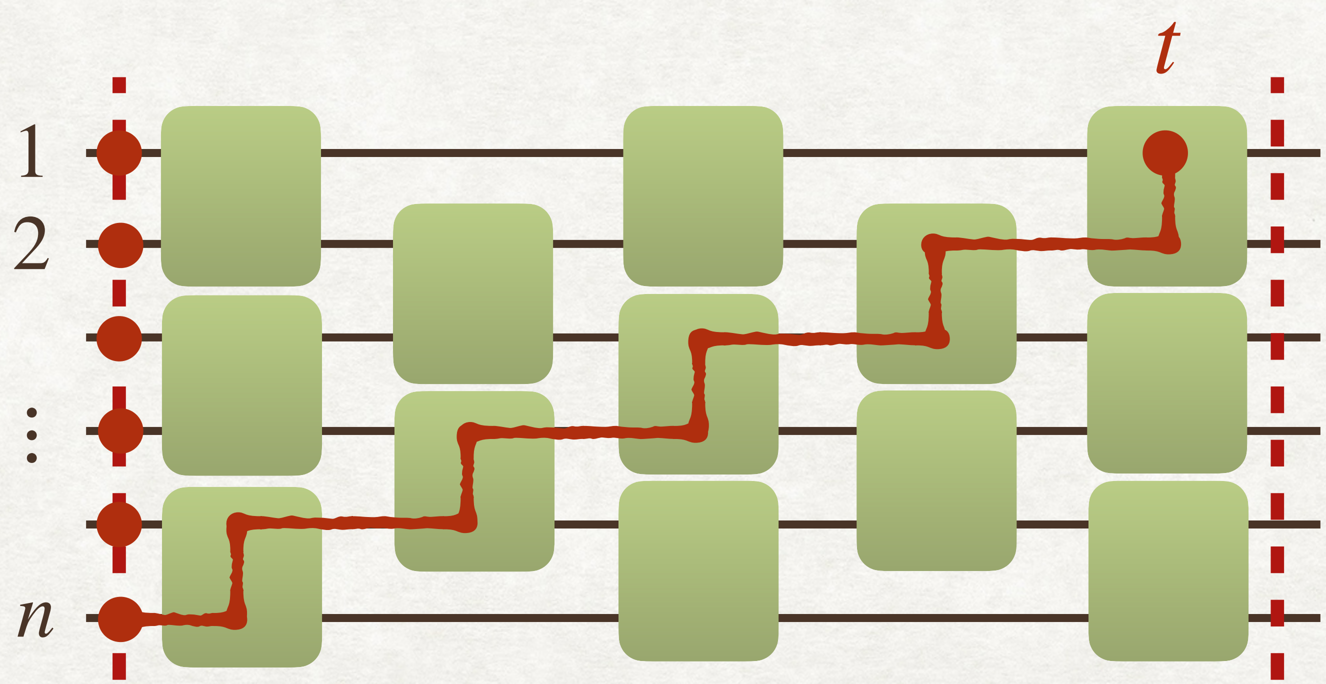
- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.



Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.

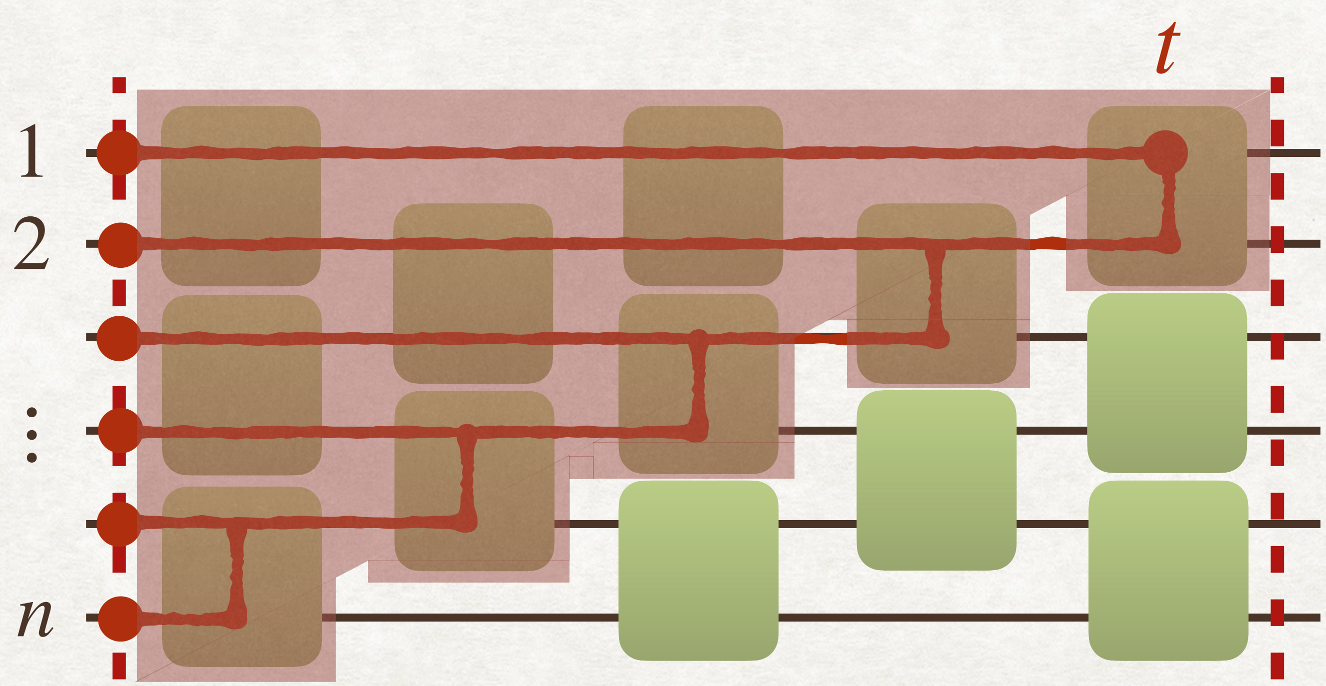
may be unique to t'



Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'.

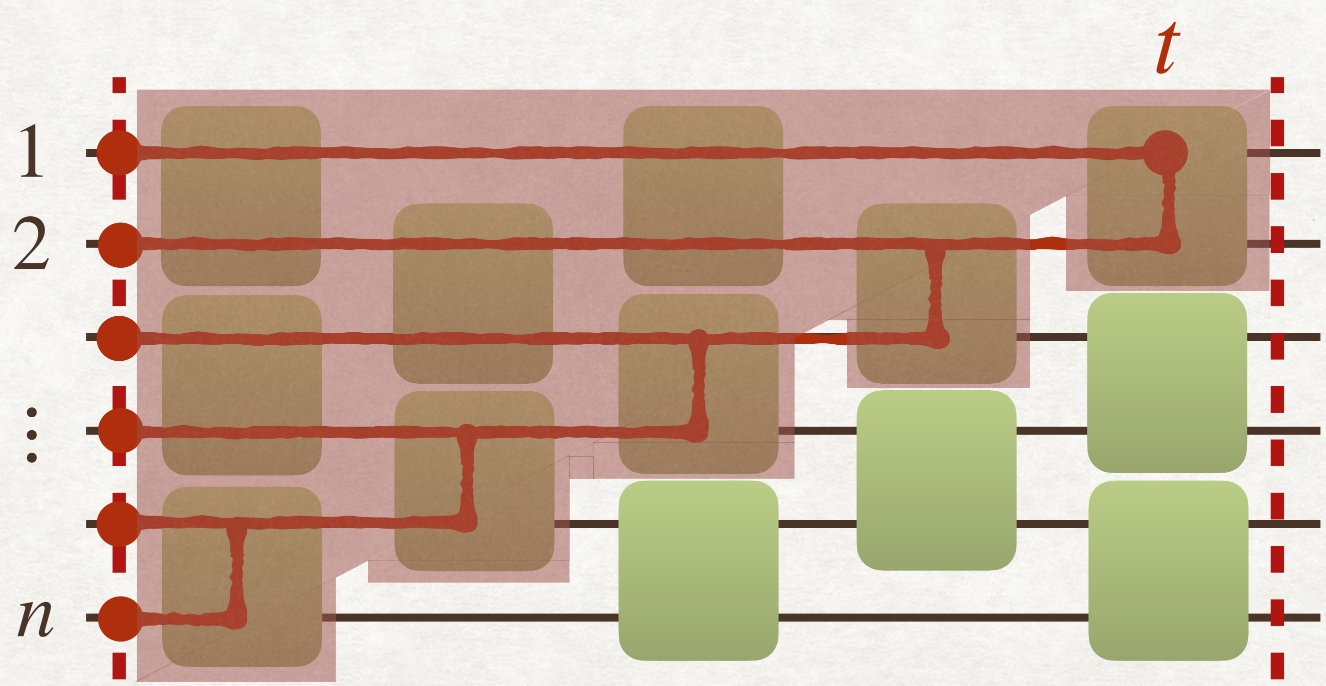
    may be unique to t'
  - Gates in paths form backward light cone



Terminology + mindset

- **Block**: the gates between 2 vertical cuts
- **Backward light cone**
  - Suppose that there exists a qubit t that connects, via a path of gates, to each beginning-of-block qubit t'. (may be unique to t')
  - Gates in paths form backward light cone

  $\longrightarrow$ Block is well-connected



Theorem: Linear growth of complexity

Theorem: Linear growth of complexity

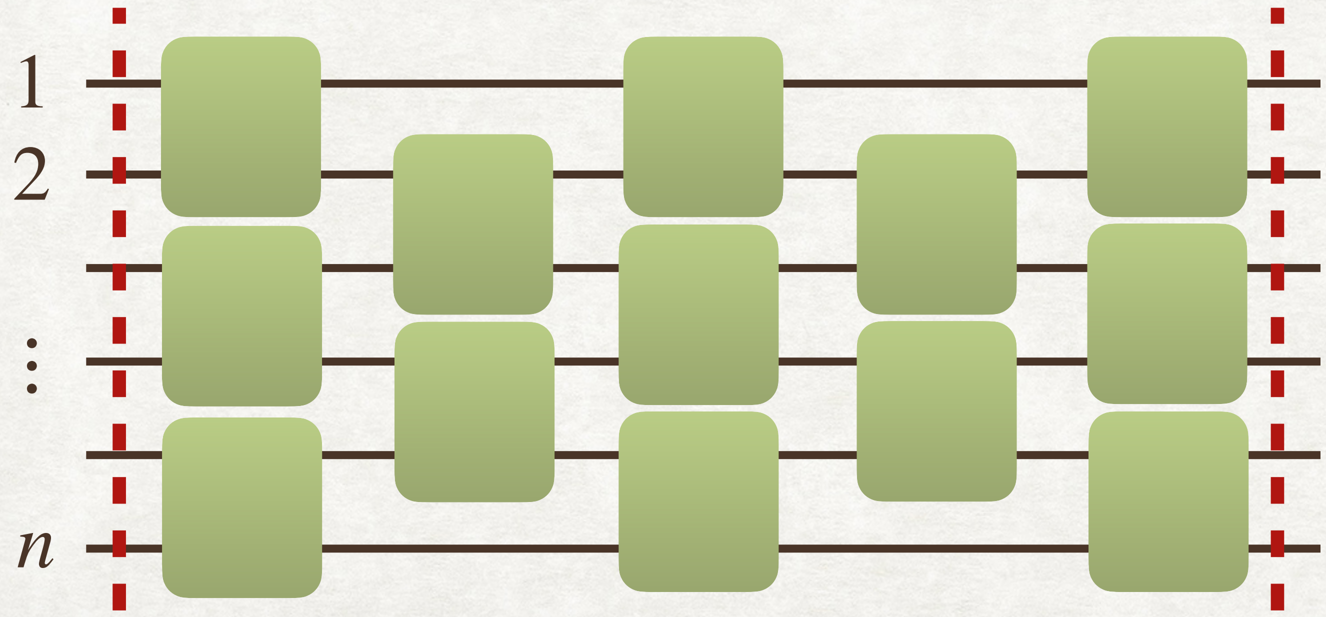
- A = any architecture formed by concatenating
  T blocks of $\leq L$ gates each,

Theorem: Linear growth of complexity

- A = any architecture formed by concatenating
  T blocks of $\leq L$ gates each,
  each block containing a backward light cone.

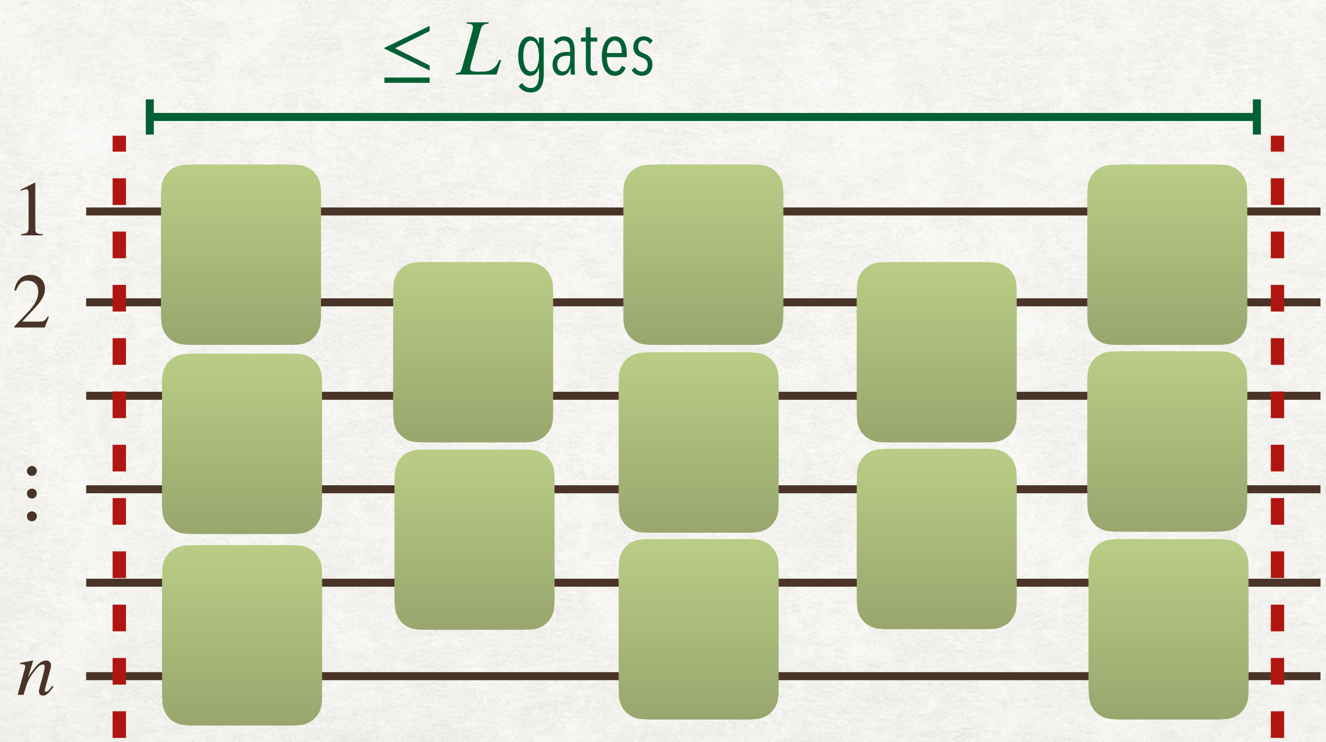
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating
  T blocks of $\leq L$ gates each,
  each block containing a backward light cone.



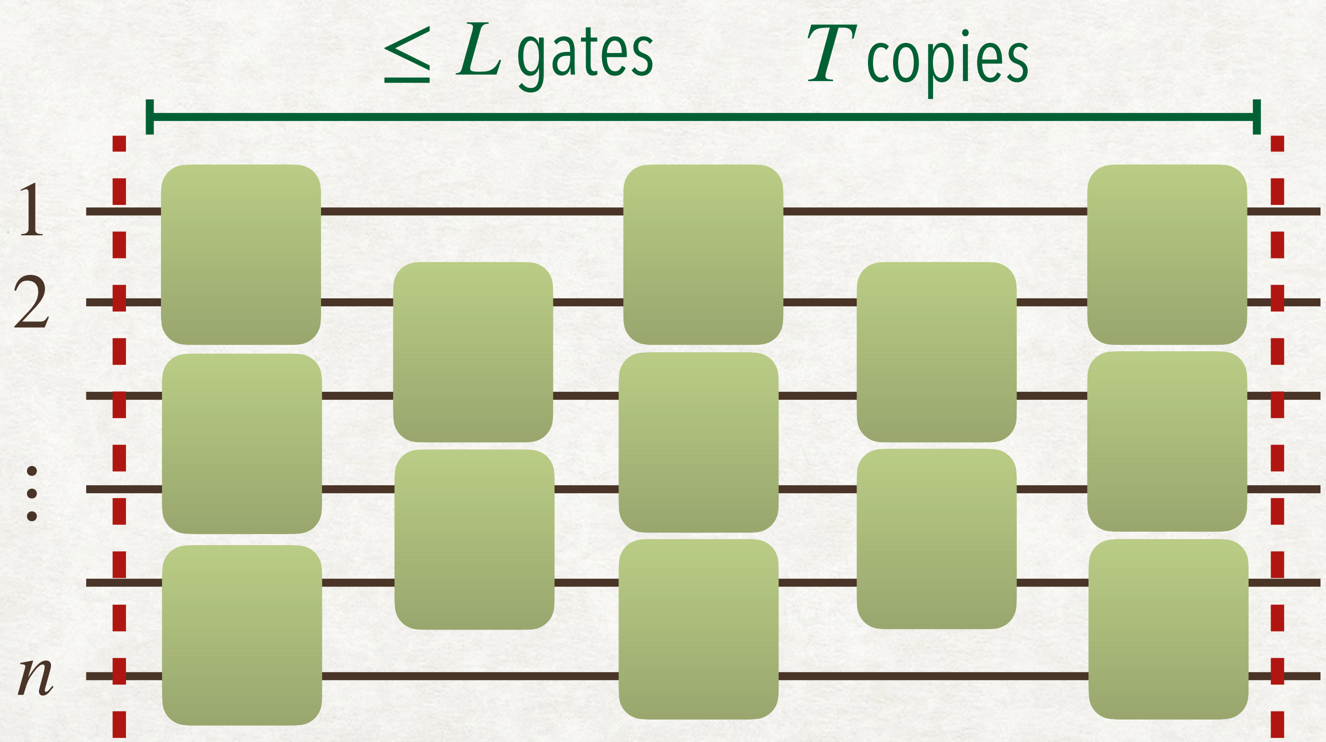
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating
  T blocks of $\leq L$ gates each,
  each block containing a backward light cone.



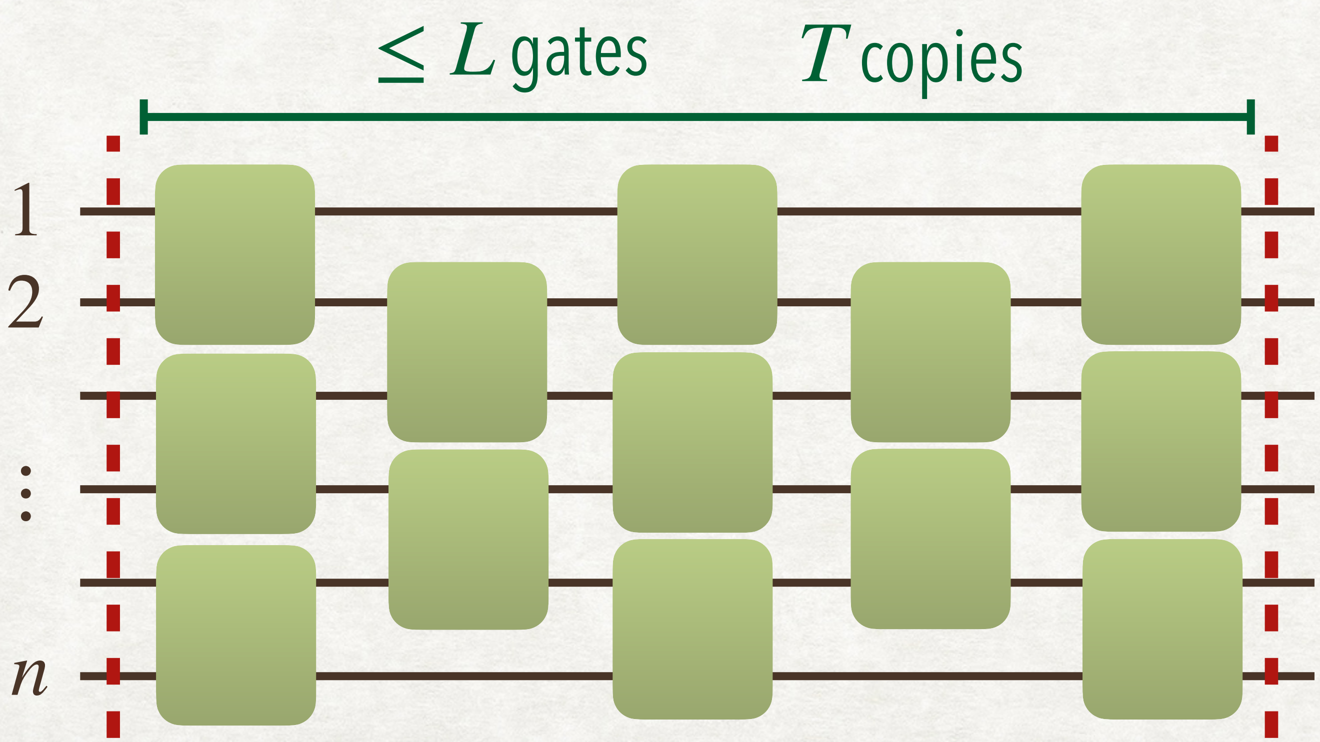
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating
  T blocks of $\leq L$ gates each,
  each block containing a backward light cone.



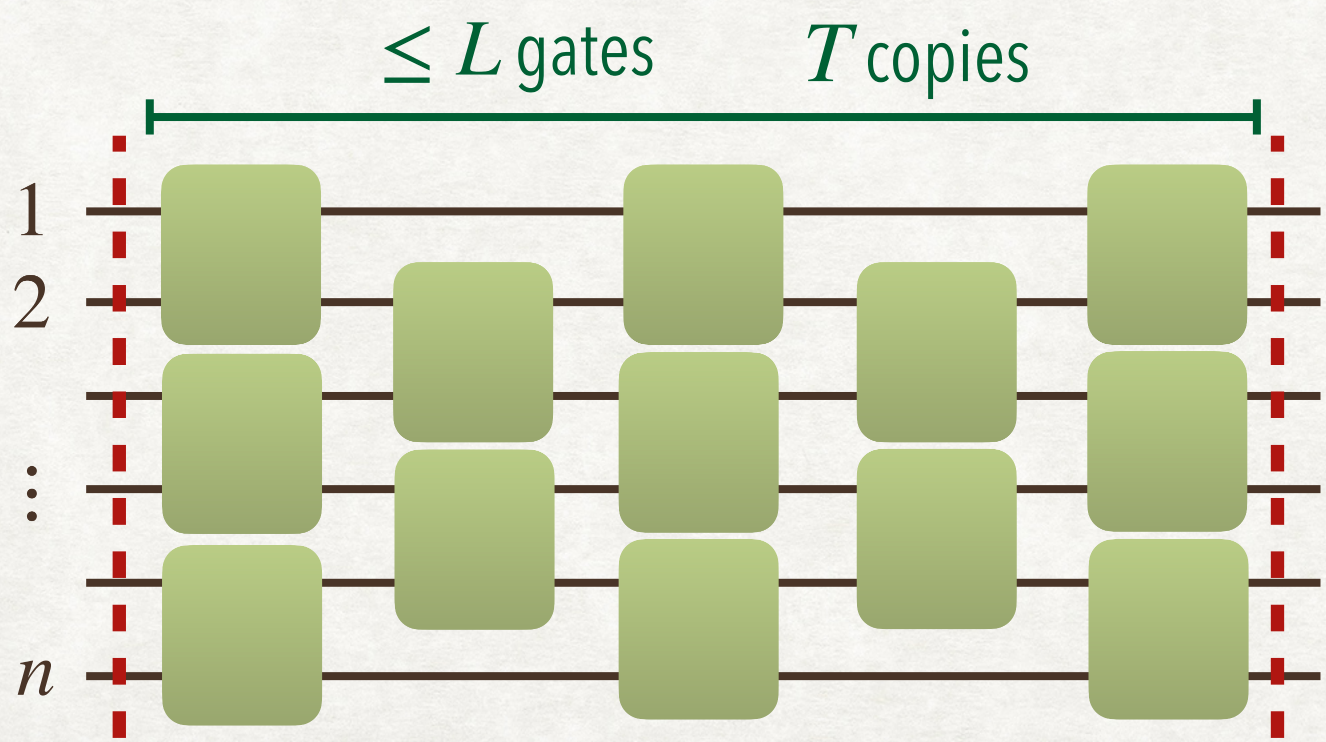
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates



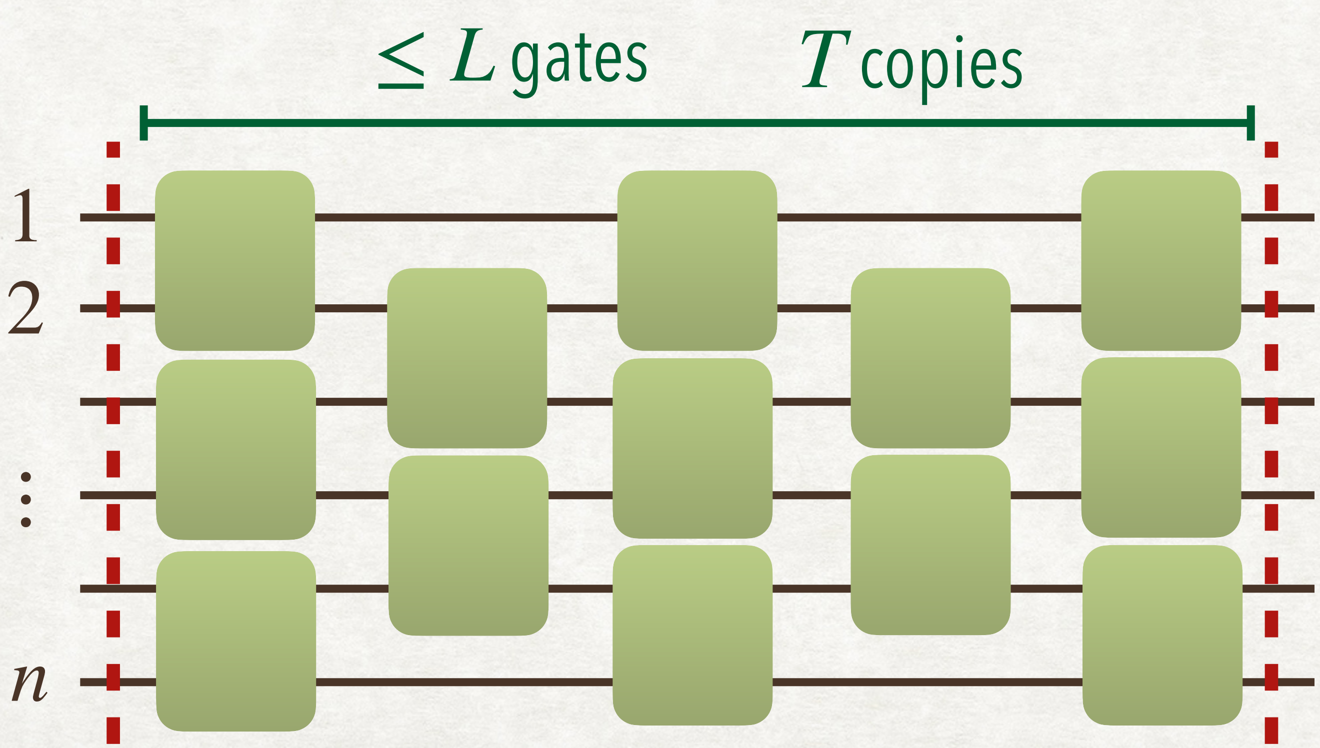
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates
- U = unitary implemented by any random quantum circuit in architecture A



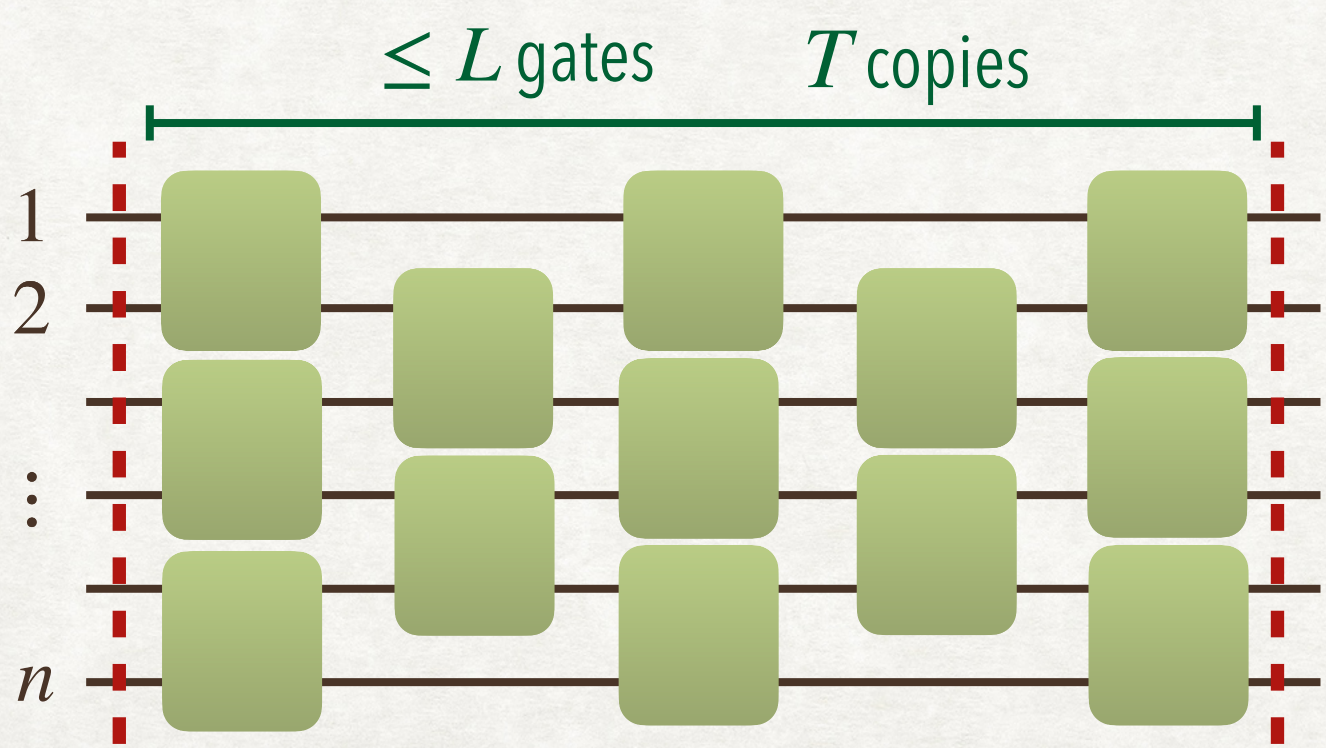
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates
- U = unitary implemented by any random quantum circuit in architecture A
- Unitary's exact complexity:



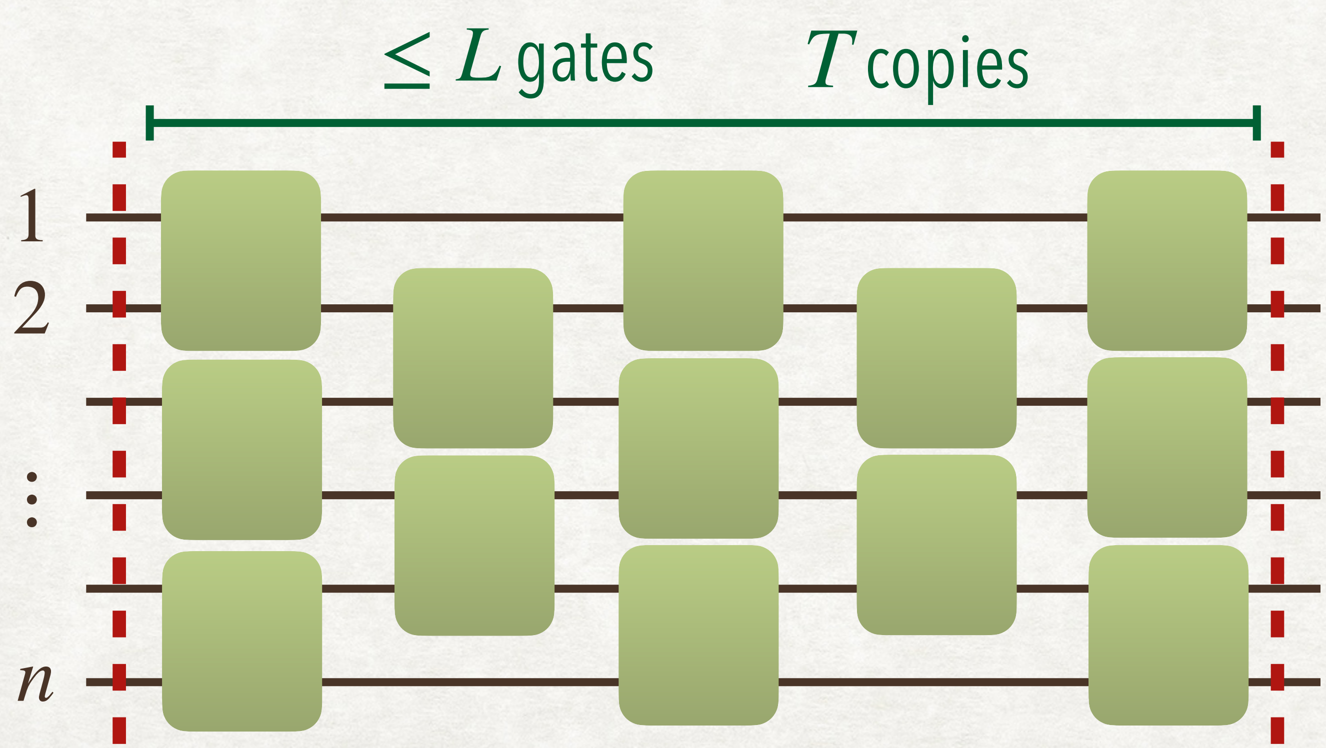
Theorem: Linear growth of complexity

- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates
- U = unitary implemented by any random quantum circuit in architecture A
- Unitary's exact complexity: $\boxed{\mathscr{C}(U) \geq \frac{R}{9L} - \frac{n}{3}, \quad \text{for all } T \leq 4^n - 1.}$



Theorem: Linear growth of complexity

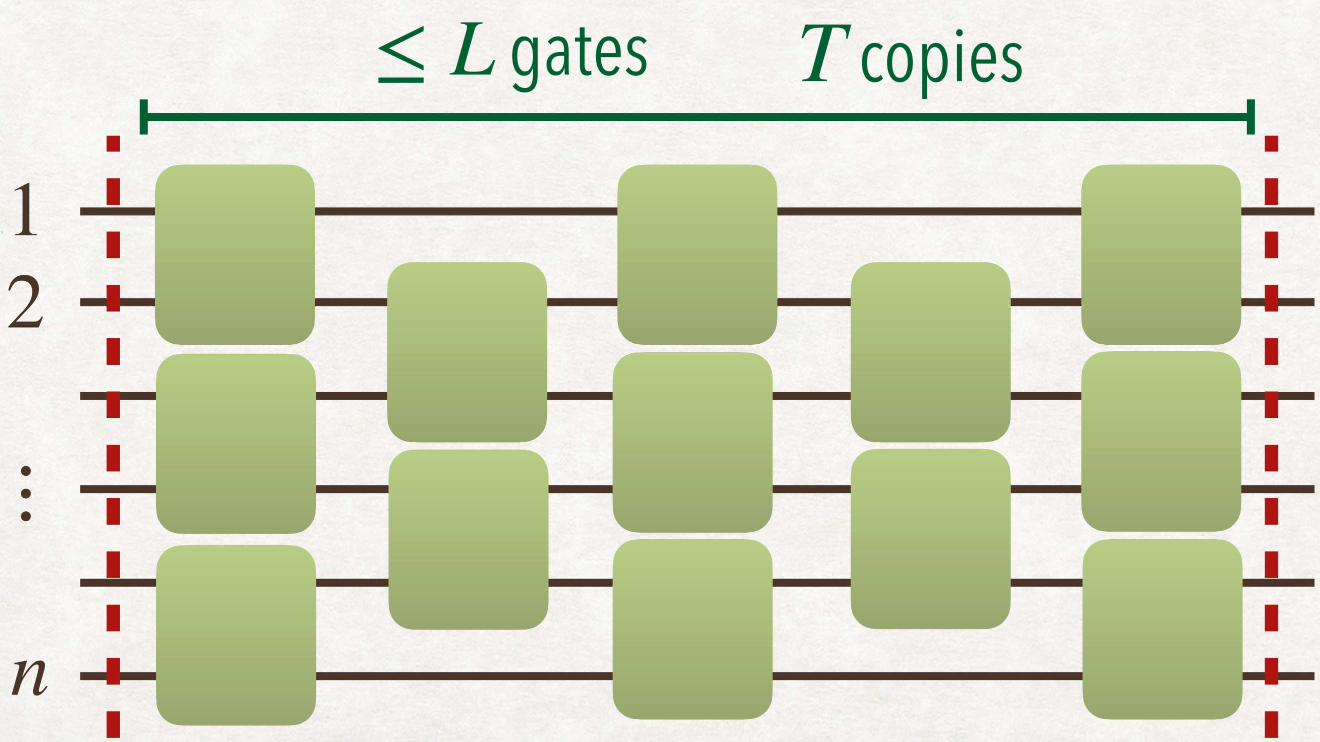
- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates
- U = unitary implemented by any random quantum circuit in architecture A
- Unitary's exact complexity: $\boxed{\mathscr{C}(U) \geq \frac{\boxed{R}}{9L} - \frac{n}{3}, \quad \text{for all } T \leq 4^n - 1.}$



Theorem: Linear growth of complexity

- A = any architecture formed by concatenating T blocks of $\leq L$ gates each, each block containing a backward light cone.
- R = total number of gates
- U = unitary implemented by any random quantum circuit in architecture A
- Unitary's exact complexity: $$\mathscr{C}(U) \geq \frac{\boxed{R}}{9L} - \frac{n}{3}, \quad \text{for all } T \leq 4^n - 1.$$

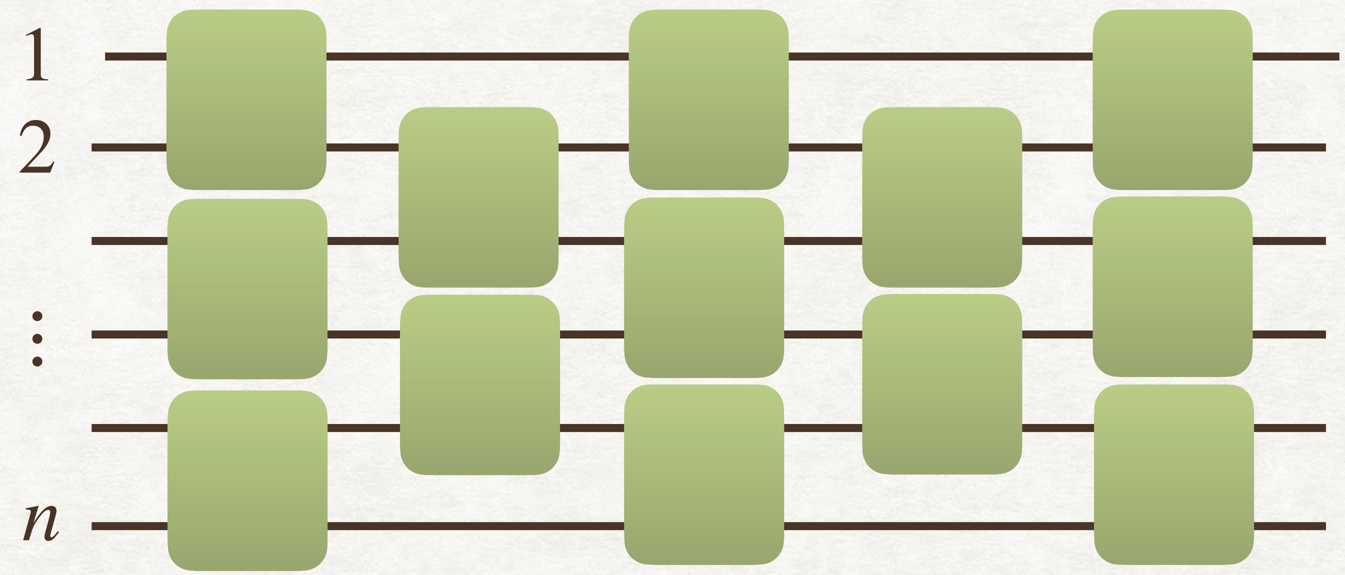
Negligible at exponential times



Key proof idea: architecture’s accessible dimension

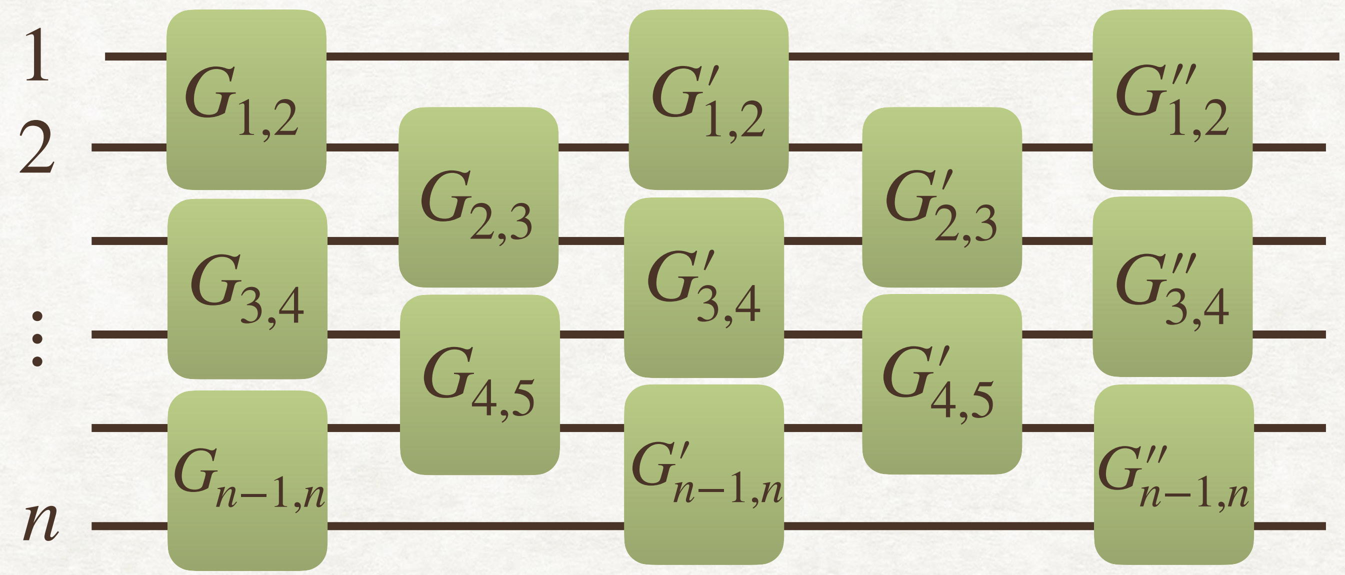
Key proof idea: architecture's accessible dimension

- Architecture A:



Key proof idea: architecture's accessible dimension

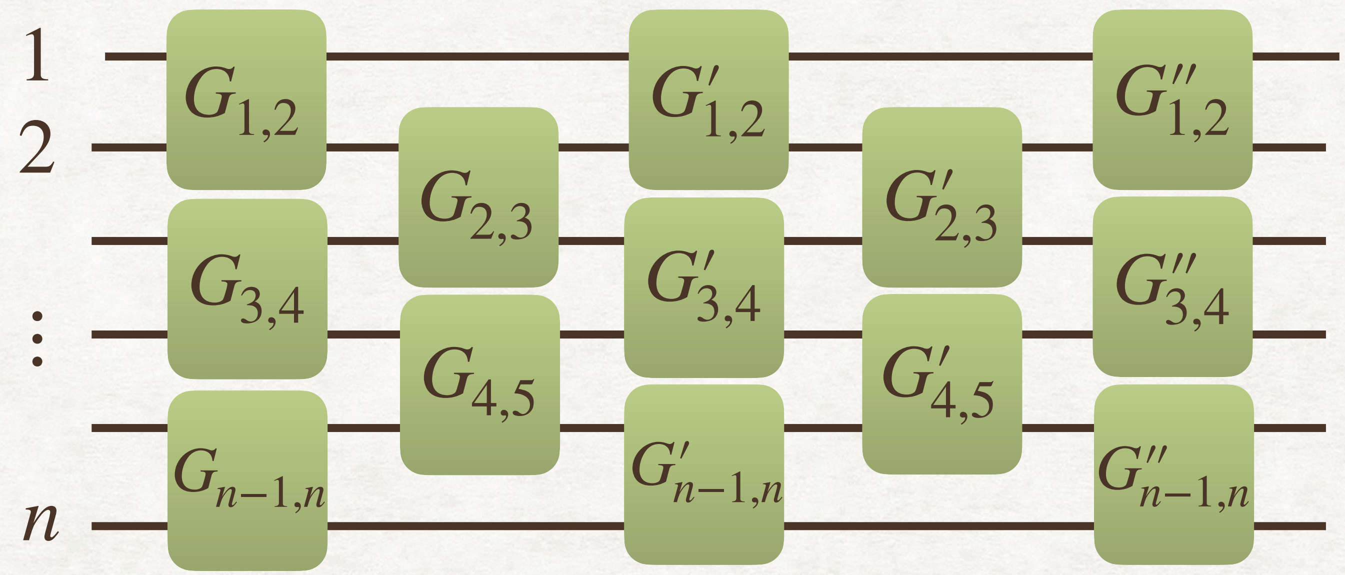
- Architecture A:



  - Slot in gates + contract (apply F^A)

Key proof idea: architecture's accessible dimension

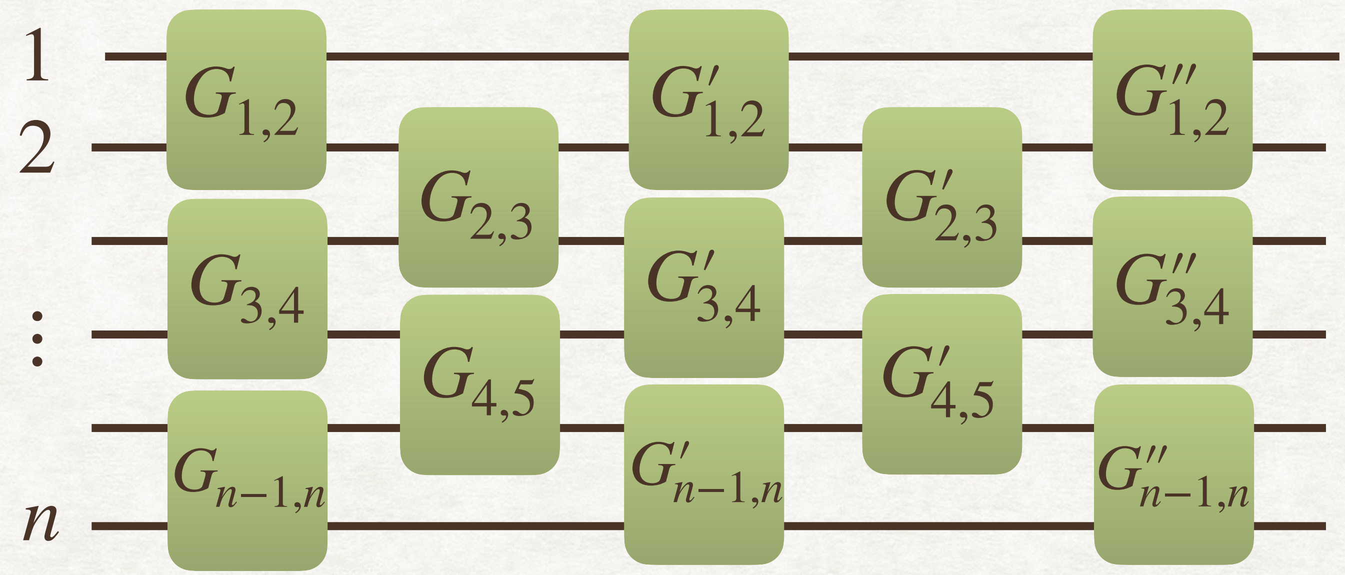
- Architecture A:



- Slot in gates + contract (apply F^A) $\longrightarrow$ unitary

Key proof idea: architecture's accessible dimension

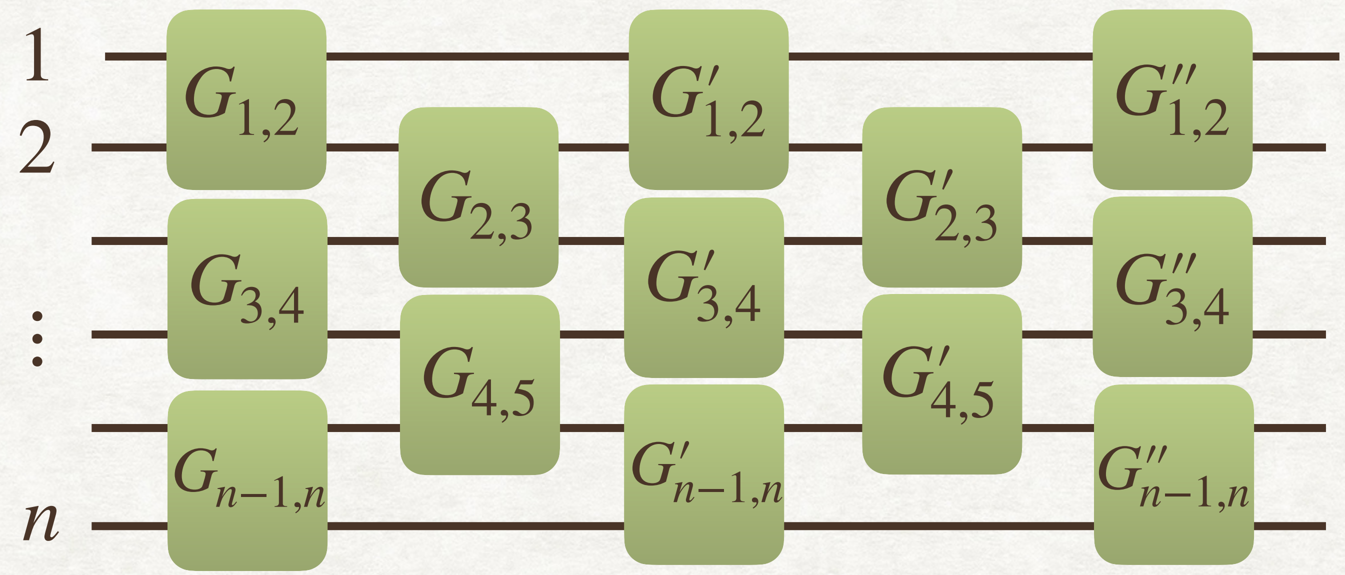
- Architecture A:



- Slot in gates + contract (apply F^A) $\longrightarrow$ unitary
- Set of all such unitaries (image of F^A): $\mathcal{U}(A)$

Key proof idea: architecture's accessible dimension

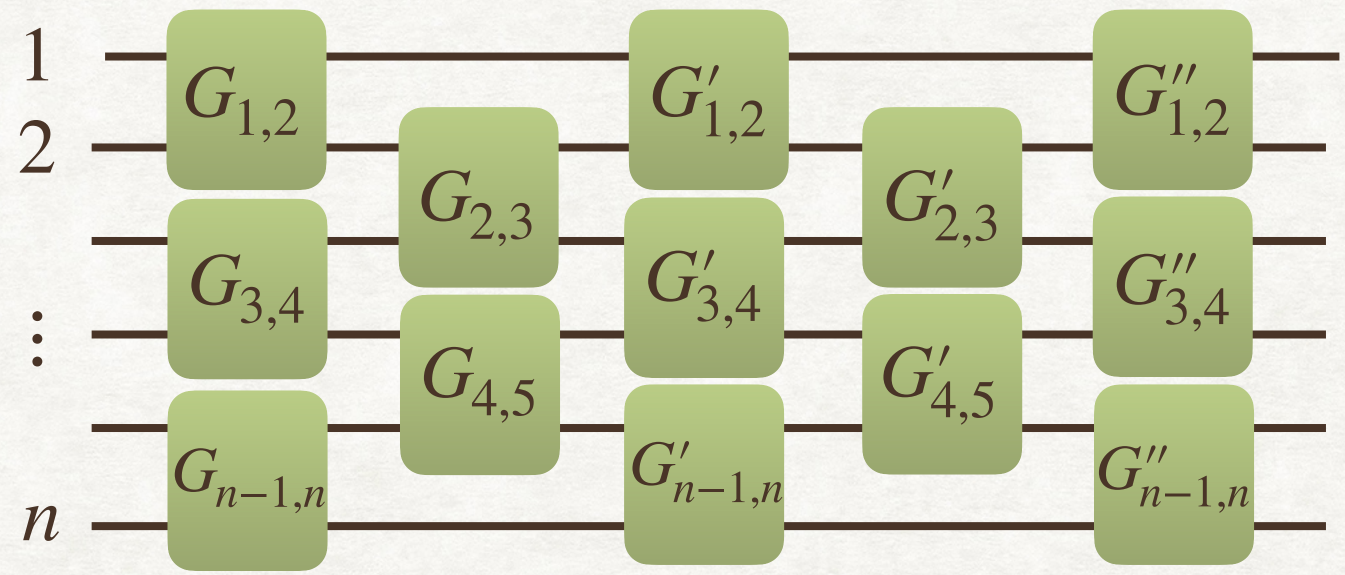
- Architecture A:



- Slot in gates + contract (apply F^A) $\longrightarrow$ unitary
- Set of all such unitaries (image of F^A): $\mathscr{U}(A)$
- **Accessible dimension**, d_A:

Key proof idea: architecture's accessible dimension

- Architecture A:



- Slot in gates + contract (apply F^A) $\longrightarrow$ unitary
- Set of all such unitaries (image of F^A): $\mathcal{U}(A)$
- **Accessible dimension**, d_A: number of degrees of freedom needed to describe $\mathcal{U}(A)$ locally

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
    - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
    - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
    - <u>Contrast</u>: Nielsen's geometry, unitary t-designs
- **Algebraic set**: the set of solutions to   a set of equations

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs
- **Algebraic set**: the set of solutions to a set of equations
  - Example: $\mathrm{SU}(4)^{\times R}$

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs
- **Algebraic set**: the set of solutions to a set of equations
  - Example: $\mathrm{SU}(4)^{\times R}$

  $\longrightarrow$ Set of equations: $\left\{ U^\dagger U = \mathbf{1},\ \det(U) = 1 \right\}$

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs
- **Algebraic set**: the set of solutions to a set of equations
  - Example: $\mathrm{SU}(4)^{\times R}$

    ⟶ Set of equations: $\left\{U^\dagger U = \mathbf{1},\ \det(U) = 1\right\}$
- Generalization: **semialgebraic set**:

Key proof idea: architecture's accessible dimension

- Rigorous definition rooted in **algebraic geometry**
  - Bochnak, Coste, and Roy, *Real algebraic geometry*, volume 36, Springer Science & Business Media (2013).
  - Contrast: Nielsen's geometry, unitary t-designs
- **Algebraic set**: the set of solutions to a set of equations
  - Example: $\mathrm{SU}(4)^{\times R}$

  ⟶ Set of equations: $\left\{ U^\dagger U = \mathbf{1},\ \det(U) = 1 \right\}$
- Generalization: **semialgebraic set**:

  the set of solutions to a set of equations and inequalities

Key proof idea: architecture's accessible dimension

- **Tarski-Seidenberg principle**

Key proof idea: architecture's accessible dimension

- **Tarski-Seidenberg principle** : If W is a semialgebraic set

Key proof idea: architecture's accessible dimension

- **Tarski-Seidenberg principle** : If W is a semialgebraic set and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,

Key proof idea: architecture's accessible dimension

- **Tarski-Seidenberg principle** : If W is a semialgebraic set and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map, then $F(W) =: W'$ is a semialgebraic set.

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map, then $F(W) =: W'$ is a semialgebraic set.

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set
  and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,
  then $F(W) =: W'$ is a semialgebraic set.

contraction map F^A

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set
  and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,
  then $F(W) =: W'$ is a semialgebraic set.

contraction map F^A

- Why F^A is a polynomial function: multiplies matrix elements together

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set

  contraction map F^A and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,

  then $F(W) =: W'$ is a semialgebraic set.

  $\mathcal{U}(A)$

  - Why F^A is a polynomial function: multiplies matrix elements together

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set

contraction map F^A

and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,
then $F(W) =: W'$ is a semialgebraic set.

$\mathcal{U}(A)$

  - Why F^A is a polynomial function: multiplies matrix elements together
- Every semialgebraic set W' decomposes into a union of manifolds: $W' = \bigcup_j M_j$.

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set

contraction map F^A

and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,
then $F(W) =: W'$ is a semialgebraic set.

$\mathscr{U}(A)$

  - Why F^A is a polynomial function: multiplies matrix elements together

- Every semialgebraic set W' decomposes into a union of manifolds: $W' = \bigcup_j M_j$.
- **Dimension of semialgebraic set** : the greatest dimension of any manifold in the decomposition
  - $\dim(W') := \max_j \{\dim(M_j)\}$

Key proof idea: architecture's accessible dimension

$\mathrm{SU}(4)^{\times R}$

- **Tarski-Seidenberg principle** : If W is a semialgebraic set
  and $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map,
  then $F(W) =: W'$ is a semialgebraic set.

contraction map F^A

$\mathcal{U}(A)$

  - Why F^A is a polynomial function: multiplies matrix elements together
- Every semialgebraic set W' decomposes into a union of manifolds: $W' = \bigcup_j M_j$.
- **Dimension of semialgebraic set** : the greatest dimension of any manifold in the decomposition
  - $\dim(W') := \max_j \{\dim(M_j)\}$
- $\dim\big(\mathcal{U}(A)\big) =$ accessible dimension of architecture A

Key proof idea: architecture's accessible dimension

Learn about d_A from
algebraic geometry and differential topology
$\Rightarrow$ infer about complexity

Proof sketch

Proof sketch

(1) Lower bound on accessible dimension

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof
  elements:

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - Algebraic geometry, differential topology

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - Algebraic geometry, differential topology
  - Construction of Clifford circuit

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - ✦ Algebraic geometry, differential topology
  - ✦ Construction of Clifford circuit

Transform the Pauli operators to the Pauli operators (to within phases)

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - Algebraic geometry, differential topology
  - Construction of Clifford circuit

    Transform the Pauli operators to the Pauli operators (to within phases)
  - Arbitrary nontrivial n-qubit Pauli string P

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - ✦ Algebraic geometry, differential topology
  - ✦ Construction of Clifford circuit

    Transform the Pauli operators to the Pauli operators (to within phases)
  - ✦ Arbitrary nontrivial n-qubit Pauli string $P \mapsto Z_n$ via Clifford circuit
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - ✦ Algebraic geometry, differential topology
  - ✦ Construction of Clifford circuit

    Transform the Pauli operators to the Pauli operators (to within phases)

  - ✦ Arbitrary nontrivial n-qubit Pauli string $P \mapsto Z_n$ via Clifford circuit
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).
  - ✦ The n-qubit Pauli strings form a basis for the space of n-qubit Hermitian operators.

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - ✦ Algebraic geometry, differential topology
  - ✦ Construction of Clifford circuit

    Transform the Pauli operators to the Pauli operators (to within phases)
  - ✦ Arbitrary nontrivial n-qubit Pauli string $P \mapsto Z_n$ via Clifford circuit
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).
  - ✦ The n-qubit Pauli strings form a basis for the space of n-qubit Hermitian operators.
  - ✦ Number of nontrivial n-qubit Pauli strings: $4^n - 1$

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

- Key proof elements:
  - Algebraic geometry, differential topology
  - Construction of Clifford circuit

    Transform the Pauli operators to the Pauli operators (to within phases)
  - Arbitrary nontrivial n-qubit Pauli string $P \mapsto Z_n$ via Clifford circuit
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016). Webb, arXiv:1510.02769 (2015). Zhu, Phys. Rev. A **96**, 062336 (2017).
  - The n-qubit Pauli strings form a basis for the space of n-qubit Hermitian operators.
  - Number of nontrivial n-qubit Pauli strings: $4^n - 1$ ⟶ Our bound holds for $T \leq 4^n - 1$.

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

(2) Upper bound on accessible dimension

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

(2) Upper bound on accessible dimension

- $\boxed{d_A \leq 9R + 3n}$

Proof sketch

(1) Lower bound on accessible dimension (the toughest step): $\boxed{d_A \geq T}$

(2) Upper bound on accessible dimension

- $\boxed{d_A \leq 9R + 3n}$

- Proof strategy: parameter counting $\longrightarrow$ Ask during Q&A

(3) Putting it all together

(3) Putting it all together

- Assume the theorem's assumptions.

(3) Putting it all together

- Assume the theorem's assumptions.
    - A = architecture formed from T blocks,

(3) Putting it all together

- Assume the theorem's assumptions.
  - $A =$ architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates

(3) Putting it all together

- Assume the theorem's assumptions.
  - A = architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates
  - Total number of gates: $R \leq TL$

(3) Putting it all together

- Assume the theorem's assumptions.
  - A = architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates
  - Total number of gates: $R \leq TL$
  - Set of corresponding unitaries: $\mathscr{U}(A)$

(3) Putting it all together

- Assume the theorem's assumptions.
    - A = architecture formed from T blocks,
      each containing a backward light cone and $\leq L$ gates
    - Total number of gates: $R \leq TL$
    - Set of corresponding unitaries: $\mathscr{U}(A)$
- Sample a unitary U from $\mathscr{U}(A)$ uniformly randomly.

(3) Putting it all together

- Assume the theorem's assumptions.
  - A = architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates
  - Total number of gates: $R \leq TL$
  - Set of corresponding unitaries: $\mathscr{U}(A)$
- Sample a unitary U from $\mathscr{U}(A)$ uniformly randomly.
- With what probability can U be implemented with $R' < \dfrac{R}{9L} - \dfrac{n}{3}$ gates?

(3) Putting it all together

- Assume the theorem's assumptions.
  - A = architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates
  - Total number of gates: $R \leq TL$
  - Set of corresponding unitaries: $\mathscr{U}(A)$
- Sample a unitary U from $\mathscr{U}(A)$ uniformly randomly.
- With what probability can U be implemented with $R' < \frac{R}{9L} - \frac{n}{3}$ gates?

does $\mathscr{C}(U) < \frac{R}{9L} - \frac{n}{3}$?

(3) Putting it all together

- Assume the theorem's assumptions.
    - A = architecture formed from T blocks,
      each containing a backward light cone and $\leq L$ gates
    - Total number of gates: $R \leq TL$
    - Set of corresponding unitaries: $\mathscr{U}(A)$
- Sample a unitary U from $\mathscr{U}(A)$ uniformly randomly.
- With what probability can U be implemented with $R' < \frac{R}{9L} - \frac{n}{3}$ gates?

does $\mathscr{C}(U) < \frac{R}{9L} - \frac{n}{3}$?

Show that the probability = 0, using lemmata (1) and (2).

(3) Putting it all together

- $R' < \frac{R}{9L} - \frac{n}{3}$

(3) <u>Putting it all together</u>

- $R' < \underset{\leq\, T/9}{\boxed{\dfrac{R}{9L}}} - \dfrac{n}{3}$

(3) Putting it all together

- $R' < \underset{\substack{\rotatebox{90}{$\geq$} \\ T/9}}{\boxed{\frac{R}{9L}}} - \frac{n}{3} \longrightarrow$ Solve for T.

(3) <u>Putting it all together</u>

- $R' < \boxed{\dfrac{R}{9L}} - \dfrac{n}{3}$ $\longrightarrow$ Solve for T. $\longrightarrow$ (3A) $T > 9R' + 3n$

$$\underset{\displaystyle T/9}{\underset{\displaystyle \geq}{\boxed{\dfrac{R}{9L}}}}$$

(3) Putting it all together

- $R' < \underset{\displaystyle \leq T/9}{\boxed{\frac{R}{9L}}} - \frac{n}{3}$ ⟶ Solve for T. ⟶ (3A) $T > 9R' + 3n$

- (1) Lower bound on accessible dimension: $d_A \geq T$

(3) Putting it all together

- $R' < \boxed{\dfrac{R}{9L}} - \dfrac{n}{3}$ $\longrightarrow$ Solve for T. $\longrightarrow$ (3A) $T > 9R' + 3n$

  $\underset{T/9}{\leq}$

- (1) Lower bound on accessible dimension: $d_A \geq T$

(3) Putting it all together

- $R' < \boxed{\dfrac{R}{9L}} - \dfrac{n}{3}$ ⟶ Solve for T. ⟶ (3A) $T > 9R' + 3n$

  $\boxed{\dfrac{R}{9L}} \geq T/9$

- (1) Lower bound on accessible dimension: $d_A \geq T$

$\therefore$ (1) + (3A) $\Rightarrow$ $d_A > 9R' + 3n$

(3) <u>Putting it all together</u>

- $R' < \underset{\substack{\rotatebox{90}{$\leq$} \\ T/9}}{\boxed{\frac{R}{9L}}} - \frac{n}{3}$ $\longrightarrow$ Solve for T. $\longrightarrow$ (3A) $T > 9R' + 3n$

- (1) <u>Lower bound on accessible dimension</u>: $d_A \geq T$

$\therefore$ <u>(1) + (3A) $\Rightarrow$</u> $d_A > 9R' + 3n$

- Consider any architecture A' of R' gates.

(3) Putting it all together

- $R' < \boxed{\dfrac{R}{9L}} - \dfrac{n}{3}$ ⟶ Solve for T. ⟶ (3A) $T > 9R' + 3n$

$$\frac{R}{9L} \geq T/9$$

- (1) Lower bound on accessible dimension: $d_A \geq T$

$\therefore$ (1) + (3A) $\Rightarrow$ $d_A > 9R' + 3n$

- Consider any architecture A' of R' gates. ⟶

  Accessible dimension from (2): $d_{A'} \leq 9R' + 3n$

(3) Putting it all together

- $R' < \boxed{\dfrac{R}{9L}} - \dfrac{n}{3}$ → Solve for T. → (3A) $T > 9R' + 3n$

  $\geq$ $T/9$

- (1) Lower bound on accessible dimension: $d_A \geq T$

$\therefore$ (1) + (3A) $\Rightarrow$ $d_A > 9R' + 3n$

- Consider any architecture A' of R' gates. →

  Accessible dimension from (2): $d_{A'} \leq 9R' + 3n$

(3) Putting it all together

- $R' < \boxed{\frac{R}{9L}} - \frac{n}{3}$ ⟶ Solve for T. ⟶ (3A) $T > 9R' + 3n$

  $\frac{R}{9L} \geq T/9$

- (1) Lower bound on accessible dimension: $d_A \geq T$

$\therefore$ (1) + (3A) $\Rightarrow$ $d_A > 9R' + 3n$

- Consider any architecture A' of R' gates. ⟶

  Accessible dimension from (2): $d_{A'} \leq 9R' + 3n$

$\therefore\ d_{A'} < d_A$

(3) Putting it all together

$$\therefore\ d_{A'} < d_A \longrightarrow$$

(3) Putting it all together

$\therefore\ d_{A'} < d_A \longrightarrow$

- Lemma: $\mathscr{U}(A')$ forms a measure-0 subset of $\mathscr{U}(A)$.

(3) Putting it all together

$\therefore\ d_{A'} < d_A \longrightarrow$

- Lemma: $\mathscr{U}(A')$ forms a measure-0 subset of $\mathscr{U}(A)$.
    - Proof tool: dimension theory of real algebraic sets

(3) Putting it all together

$\therefore\ d_{A'} < d_A \longrightarrow$

- Lemma: $\mathcal{U}(A')$ forms a measure-0 subset of $\mathcal{U}(A)$.
  - Proof tool: dimension theory of real algebraic sets

$\therefore$ If you randomly pick a U effected with a sufficiently connected R-gate circuit,

(3) Putting it all together

$\therefore\ d_{A'} < d_A \longrightarrow$

- Lemma: $\mathscr{U}(A')$ forms a measure-0 subset of $\mathscr{U}(A)$.
  - Proof tool: dimension theory of real algebraic sets

$\therefore$ If you randomly pick a U effected with
a sufficiently connected R-gate circuit,
you can implement U with a smaller R'-gate circuit
with probability 0.

(3) Putting it all together

$\therefore\ d_{A'} < d_A \longrightarrow$

- Lemma: $\mathscr{U}(A')$ forms a measure-0 subset of $\mathscr{U}(A)$.
  - Proof tool: dimension theory of real algebraic sets

$\therefore$ If you randomly pick a U effected with
a sufficiently connected R-gate circuit,
you can implement U with a smaller R'-gate circuit
with probability 0. 👍

Extensions

Extensions

(1) Lower bound on exact state complexity

Extensions

(1) Lower bound on exact state complexity

- $|\psi\rangle = n$-qubit pure state

Extensions

(1) Lower bound on exact state complexity

- $|\psi\rangle = n$-qubit pure state
- $\mathscr{C}_s(|\psi\rangle) =$ least number of two-qubit gates required to prepare $|\psi\rangle$ from $|0^n\rangle$

Extensions

(1) Lower bound on exact state complexity

- $|\psi\rangle = n$-qubit pure state
- $\mathscr{C}_{\mathrm{s}}(|\psi\rangle)$ = least number of two-qubit gates required
  to prepare $|\psi\rangle$ from $|0^n\rangle$
- If $|\psi\rangle = U|0^n\rangle$, for some U that satisfies our theorem's assumptions, then

Extensions

(1) Lower bound on exact state complexity

- $|\psi\rangle = n$-qubit pure state
- $\mathscr{C}_{\mathrm{s}}(|\psi\rangle)$ = least number of two-qubit gates required to prepare $|\psi\rangle$ from $|0^n\rangle$
- If $|\psi\rangle = U|0^n\rangle$, for some U that satisfies our theorem's assumptions, then

$$\mathscr{C}_{\mathrm{s}}(|\psi\rangle) \geq \frac{R}{9L} - \frac{n}{3}, \text{ until } T \leq 2^{n+1} - 1 .$$

Extensions

(1) Lower bound on exact state complexity

- $|\psi\rangle = n$-qubit pure state
- $\mathscr{C}_s(|\psi\rangle) =$ least number of two-qubit gates required to prepare $|\psi\rangle$ from $|0^n\rangle$
- If $|\psi\rangle = U|0^n\rangle$, for some U that satisfies our theorem's assumptions, then

$$\mathscr{C}_s(|\psi\rangle) \geq \frac{R}{9L} - \frac{n}{3}, \text{ until } T \leq 2^{n+1} - 1 .$$

- Applications to resource theory: NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Extensions

(2) Random architecture

Extensions

(2) Random architecture

Remove backward light cone from assumptions

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

Extensions

(2) Random architecture → probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

Extensions

(2) Random architecture → probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

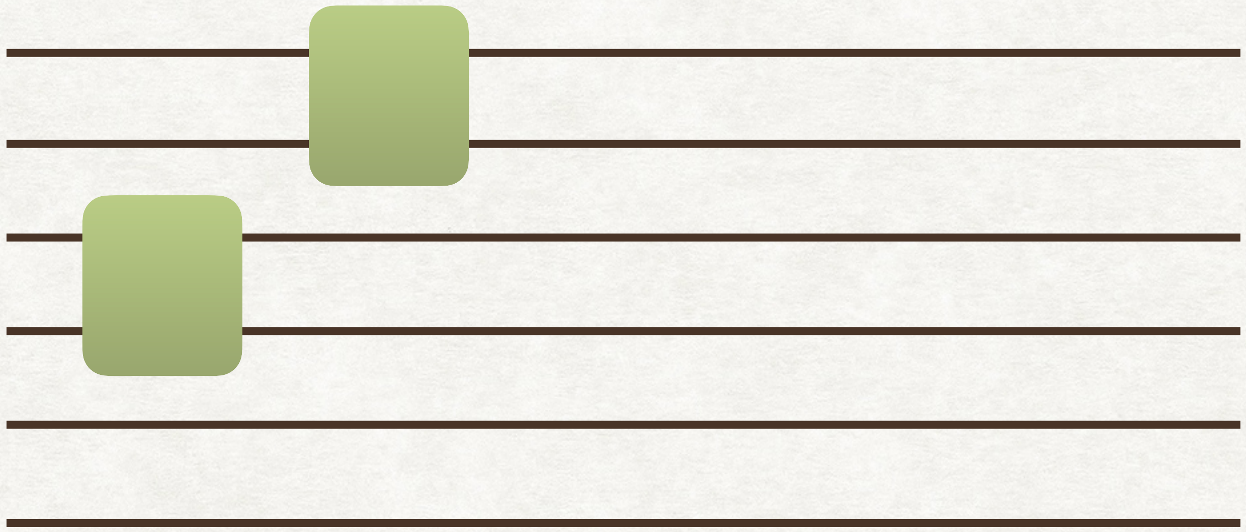
- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

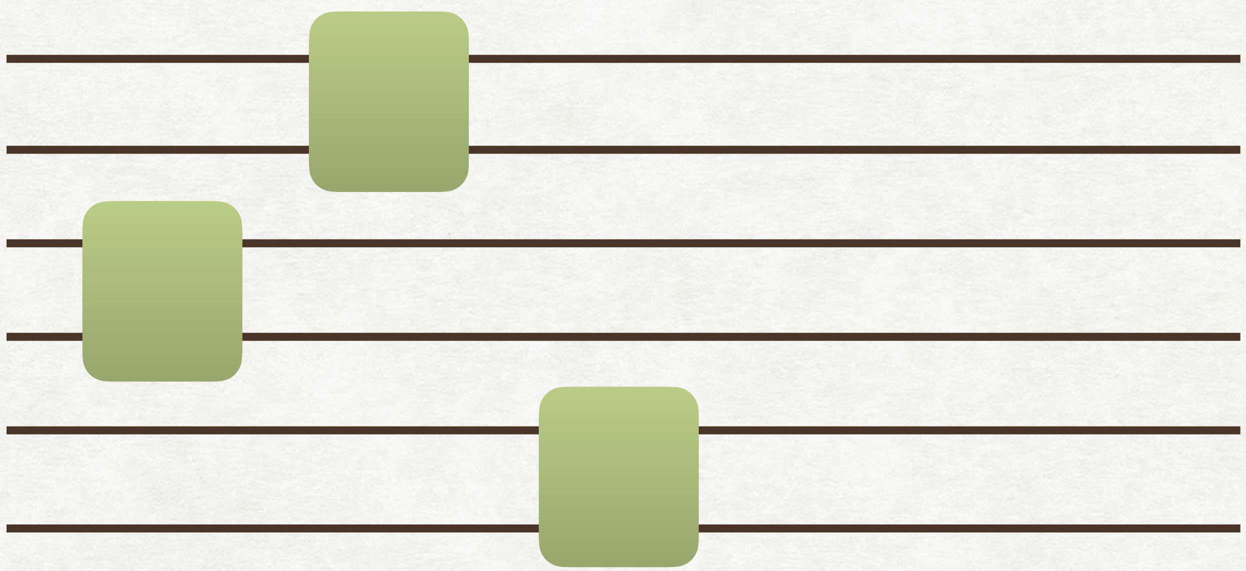


Extensions

(2) Random architecture ⟶ probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

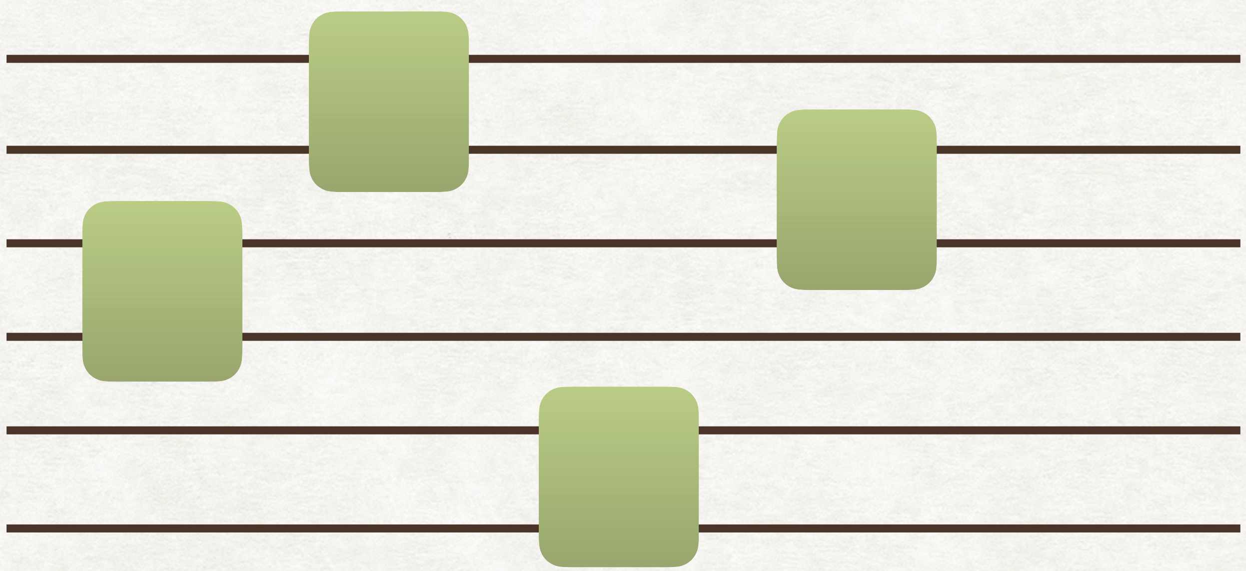


Extensions

(2) Random architecture → probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

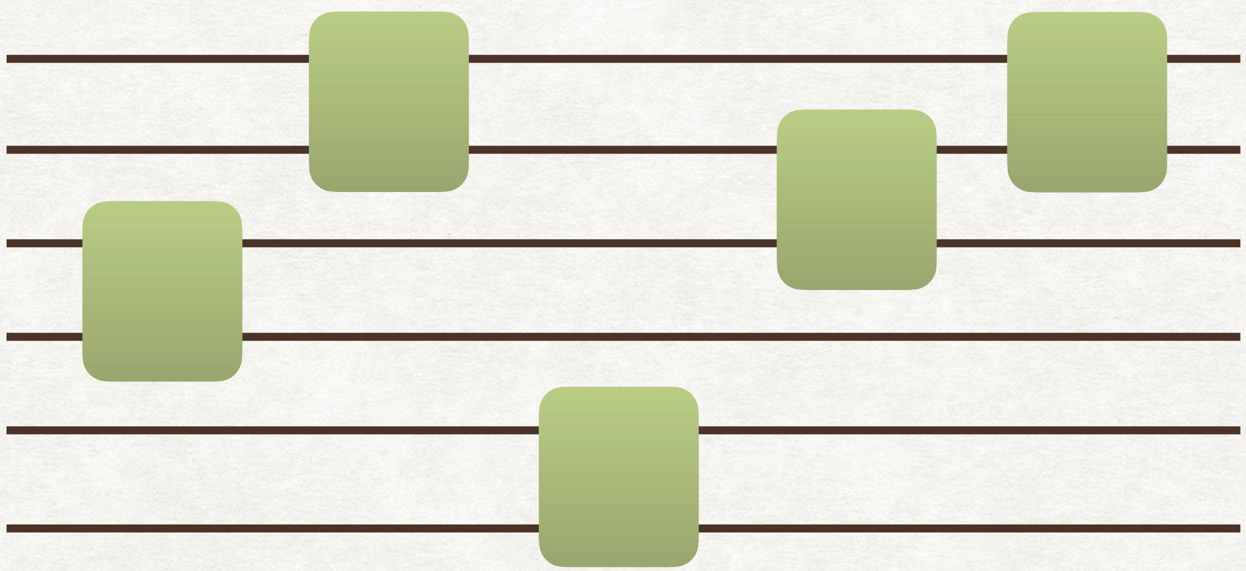


Extensions

(2) Random architecture → probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.

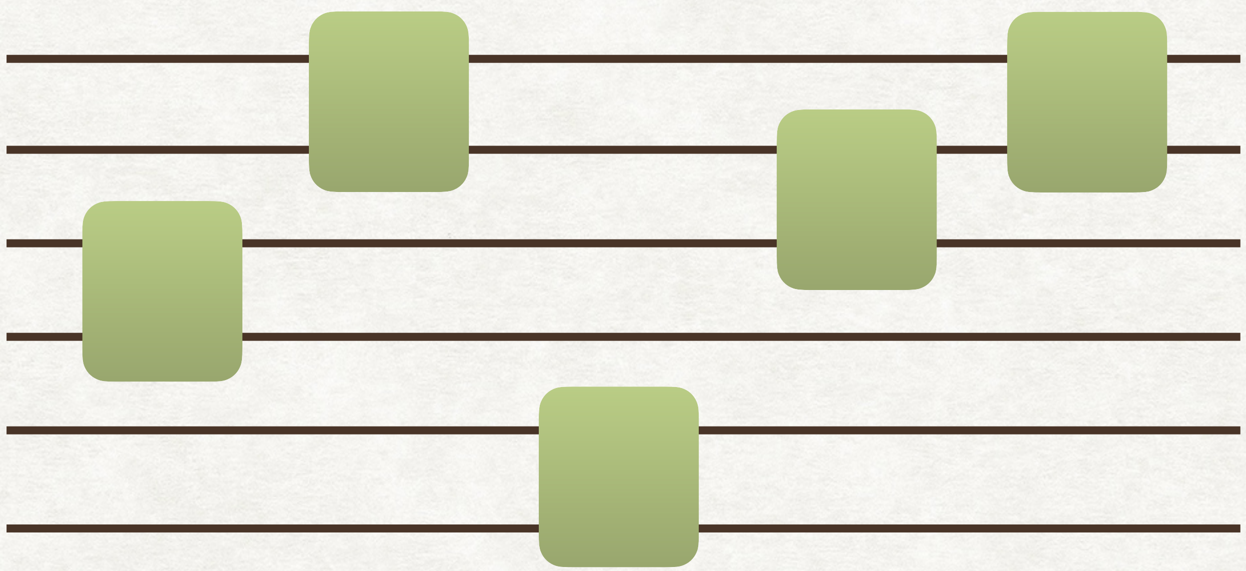


Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.



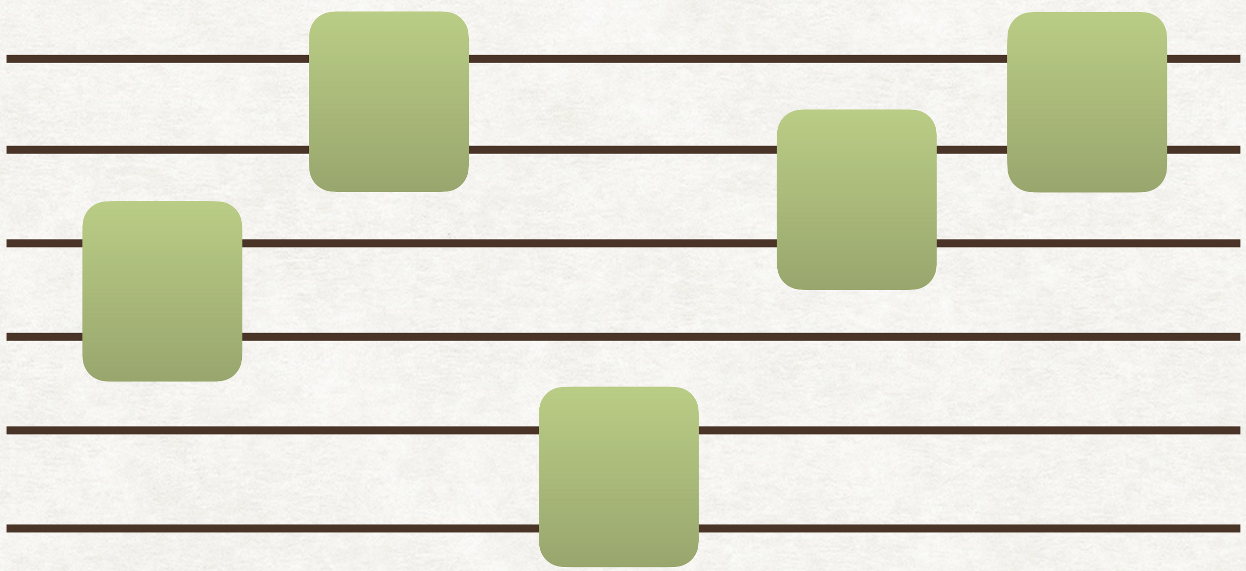
- With high probability, the gates form backward light cones.

Extensions

(2) Random architecture ⟶ probabilistic lower bound on exact circuit complexity

↖ Remove backward light cone from assumptions

- Example: At each time step, randomly pick a nearest-neighbor pair and a gate.



- With high probability, the gates form backward light cones. ⟶ $\mathscr{C}(U)$ obeys a linear lower bound.

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\left(\mathscr{C}(U) \geq \alpha \, \frac{R}{9n(n-1)^2} - \frac{n}{3}\right) \geq 1 - \frac{1}{1-\alpha}\,(n-1)\, e^{-n}$$

  $\forall \alpha \in [0, 1)\,.$

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\Big(\underbrace{\mathscr{C}(U) \geq \alpha \frac{R}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\Big) \geq 1 - \frac{1}{1-\alpha}(n-1)\,e^{-n}$$

$\forall \alpha \in [0, 1)$.

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\Bigg(\underbrace{\mathscr{C}(U) \geq \alpha \frac{\boxed{R}}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\Bigg) \geq 1 - \frac{1}{1-\alpha}(n-1)\, e^{-n}$$

  $\forall \alpha \in [0, 1)\,.$

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\left(\underbrace{\mathscr{C}(U) \geq \alpha\, \frac{\boxed{R}}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\right) \geq \underbrace{1 - \frac{1}{1-\alpha}\,(n-1)\, e^{-n}}_{\text{High probability of being obeyed}}$$

$\forall \alpha \in [0, 1)$.

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\left(\underbrace{\mathscr{C}(U) \geq \alpha \frac{R}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\right) \geq \underbrace{1 - \frac{1}{1-\alpha}(n-1)\,e^{-n}}_{\text{High probability of being obeyed}}$$

$\forall \alpha \in [0, 1)$.

α large

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $$\Pr\left(\mathscr{C}(U) \geq \boxed{\alpha}\,\underbrace{\frac{R}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\right) \geq \underbrace{1 - \frac{1}{1-\alpha}(n-1)\,e^{-n}}_{\text{High probability of being obeyed}}$$

$\forall \alpha \in [0, 1)$.

α large $\Rightarrow$ lower bound not suppressed

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

- $\Pr\left(\underbrace{\mathscr{C}(U) \geq \boxed{\alpha}\,\frac{R}{9n(n-1)^2} - \frac{n}{3}}_{\text{Lower bound on complexity}}\right) \geq \underbrace{1 - \frac{1}{1-\boxed{\alpha}}(n-1)\,e^{-n}}_{\text{High probability of being obeyed}}$

  $\forall \alpha \in [0, 1)\,.$

α large $\Rightarrow$ lower bound not suppressed $\Rightarrow$ probability bound is low

Extensions

(2) Random architecture $\longrightarrow$ probabilistic lower bound on exact circuit complexity

$$\bullet\ \underbrace{\Pr\left(\mathscr{C}(U) \geq \boxed{\alpha}\frac{R}{9n(n-1)^2} - \frac{n}{3}\right)}_{\text{Lower bound on complexity}} \geq \underbrace{1 - \frac{1}{1-\boxed{\alpha}}(n-1)\,e^{-n}}_{\text{High probability of being obeyed}}$$

$\forall \alpha \in [0,\, 1)\,.$

α large $\Rightarrow$ lower bound not suppressed $\Rightarrow$ probability bound is low

- Key proof tool: Chebyshev's/Markov's inequality

Extensions

(3) Lower bound on approximate circuit complexity

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
    - Suppose that U satisfies our theorem's assumptions.

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U:

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U:

    $\forall \delta \in (0, 1]$,

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U: $\forall \delta \in (0, 1]$, there exists an $\varepsilon := \varepsilon(A, \delta) > 0$

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U: $\forall \delta \in (0, 1]$, there exists an $\varepsilon := \varepsilon(A, \delta) > 0$ such that, with probability $1 - \delta$,

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U: $\forall \delta \in (0, 1]$, there exists an $\varepsilon := \varepsilon(A, \delta) > 0$ such that, with probability $1 - \delta$, $||U - U'||_{\mathrm{F}} \geq \varepsilon$.

Extensions

(3) Lower bound on approximate circuit complexity

- Synopsis
  - Suppose that U satisfies our theorem's assumptions.
  - Consider trying to approximate U with a U' implemented by a short, bound-violating circuit.
  - U' probably has a large Frobenius distance from U: $\forall \delta \in (0, 1]$, there exists an $\varepsilon := \varepsilon(A, \delta) > 0$ such that, with probability $1 - \delta$, $||U - U'||_{\mathrm{F}} \geq \varepsilon$.
- Shortcoming: ε can be uncontrollably small.

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

- We're extending $\mathcal{U}(A')$ to include all the unitaries close in some matrix norm.

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

- We're extending $\mathscr{U}(A')$ to include all the unitaries close in some matrix norm.

- The set grows in all directions in unitary space.

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

- We're extending $\mathscr{U}(A')$ to include all the unitaries close in some matrix norm.

- The set grows in all directions in unitary space.

$\Rightarrow$ The accessible dimension leaps to its maximum, 4^n.

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

- We're extending $\mathcal{U}(A')$ to include all the unitaries close in some matrix norm.

- The set grows in all directions in unitary space.

$\Rightarrow$ The accessible dimension leaps to its maximum, 4^n.

$\Rightarrow$ $d_{A'} < d_A$ not generally satisfied

Extensions

(3) Lower bound on approximate circuit complexity

Why ε can be uncontrollably small

- We're extending $\mathscr{U}(A')$ to include all the unitaries close in some matrix norm.

- The set grows in all directions in unitary space.

$\Rightarrow$ The accessible dimension leaps to its maximum, 4^n.

$\Rightarrow$ $d_{A'} < d_A$ not generally satisfied

$\Rightarrow$ The accessible dimension is too crude a tool.


OPPORTUNITY

(1) Strongly lower-bound approximate circuit complexity.

(1) Strongly lower-bound approximate circuit complexity.

- Strategy: try to prove conjectures about $\mathscr{U}(A)$ and overlapping $\mathscr{U}(A')$

(1) Strongly lower-bound approximate circuit complexity.

- Strategy: try to prove conjectures about $\mathscr{U}(A)$ and overlapping $\mathscr{U}(A')$ $\longrightarrow$

(1) Strongly lower-bound approximate circuit complexity.

- Strategy: try to prove conjectures about $\mathscr{U}(A)$ and overlapping $\mathscr{U}(A')$ $\longrightarrow$

(2) Lower-bound Nielsen's complexity.

(1) Strongly lower-bound approximate circuit complexity.

- Strategy: try to prove conjectures about $\mathscr{U}(A)$ and overlapping $\mathscr{U}(A')$ $\longrightarrow$

(2) Lower-bound Nielsen's complexity.

- Nielsen's complexity $\geq$ approximate circuit complexity

Nielsen *et al.*, Science **311**, 1133 (2006).
Nielsen *et al.*, Phys. Rev. A **73**, 062323 (2006).

(1) Strongly lower-bound approximate circuit complexity.

- Strategy: try to prove conjectures about $\mathscr{U}(A)$ and overlapping $\mathscr{U}(A')$ $\longrightarrow$

(2) Lower-bound Nielsen's complexity.

- Nielsen's complexity $\geq$ approximate circuit complexity $\geq$ lower bound (1)

Nielsen *et al.*, Science **311**, 1133 (2006).

Nielsen *et al.*, Phys. Rev. A **73**, 062323 (2006).

(3) Accessible dimension as a new mathematical tool in many-body quantum physics

(3) Accessible dimension as a new mathematical tool in many-body quantum physics

- Distinct from Nielsen's geometry, t-designs

(3) Accessible dimension as a new mathematical tool in many-body quantum physics

- Distinct from Nielsen's geometry, t-designs
- Possible applications: Brownian circuits, hybrid circuits, …

(3) Accessible dimension as a new mathematical tool in many-body quantum physics

- Distinct from Nielsen's geometry, t-designs
- Possible applications: Brownian circuits, hybrid circuits, …

(4) Resource-theory opportunities

- NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

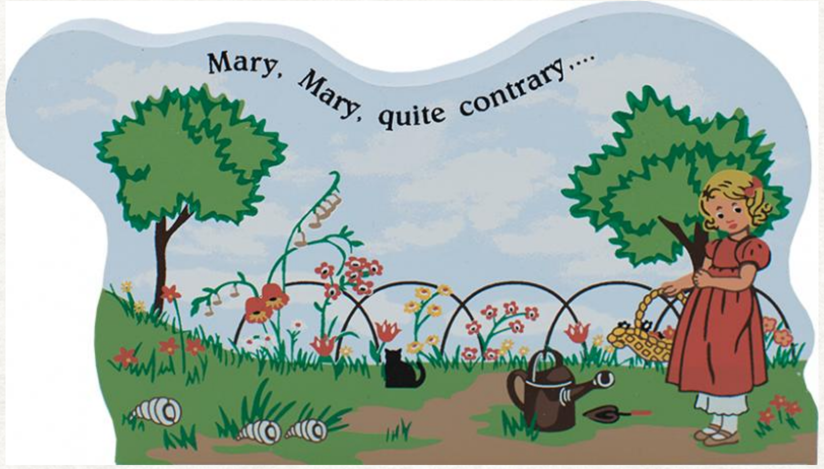
- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

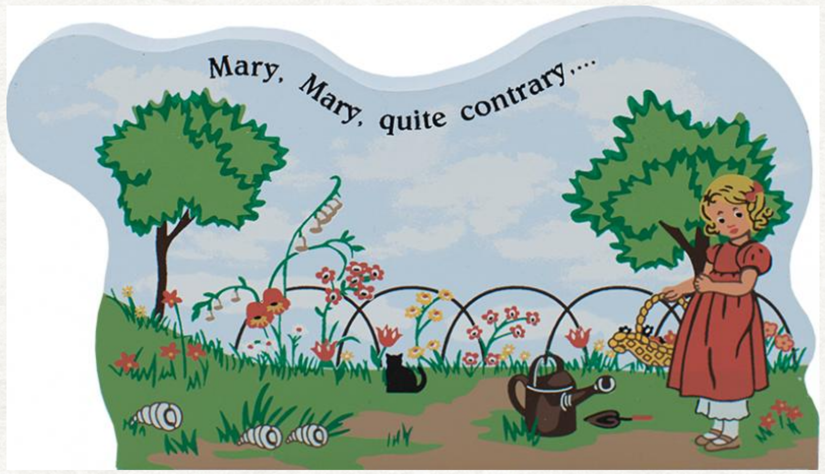
(1) 

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

(1) 

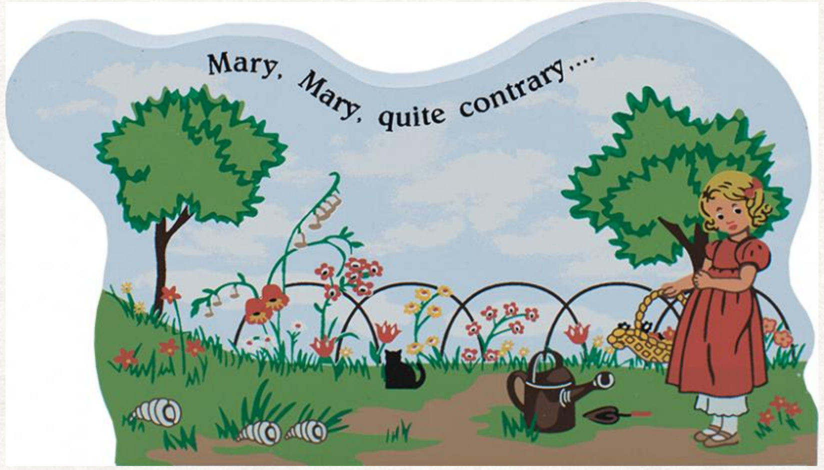
(2) Can define a resource theory for quantum uncomplexity

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

(1) 

(2) Can define a resource theory for quantum uncomplexity

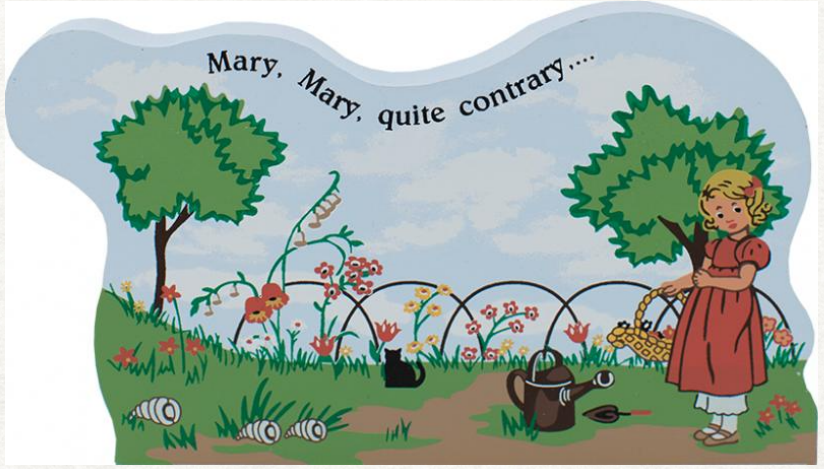
- 2 proofs

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

(1) 

(2) Can define a resource theory for quantum uncomplexity

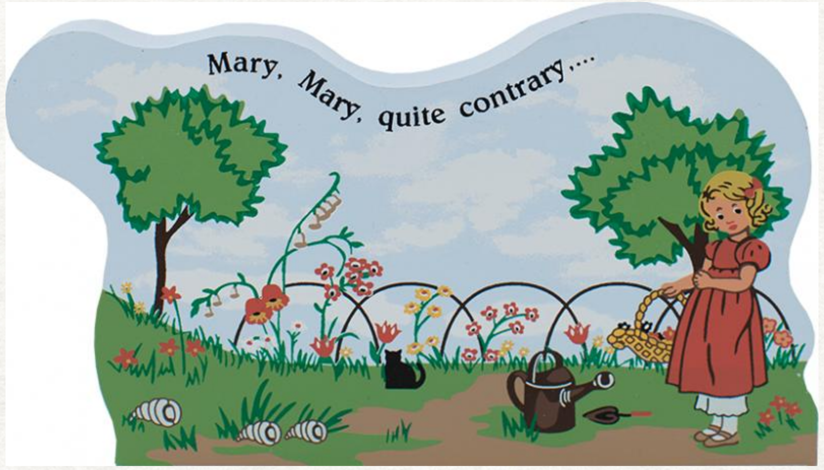
- 2 proofs
- Toolkits: algebraic geometry, differential topology

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

(1) 

(2) Can define a resource theory for quantum uncomplexity

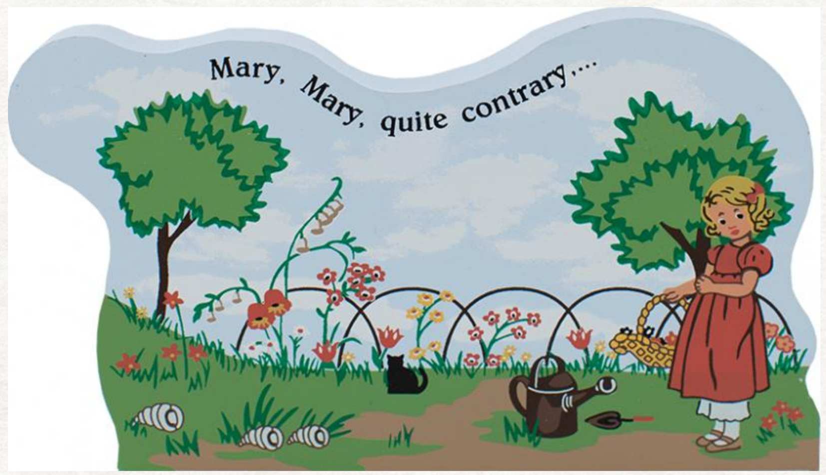
- 2 proofs
- Toolkits: algebraic geometry, differential topology ⟶ accessible dimension

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Recap

- Quantum complexity as a relevant tool across many-body physics 
- Two 2017/2018 complexity conjectures by Brown and Susskind

(1) 

(2) Can define a resource theory for quantum uncomplexity

- 2 proofs
- Toolkits: algebraic geometry, differential topology ⟶ accessible dimension
- 

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Thanks for your time!

Haferkamp, Faist, Kothakonda, Eisert, and NYH, accepted by *Nat. Phys.* (in press) arXiv:2106.05305.

NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

- $U \in \mathrm{SU}(2^n)$

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

- $U \in \mathrm{SU}(2^n)$
- $2^n =: N$

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

- $U \in \mathrm{SU}(2^n)$
- $2^n =: N$

Strategy

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

- $U \in \mathrm{SU}(2^n)$
- $2^n =: N$

<u>Strategy</u>

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

- $U \in \mathrm{SU}(2^n)$
- $2^n =: N$

Strategy

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

(2) Maximal complexity $\sim$ log(# unitaries)

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) <u>"Count" the unitaries in $\mathrm{SU}(N)$.</u>

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Unitaries form continuous set $\longrightarrow$ can't really count

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Unitaries form continuous set $\longrightarrow$ can't really count
- But we might distinguish/implement unitaries only with finite precision.
    - NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Unitaries form continuous set $\longrightarrow$ can't really count
- But we might distinguish/implement unitaries only with finite precision.
  - NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

$\longrightarrow$ Regularize $\mathrm{SU}(N)$ – attribute to each U a radius-ϵ ball

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Unitaries form continuous set $\longrightarrow$ can't really count
- But we might distinguish/implement unitaries only with finite precision.
  - NYH, Kothakonda, Haferkamp, Munson, Eisert, and Faist, arXiv:2110.11371 (2021).

$\longrightarrow$ Regularize $\mathrm{SU}(N)$ – attribute to each U a radius-ϵ ball

- How many ϵ-balls in $\mathrm{SU}(N)$?

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Dimension of $\mathrm{SU}(N)$: $N^2 - 1$

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Dimension of $\mathrm{SU}(N)$: $N^2 - 1$
- Volume of $\mathrm{SU}(N)$: $V_{\mathrm{SU}(N)} = \dfrac{2\pi^{(N+2)(N-1)/2}}{1!2!\ldots(N-1)!}$
    - Boya, Sudarshan, and Tilma, Rep. Math. Phys. **52**, 3 (2003).

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Dimension of $\mathrm{SU}(N)$: $N^2 - 1$
- Volume of $\mathrm{SU}(N)$: $V_{\mathrm{SU}(N)} = \dfrac{2\pi^{(N+2)(N-1)/2}}{1!2!\dots(N-1)!}$
    - Boya, Sudarshan, and Tilma, Rep. Math. Phys. **52**, 3 (2003).
- Volume of ϵ-ball of dimension $N^2 - 1$: $V_{\mathrm{ball}} = \dfrac{\pi^{(N^2-1)/2}\,\epsilon^{N^2-1}}{\left[(N^2-1)/2\right]!}$

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Dimension of $\mathrm{SU}(N)$: $N^2 - 1$
- Volume of $\mathrm{SU}(N)$: $V_{\mathrm{SU}(N)} = \dfrac{2\pi^{(N+2)(N-1)/2}}{1!2!\ldots(N-1)!}$
    - Boya, Sudarshan, and Tilma, Rep. Math. Phys. **52**, 3 (2003).
- Volume of ϵ-ball of dimension $N^2 - 1$: $V_{\mathrm{ball}} = \dfrac{\pi^{(N^2-1)/2}\,\epsilon^{N^2-1}}{\left[(N^2-1)/2\right]!}$
- Number of ϵ-balls in $\mathrm{SU}(N)$: $\dfrac{V_{\mathrm{SU}(N)}}{V_{\mathrm{ball}}}$

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathrm{SU}(N)$.

- Dimension of $\mathrm{SU}(N)$: $N^2 - 1$
- Volume of $\mathrm{SU}(N)$: $V_{\mathrm{SU}(N)} = \dfrac{2\pi^{(N+2)(N-1)/2}}{1!2!\ldots(N-1)!}$
  - Boya, Sudarshan, and Tilma, Rep. Math. Phys. **52**, 3 (2003).
- Volume of ϵ-ball of dimension $N^2 - 1$: $V_{\mathrm{ball}} = \dfrac{\pi^{(N^2-1)/2}\,\epsilon^{N^2-1}}{\left[(N^2-1)/2\right]!}$
- Number of ϵ-balls in $\mathrm{SU}(N)$: $\dfrac{V_{\mathrm{SU}(N)}}{V_{\mathrm{ball}}} \quad \sim$

Stirling's approximation

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(1) "Count" the unitaries in $\mathbf{SU}(N)$.

- Dimension of $\mathbf{SU}(N)$: $N^2 - 1$
- Volume of $\mathbf{SU}(N)$: $V_{\mathrm{SU}(N)} = \dfrac{2\pi^{(N+2)(N-1)/2}}{1!2!\ldots(N-1)!}$
  - Boya, Sudarshan, and Tilma, Rep. Math. Phys. **52**, 3 (2003).
- Volume of ϵ-ball of dimension $N^2 - 1$: $V_{\mathrm{ball}} = \dfrac{\pi^{(N^2-1)/2}\,\epsilon^{N^2-1}}{\left[(N^2-1)/2\right]!}$
- Number of ϵ-balls in $\mathbf{SU}(N)$: $\dfrac{V_{\mathrm{SU}(N)}}{V_{\mathrm{ball}}} \sim \left(\dfrac{2^n}{\epsilon^2}\right)^{4^n/2}$

Stirling's approximation

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(2) Maximal complexity = log(# unitaries)

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(2) Maximal complexity = log(# unitaries)

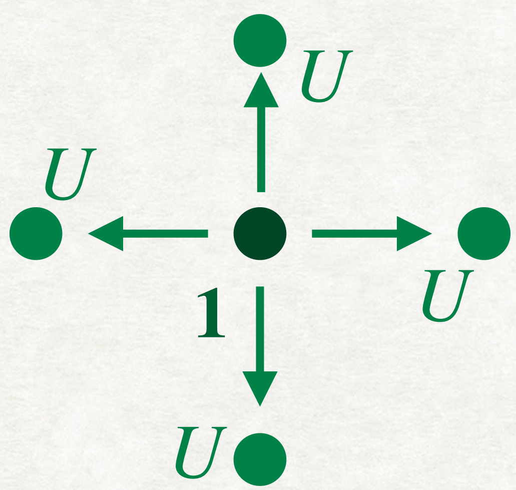
●

1

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

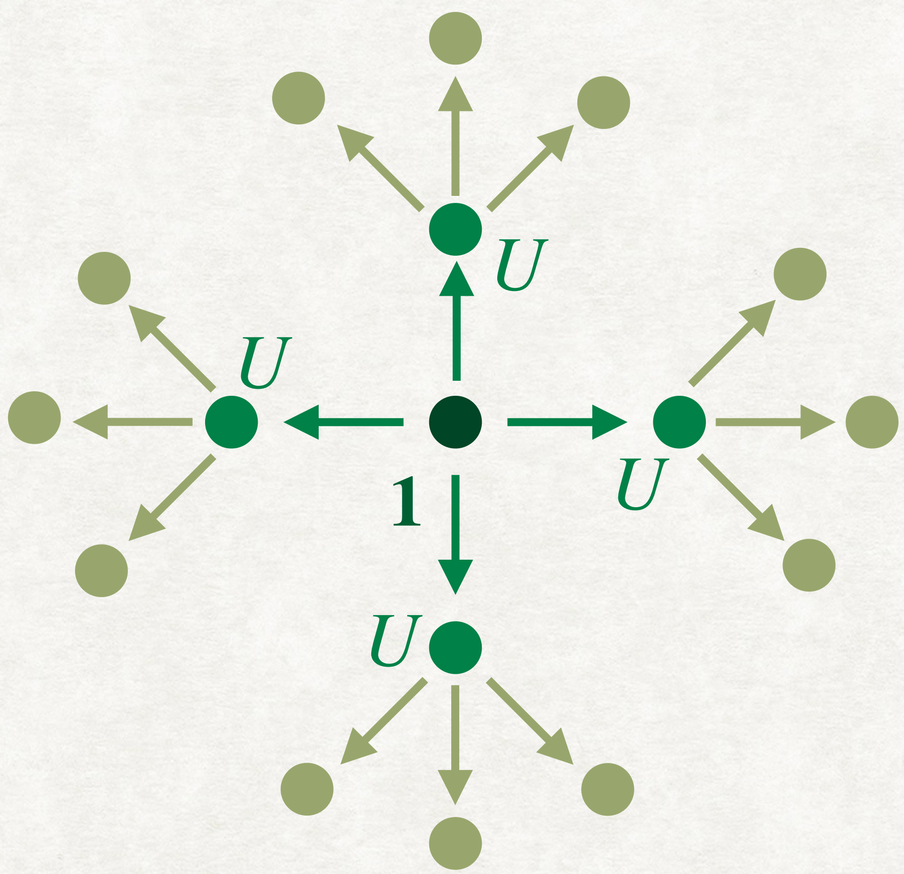
(2) Maximal complexity = log(# unitaries)



Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

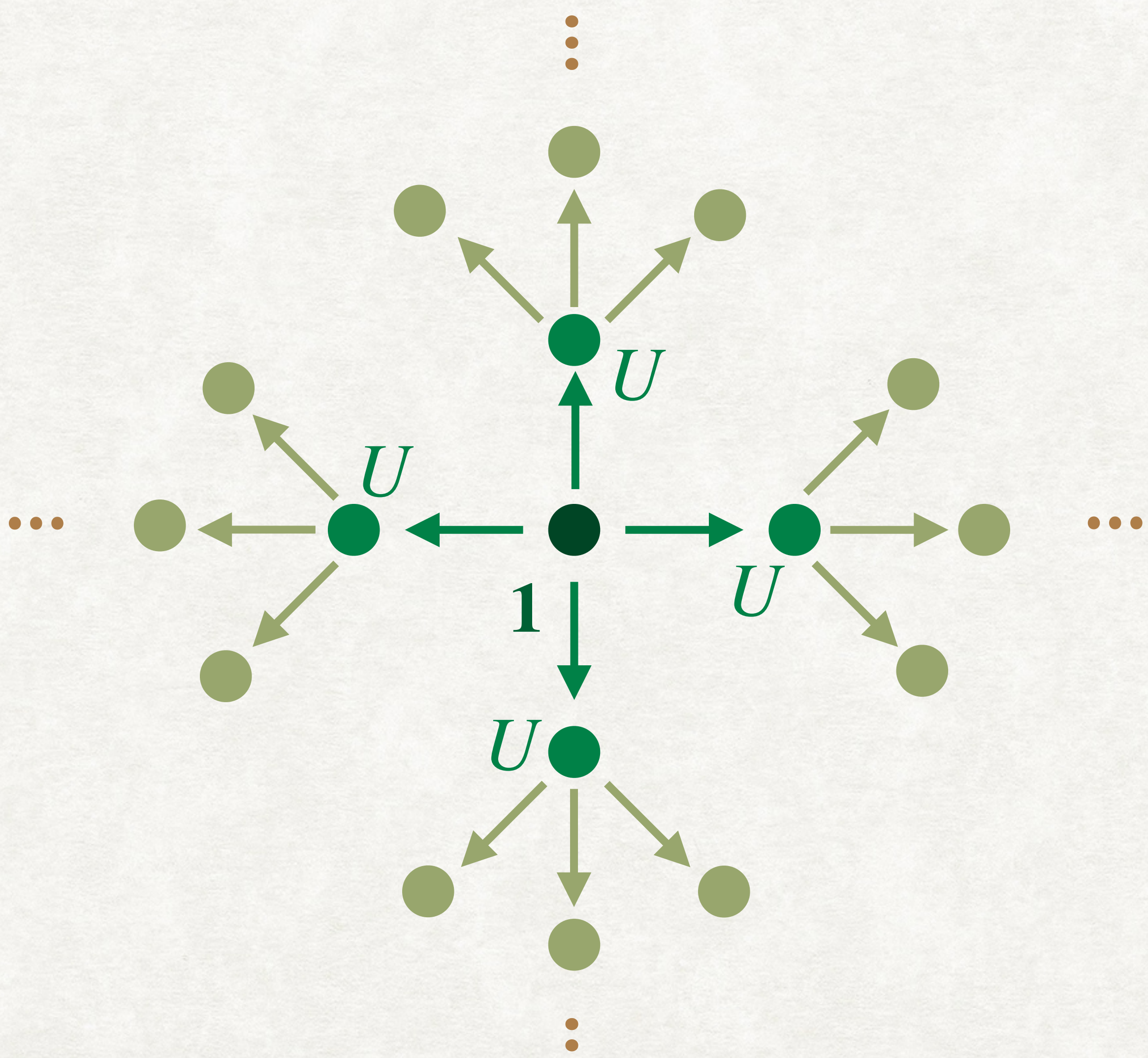
(2) Maximal complexity = log(# unitaries)



Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

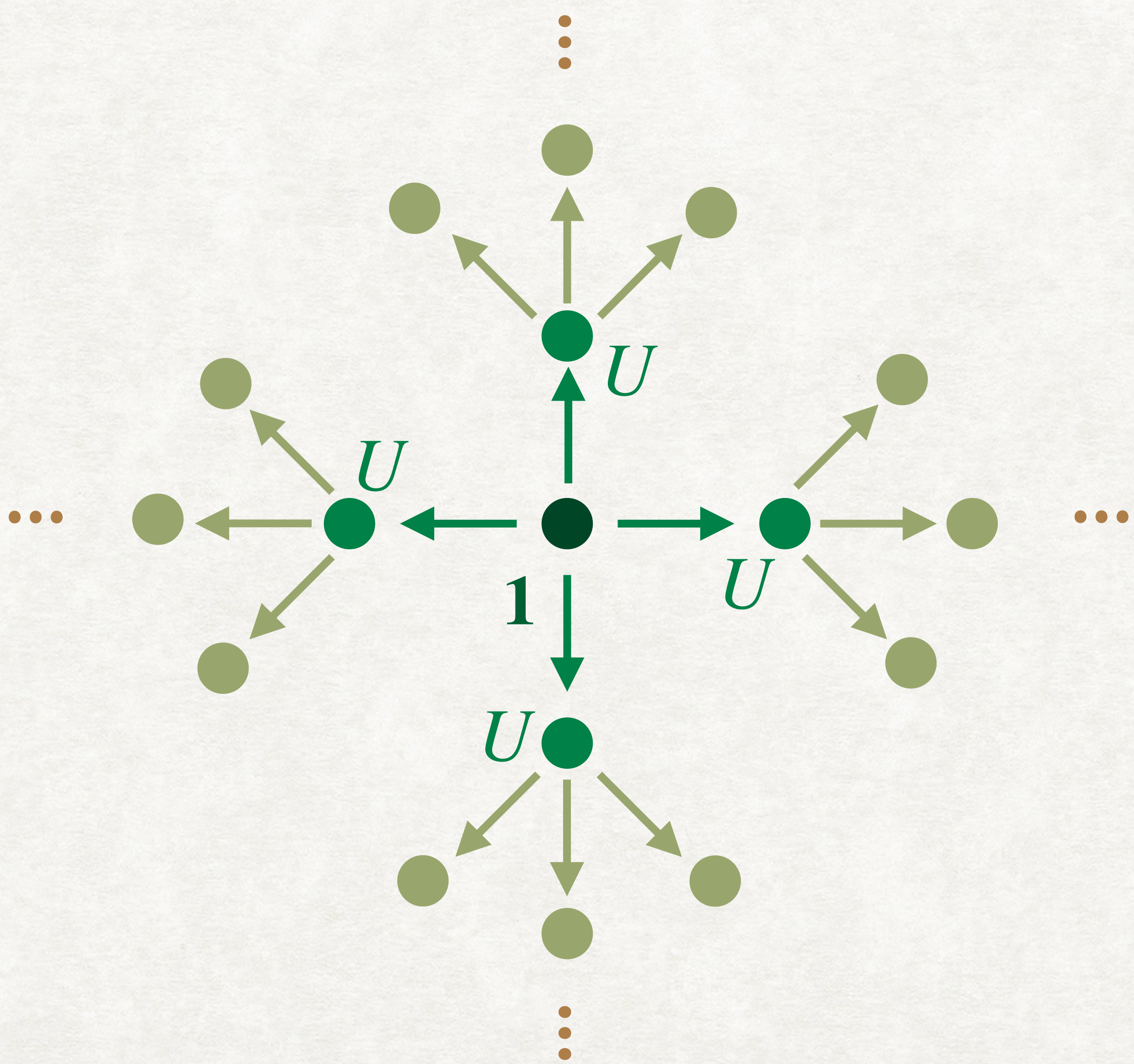
(2) Maximal complexity = log(# unitaries)



Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(2) Maximal complexity = log(# unitaries)

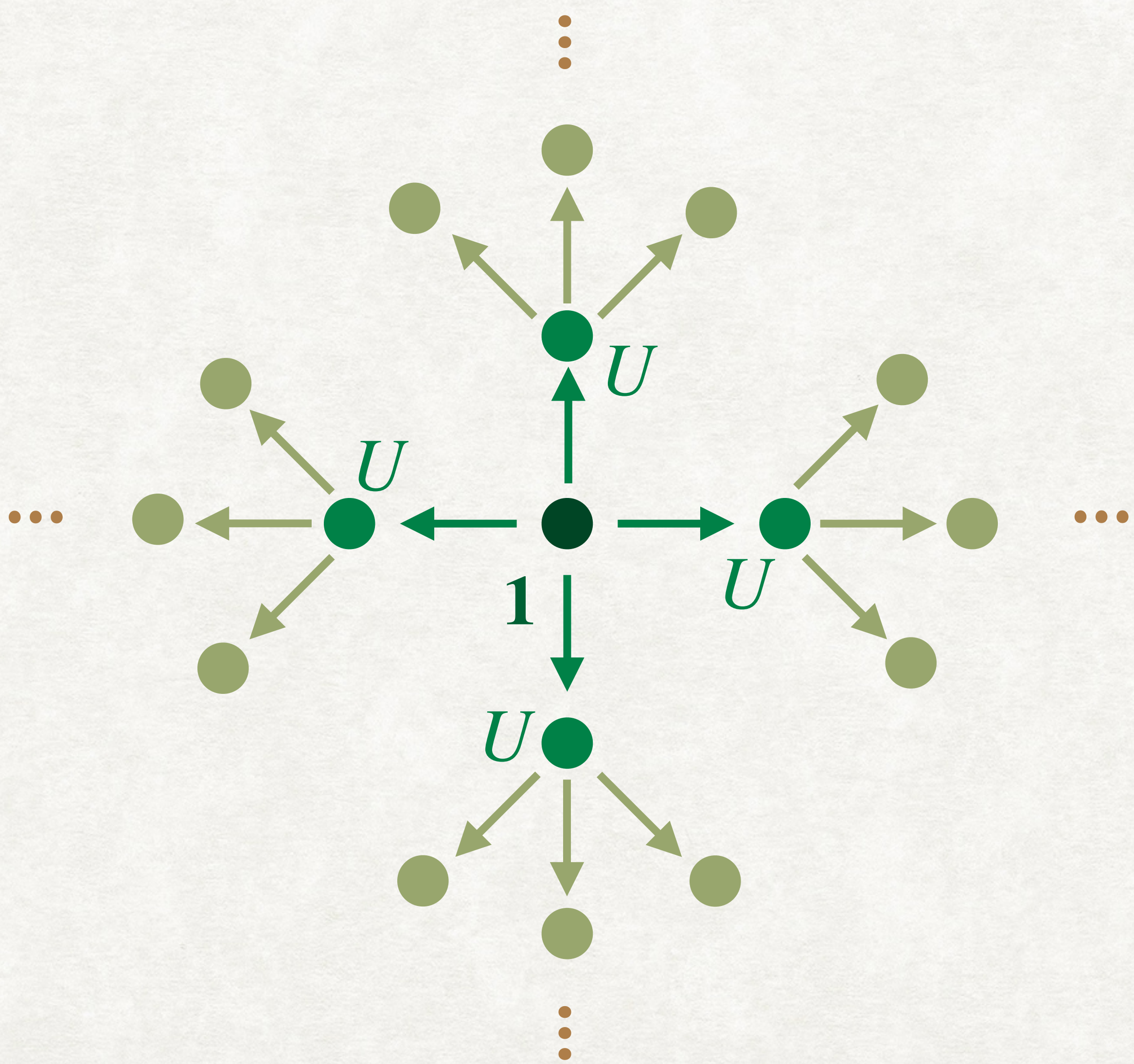


- Assumption: No unitaries collide.

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(2) Maximal complexity = log(# unitaries)

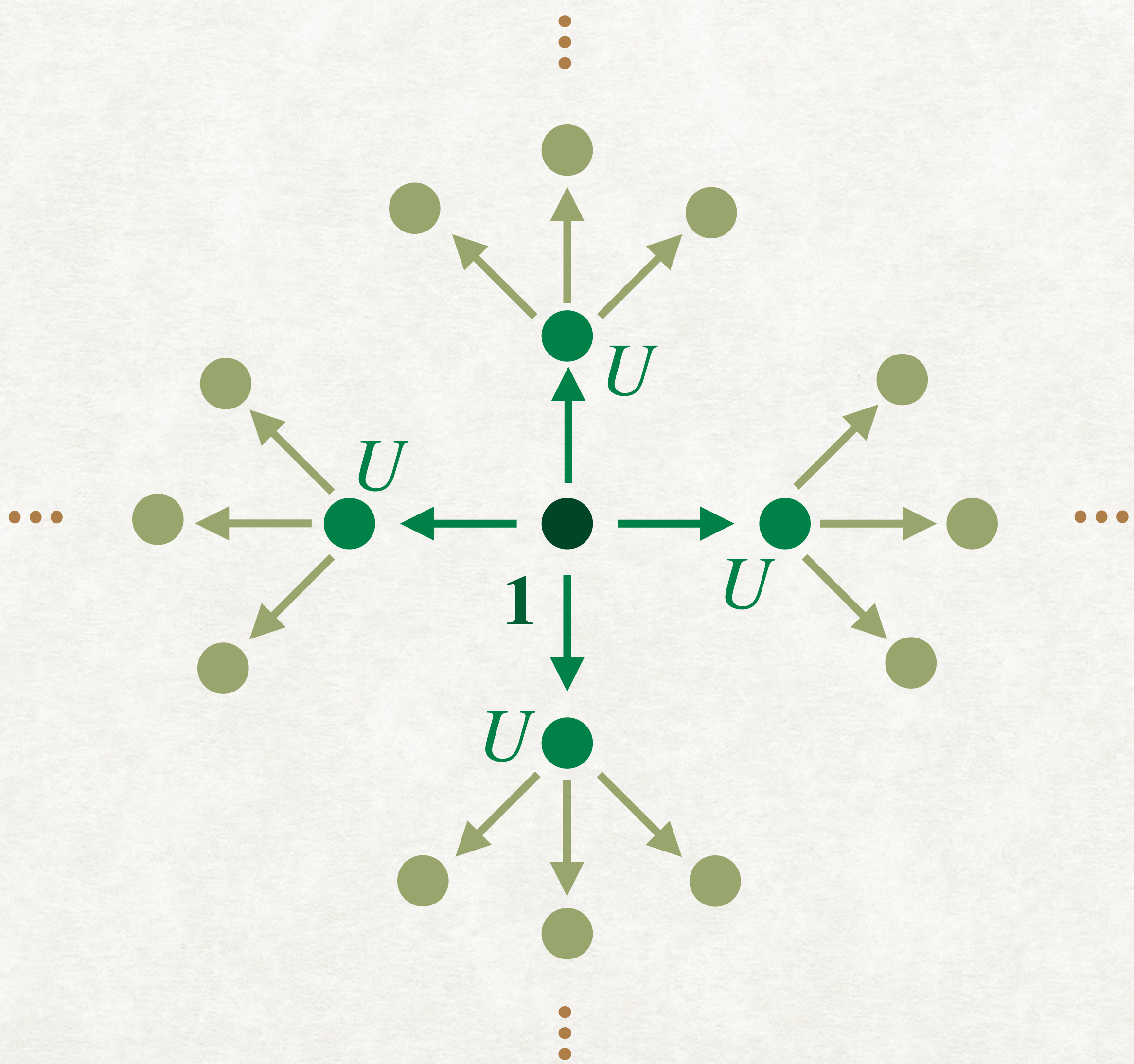


- Assumption: No unitaries collide.
  - # steps needed to reach edge from center: log(# dots)

Why an n-qubit unitary's maximal complexity $\sim 4^n$

Susskind, arXiv:1810.11563 (2018).

(2) Maximal complexity = log(# unitaries)



- Assumption: No unitaries collide.
  - # steps needed to reach edge from center: log(# dots)

Complexity = log(# unitaries)

Proof of lower bound on accessible dimension, $d_A \geq T$

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks, each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity $\longrightarrow$ accessible dimension

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity $\longrightarrow$ accessible dimension $\longrightarrow$ rank of F^A

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity $\longrightarrow$ accessible dimension $\longrightarrow$ rank of F^A

- Defined at a point $x \in \mathrm{SU}(4)^{\times \mathrm{R}}$

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity $\longrightarrow$ accessible dimension $\longrightarrow$ rank of F^A

- Defined at a point $x \in \mathrm{SU}(4)^{\times \mathrm{R}}$
  - Rank of the Jacobian of F^A

Proof of lower bound on accessible dimension, $d_A \geq T$

Setup

- A = architecture formed from T blocks,
  each containing a backward light cone and $\leq L$ gates
- Total number of gates: $R \leq TL$

Recasting

Complexity $\longrightarrow$ accessible dimension $\longrightarrow$ rank of F^A

- Defined at a point $x \in \mathrm{SU}(4)^{\times \mathrm{R}}$
  - Rank of the Jacobian of F^A
- If you perturb x, along how many directions can U spread?

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
    - Proof toolkit: differential topology

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
    - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
  - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates, $r \geq T$.

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
  - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates, $d_A = r_{\max} \geq$ $\wedge$ $r \geq T$.

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
    - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates, $r \geq \boxed{T}$.

$\boxed{d_A} = r_{\max} \boxed{\geq}$

$\wedge$

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
    - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates, $r \geq \boxed{T}$. 👍

$\boxed{d_A} = r_{\max} \boxed{\geq}$

$\wedge$

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
  - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r .
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates,

$\boxed{d_A} = r_{\max} \boxed{\geq}$

$\wedge$

$r \geq \boxed{T}$. 👍

  - Proof: by construction of x from Clifford circuits

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- Lemma: $d_A = r_{\max}$
  - Proof toolkit: differential topology
- $r_{\max} \geq$ every possible r.
- Lemma: For some set $x \in \mathrm{SU}(4)^{\times R}$ of gates, $\boxed{d_A} = r_{\max} \boxed{\geq} r \geq \boxed{T}$. 👍
  - Proof: by construction of x from Clifford circuits $\longrightarrow$

$\wedge$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- r = rank of F^A

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- r = rank of F^A

  = number of directions in which U spreads if x is perturbed

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- r = rank of F^A

  = number of directions in which U spreads if x is perturbed

  = number of parameters needed to parameterize
  the $\tilde{U}$ that results from perturbing x

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- r = rank of F^A

  = number of directions in which U spreads if x is perturbed

  = number of parameters needed to parameterize the $\tilde{U}$ that results from perturbing x

Goal: lower-bound

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
    - $x = (U_1, U_2, \ldots, U_R)$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- <u>Parameterization</u>
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary})U_j$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary}) U_j$

$$= \exp(i\epsilon H)\ \tilde{U}_j$$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j$ = (infinitesimal unitary)U_j

$$= \exp\left(i\epsilon H\right)\ \tilde{U}_j$$

Parameterized by
the 15 nontrivial
two-qubit Pauli strings
$S_k = 1X, X1, XX, \ \ldots$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- <u>Parameterization</u>
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - <u>Perturbed gate</u>: $\tilde{U}_j = (\text{infinitesimal unitary}) U_j$

$$= \exp(i\epsilon H)\ \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings

$S_k = 1X, X1, XX, \ldots$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k\right) \tilde{U}_j$$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary}) U_j$

$$= \exp\left(i\epsilon H\right)\ \tilde{U}_j$$

$$= \exp\left(i\sum_{k=1}^{15}\epsilon_{j,k}S_k\right)\ \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings

$S_k = 1X, X1, XX, \ \ldots$

- Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary}) U_j$

$$= \exp\left(i\epsilon H\right)\ \tilde{U}_j$$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k\right) \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings

$S_k = 1X, X1, XX, \ldots$

- Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

$U = F^A(x) \quad \mapsto \quad \tilde{U} = F^A(\tilde{x})$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j \; = (\text{infinitesimal unitary}) U_j$

$$= \exp\left(i\epsilon H\right)\; \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings

$S_k = 1X, X1, XX, \ \ldots$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k \right) \tilde{U}_j$$

  - Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

    $U = F^A(x) \quad \mapsto \quad \tilde{U} = F^A(\tilde{x})$

  - Perturbation to image:

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary})U_j$

$$= \exp\left(i\epsilon H\right)\, \tilde{U}_j$$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k\right) \tilde{U}_j$$

Parameterized by
the 15 nontrivial
two-qubit Pauli strings
$S_k = 1X, X1, XX, \ldots$

- Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

$U = F^A(x) \quad \mapsto \quad \tilde{U} = F^A(\tilde{x})$

- Perturbation to image: $\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0}$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary}) U_j$

$$= \exp\left(i\epsilon H\right) \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings

$S_k = 1X, X1, XX, \ldots$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k\right) \tilde{U}_j$$

- Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

$$U = F^A(x) \quad \mapsto \quad \tilde{U} = F^A(\tilde{x})$$

- Perturbation to image: $\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = \underbrace{K_{j,k}}_{\text{Hermitian operator}} F^A(x)$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Parameterization
  - $x = (U_1, U_2, \ldots, U_R) \quad \mapsto \quad \tilde{x} = (\tilde{U}_1, \tilde{U}_2, \ldots, \tilde{U}_R)$
  - Perturbed gate: $\tilde{U}_j = (\text{infinitesimal unitary})U_j$

$$= \exp(i\epsilon H)\ \tilde{U}_j$$

Parameterized by the 15 nontrivial two-qubit Pauli strings $S_k = 1X, X1, XX, \ldots$

$$= \exp\left(i \sum_{k=1}^{15} \epsilon_{j,k} S_k\right) \tilde{U}_j$$

- Perturbation to input $x \quad \Rightarrow \quad$ perturbation to image

$$U = F^A(x) \quad \mapsto \quad \tilde{U} = F^A(\tilde{x})$$

- Perturbation to image: $\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = \underbrace{K_{j,k}}_{\text{Hermitian operator}} F^A(x)$ → Closer look

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

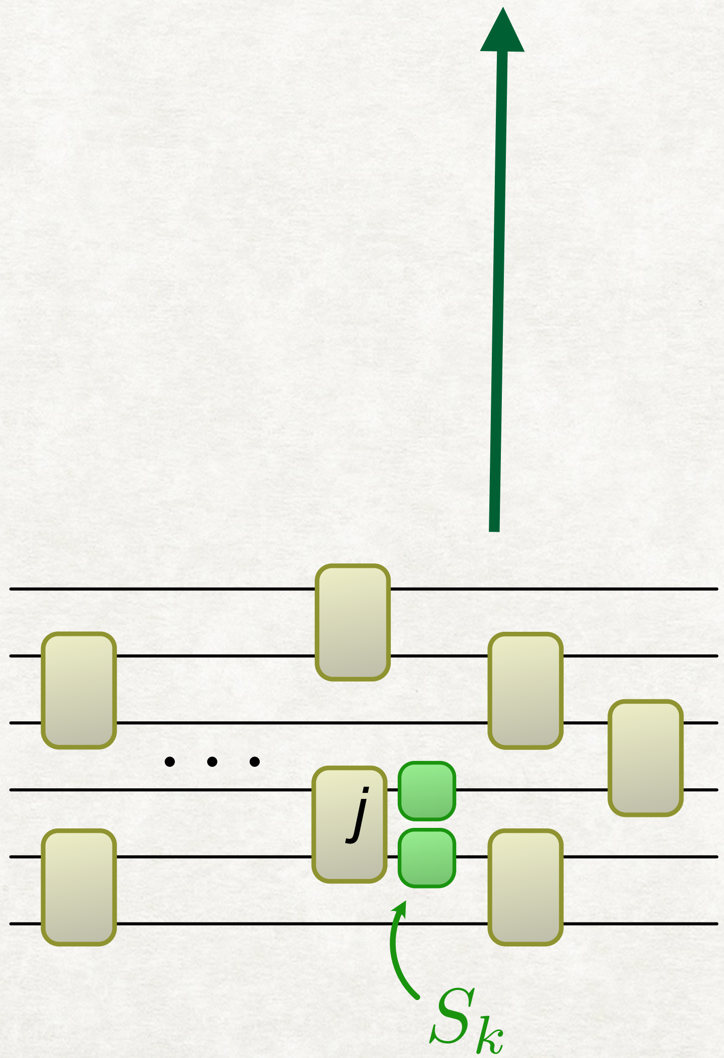
- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0}$$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

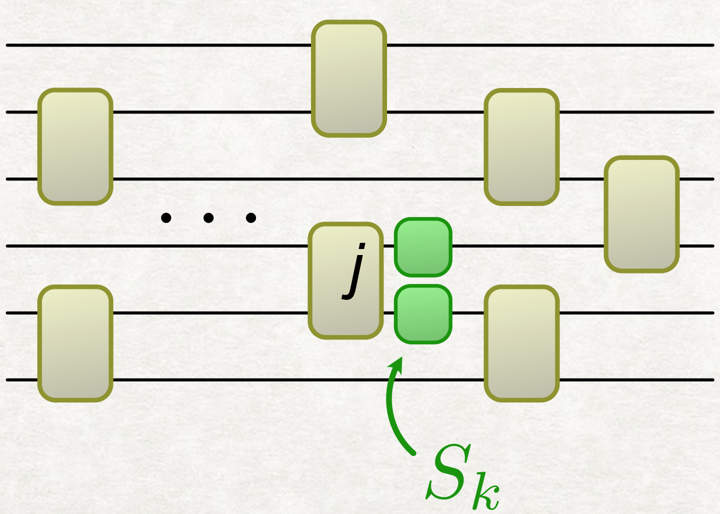
$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0}$$



Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = K_{j,k} F^A(x)$$

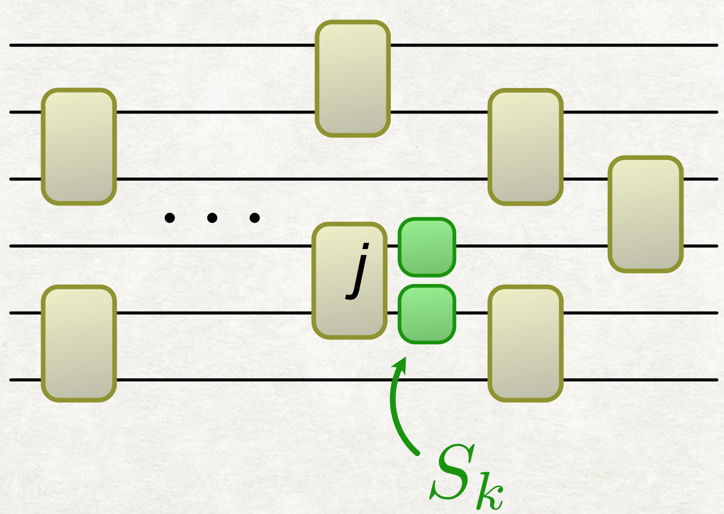


Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = K_{j,k} F^A(x)$$

- Hermitian operator

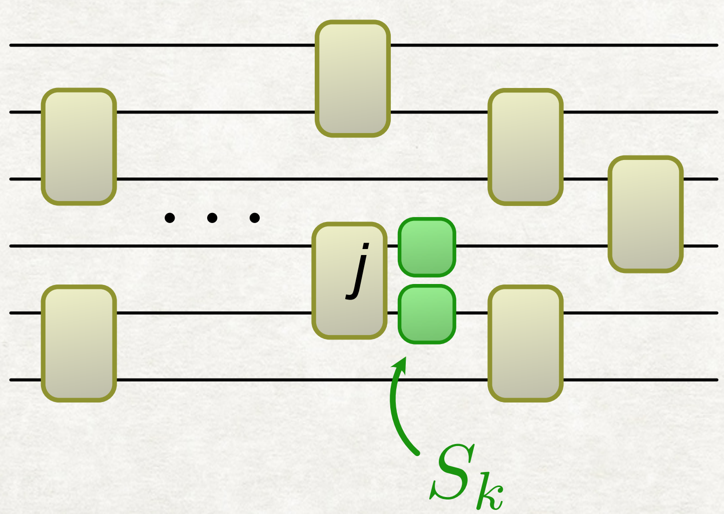


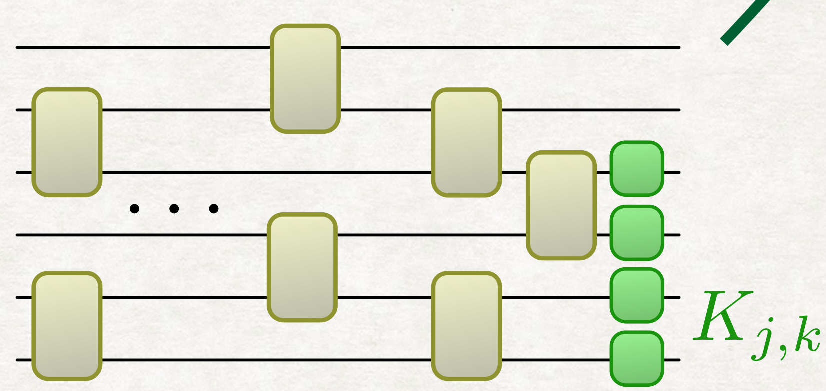
Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k, by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = K_{j,k} F^A(x)$$

- Hermitian operator
- Results from conjugating S_k with the gates that follow U_j



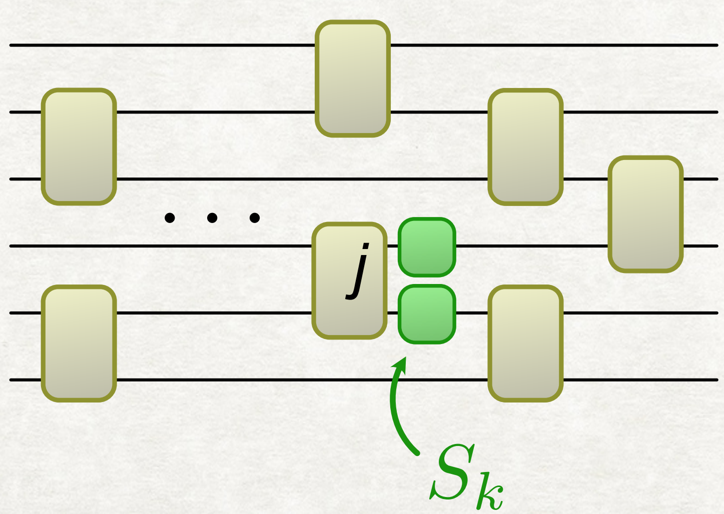


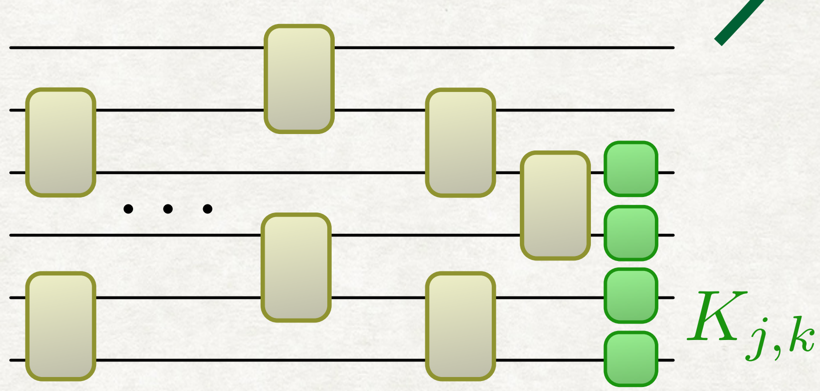
Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- Say that we perturb gate U_j in the "direction" S_k , by an amount $\epsilon_{j,k}$.
- Resulting perturbation to the whole unitary:

$$\partial_{\epsilon_{j,k}} F^A(\tilde{x}) \Big|_{\epsilon_{j,k}=0} = \underbrace{K_{j,k}} F^A(x)$$

- Hermitian operator
- Results from conjugating S_k with the gates that follow U_j
- Direction, in $\mathrm{SU}(2^n)$, in which $F^A(x)$ is perturbed





Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j.
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j .
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j.
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger = e^{i\phi} P'$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j.
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger = e^{i\phi} P'$

$\longrightarrow$ $K_{j,k}$ = (phase) (Pauli string)

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j.
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger = e^{i\phi} P'$

$\longrightarrow$ $K_{j,k}$ = (phase) (Pauli string)

- The Pauli strings are linearly independent.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j .
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger = e^{i\phi} P'$

$\longrightarrow$ $K_{j,k}$ = (phase) (Pauli string)

- The Pauli strings are linearly independent.

$\longrightarrow$ To show that $F^A(x)$ is perturbed in $\geq T$ directions, show that

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- To lower-bound r, choose Clifford gates U_j .
- **Clifford gates**
  - The gates that map the Pauli strings to the Pauli strings (to within a phase)
  - $CPC^\dagger = e^{i\phi} P'$

⟶ $K_{j,k}$ = (phase) (Pauli string)

- The Pauli strings are linearly independent.

⟶ To show that $F^A(x)$ is perturbed in $\geq T$ directions, show that perturbations lead to $\geq T$ different Pauli strings $K_{j,k}$.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks
  - Each block contains a backward light cone.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks
  - Each block contains a backward light cone.
- Assumption: There exists a list x' of Clifford gates

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks
  - Each block contains a backward light cone.
- Assumption: There exists a list x' of Clifford gates such that $F^{A'}$ has a rank $r' \geq T'$ at x'. $\Rightarrow$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks
  - Each block contains a backward light cone.
- Assumption: There exists a list x' of Clifford gates such that $F^{A'}$ has a rank $r' \geq T'$ at x'. $\Rightarrow$
- Perturb x'. $\longrightarrow$ $F^{A'}$ perturbed in $\geq T'$ directions,

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

Recursive argument

- Begin with circuit of architecture A'
  - Consists of $T' < 4^n - 1$ blocks
  - Each block contains a backward light cone.
- Assumption: There exists a list x' of Clifford gates such that $F^{A'}$ has a rank $r' \geq T'$ at x'. $\Rightarrow$
- Perturb x'. $\longrightarrow$ $F^{A'}$ perturbed in $\geq T'$ directions, represented by Pauli strings K'_{j_m, k_m}

$(m = 1, 2, \ldots, T')$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P.
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - <u>Reason</u>: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,

  and choose Clifford gates to fill that block,

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,

  and choose Clifford gates to fill that block,

  such that, if we perturb the augmented set x'' of gates,

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,

  and choose Clifford gates to fill that block,

  such that, if we perturb the augmented set x'' of gates,

  $F^{A''}(x'')$ will spread in that direction.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,
  and choose Clifford gates to fill that block,
  such that, if we perturb the augmented set x'' of gates,
  $F^{A''}(x'')$ will spread in that direction.
  - Proof tools:

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,
  and choose Clifford gates to fill that block,
  such that, if we perturb the augmented set x'' of gates,
  $F^{A''}(x'')$ will spread in that direction.
  - Proof tools: (1) Backward light cone's structure

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,
  and choose Clifford gates to fill that block,
  such that, if we perturb the augmented set x'' of gates,
  $F^{A''}(x'')$ will spread in that direction.
  - Proof tools: (1) Backward light cone's structure
    (2) We can construct a Clifford circuit : $P \mapsto Z_n$.
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{K'_{j_m, k_m}\right\}$ lacks some Pauli string P.
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block, and choose Clifford gates to fill that block, such that, if we perturb the augmented set x'' of gates, $F^{A''}(x'')$ will spread in that direction.
  - Proof tools: (1) Backward light cone's structure

    (2) We can construct a Clifford circuit : $P \mapsto Z_n$.
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- $\left\{ K'_{j_m, k_m} \right\}$ lacks some Pauli string P .
  - Reason: $T' < 4^n - 1$

    $\Rightarrow$ The perturbation fails to spread $F^{A'}(x')$ in some direction.
- We can augment the architecture with another block,
  and choose Clifford gates to fill that block,
  such that, if we perturb the augmented set x'' of gates,
  $F^{A''}(x'')$ will spread in that direction. $\Rightarrow$ A'' has a rank $\geq T' + 1$.
  - Proof tools: (1) Backward light cone's structure
    (2) We can construct a Clifford circuit : $P \mapsto Z_n$.
    - Cleve *et al.*, Quant. Inf. Comp. **16**, 0721 (2016).
      Webb, arXiv:1510.02769 (2015).
      Zhu, Phys. Rev. A **96**, 062336 (2017).

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- We can keep growing (a lower bound on) the contraction map's rank

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- We can keep growing (a lower bound on) the contraction map's rank by appending blocks with backward light cones.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- We can keep growing (a lower bound on) the contraction map's rank by appending blocks with backward light cones.
  - Start at $T' = 0$.

Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- We can keep growing (a lower bound on) the contraction map's rank by appending blocks with backward light cones.
  - Start at $T' = 0$.
  - Augment T times.

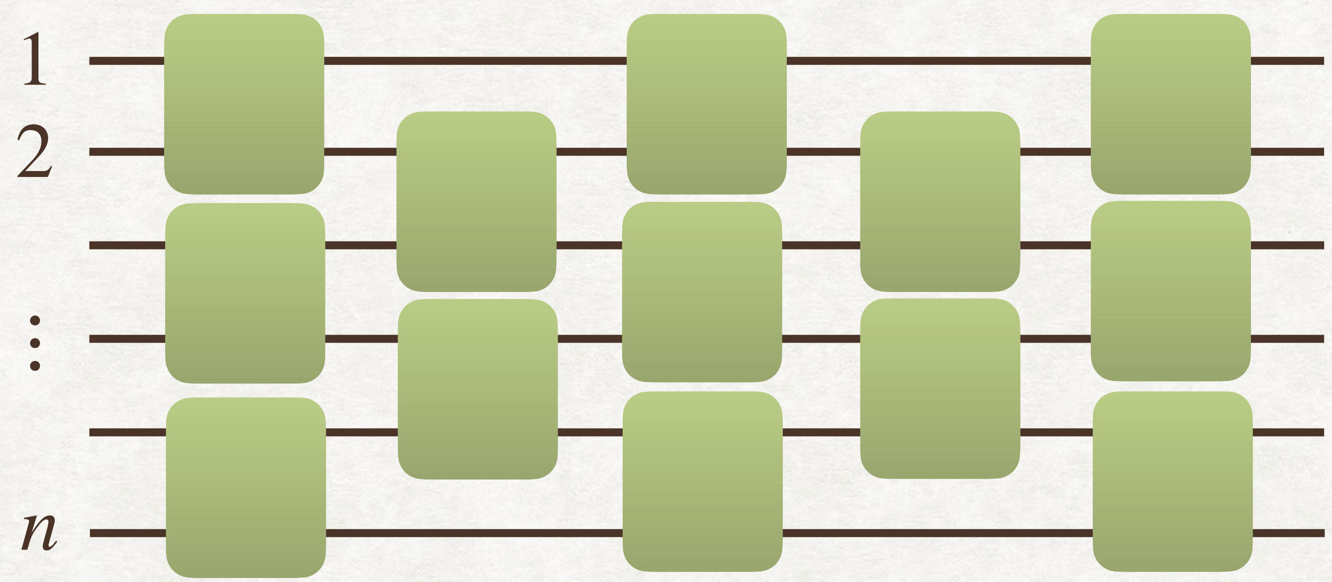
Construction of $x \in \mathrm{SU}(4)^{\times R}$ for which $r \geq T$

- We can keep growing (a lower bound on) the contraction map's rank by appending blocks with backward light cones.
  - Start at $T' = 0$.
  - Augment T times.

$\longrightarrow r \geq T$ 👍

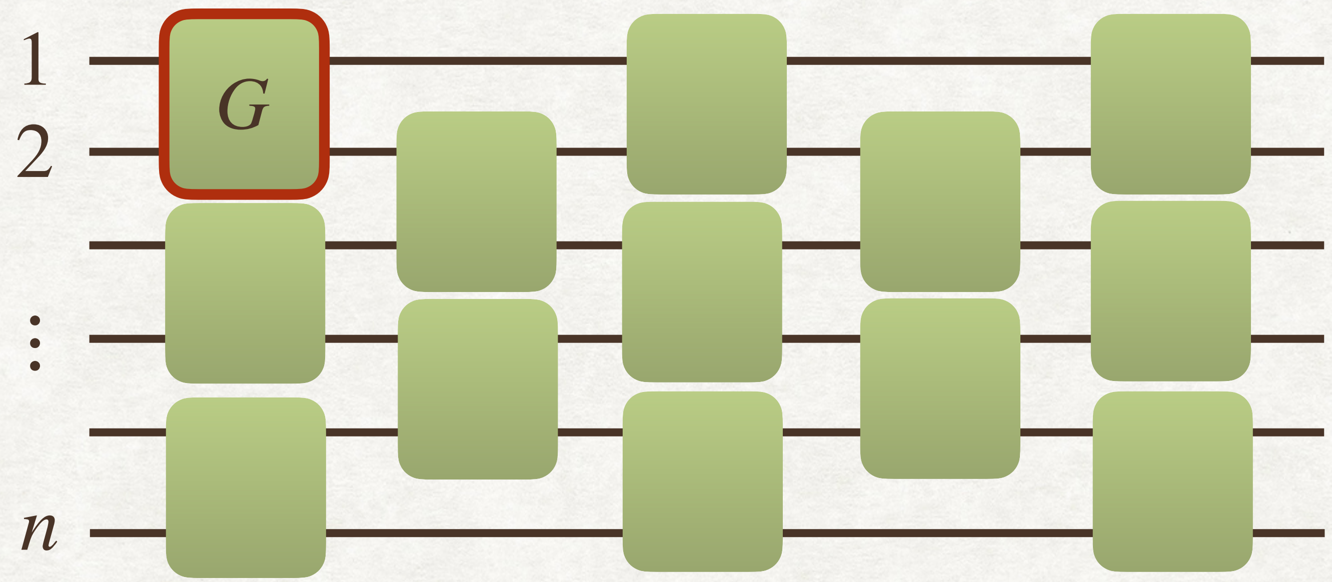
Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

- A = arbitrary n-qubit architecture of R gates



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

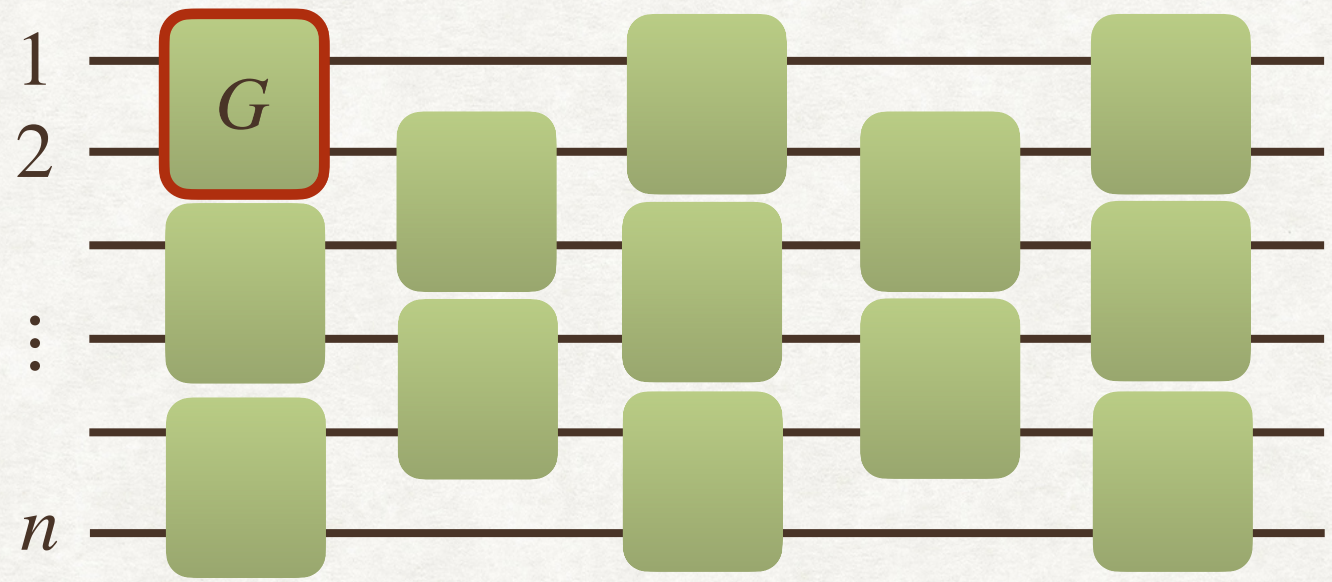
- A = arbitrary n-qubit architecture of R gates
- # of parameters needed to specify one 2-qubit gate



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

- A = arbitrary n-qubit architecture of R gates
- <u># of parameters needed to specify one 2-qubit gate</u>

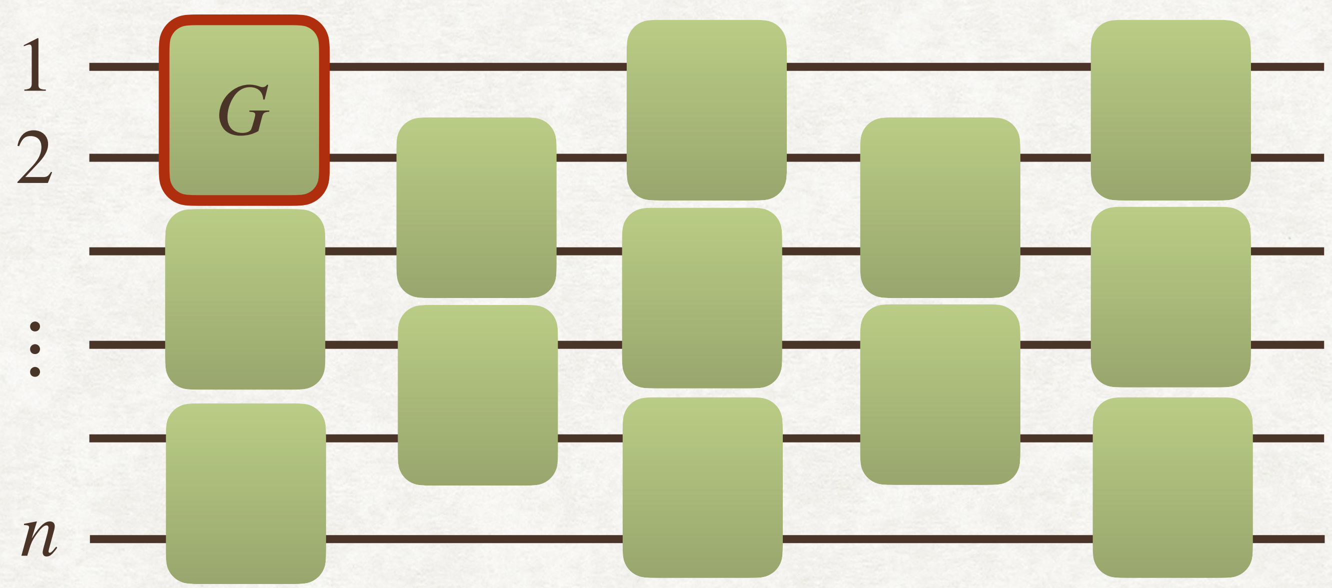
  = dimension of $SU(4)$



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

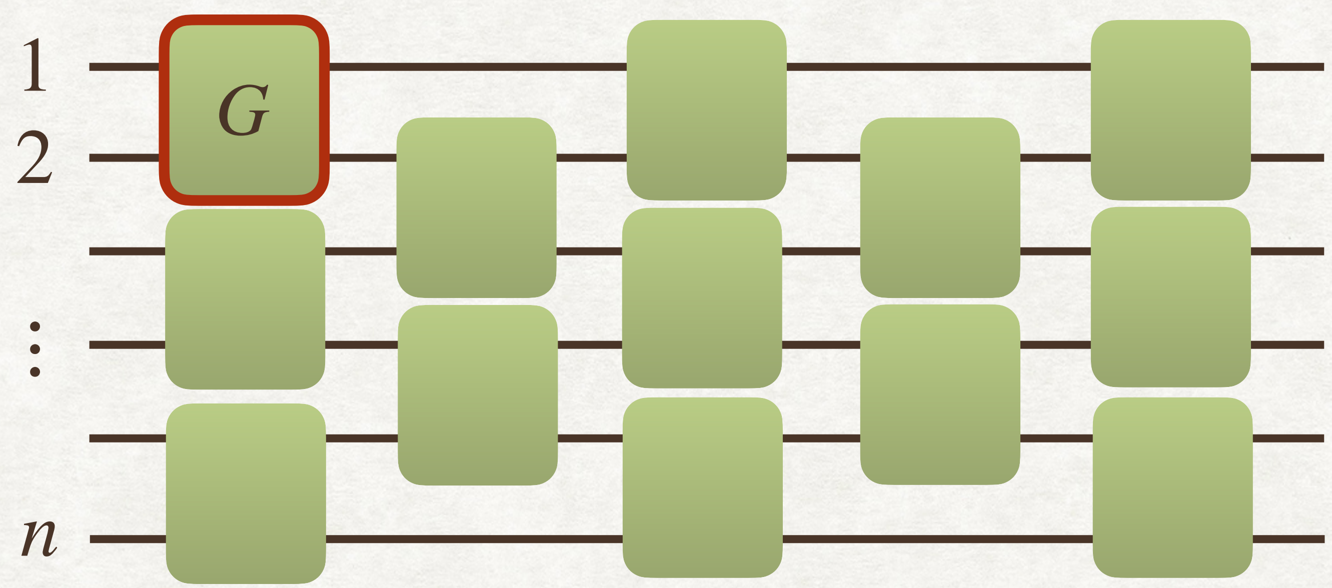
- A = arbitrary n-qubit architecture of R gates
- # of parameters needed to specify one 2-qubit gate

  = dimension of $SU(4)$ $= 4^2 - 1$



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

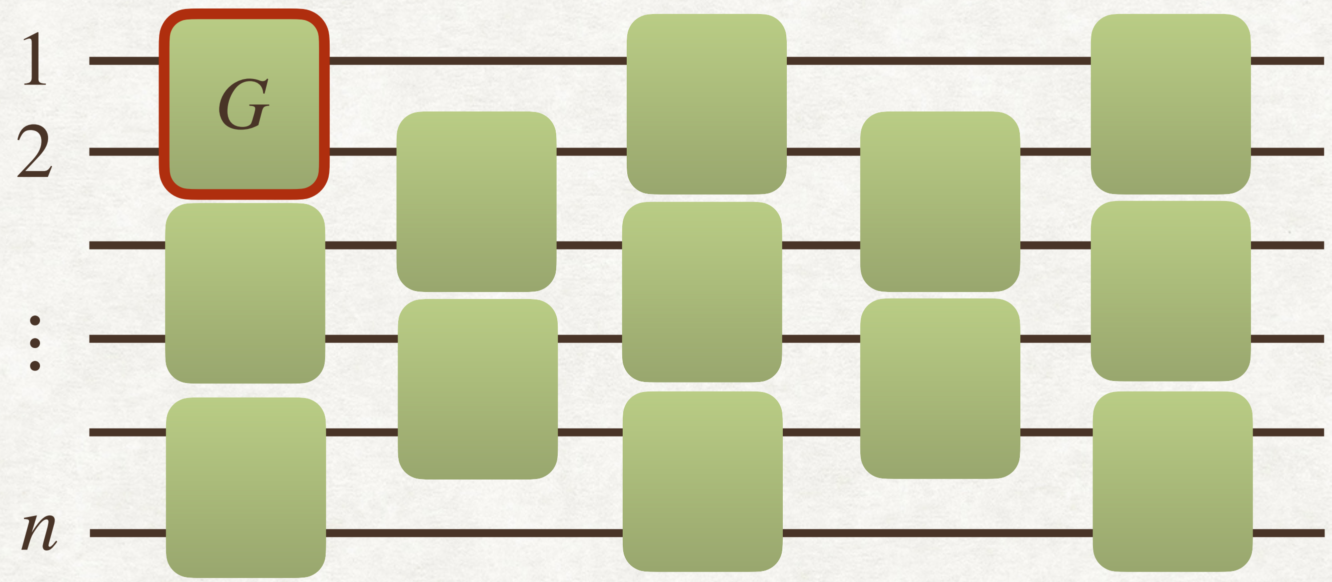
- A = arbitrary n-qubit architecture of R gates
- # of parameters needed to specify one 2-qubit gate
  = dimension of $SU(4) = 4^2 - 1 = 15$



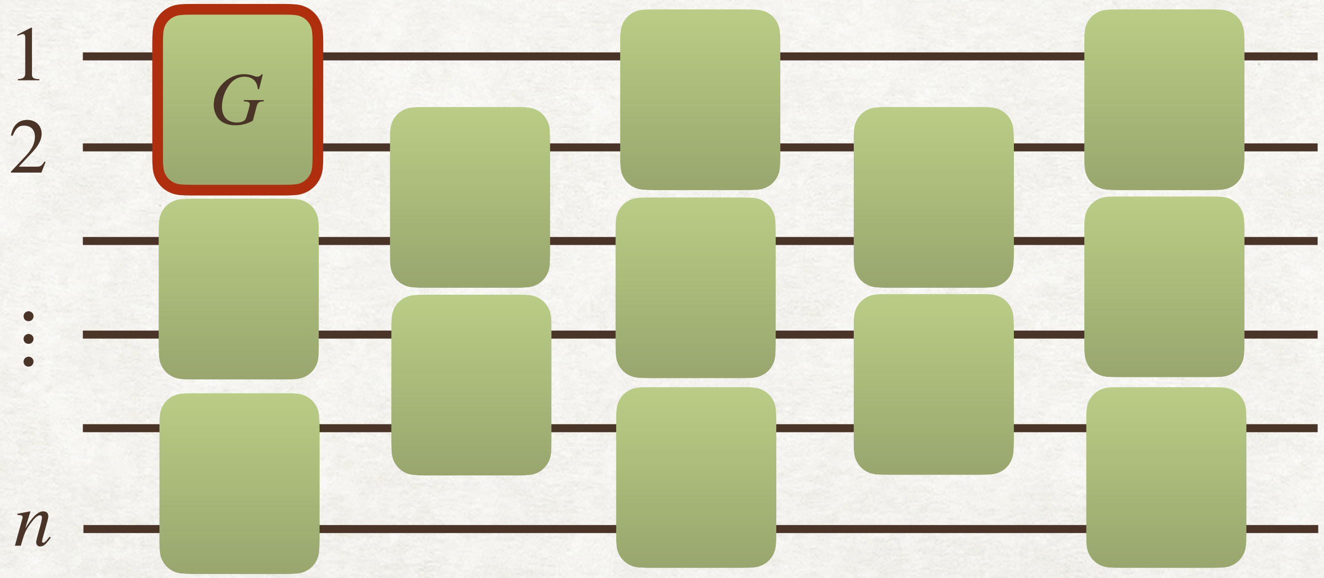
Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

- A = arbitrary n-qubit architecture of R gates
- <u># of parameters needed to specify one 2-qubit gate</u>
  = dimension of $SU(4) = 4^2 - 1 = 15$

$\longrightarrow$ <u>Naïve guess</u>: # of parameters needed to specify circuit $= 15R$

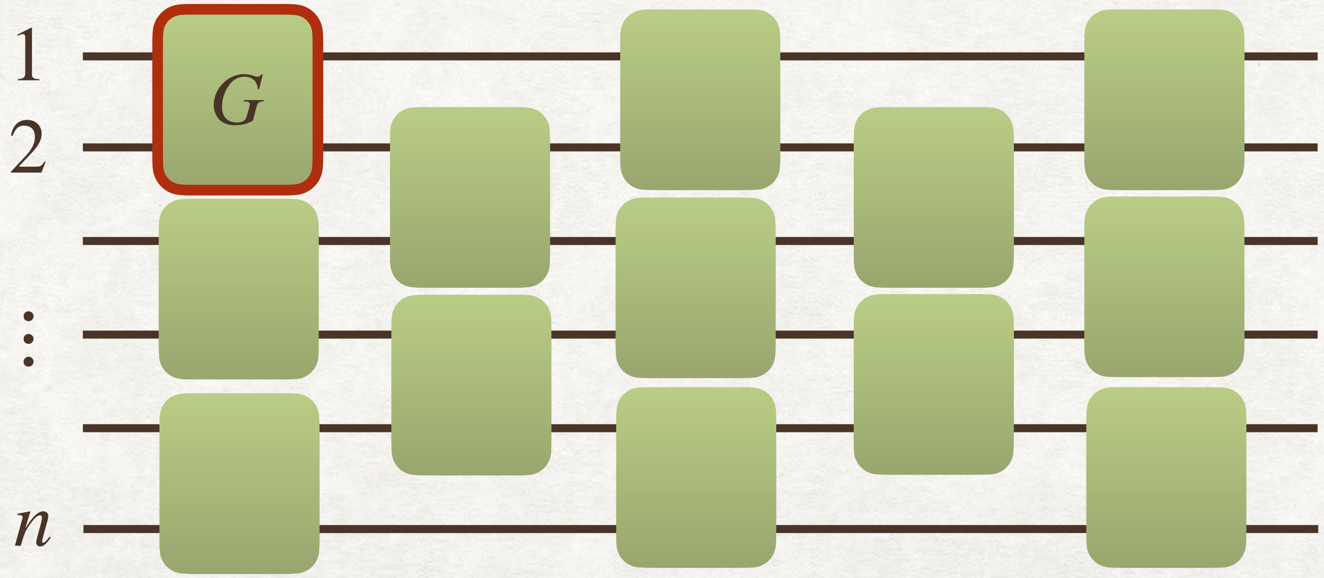


Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

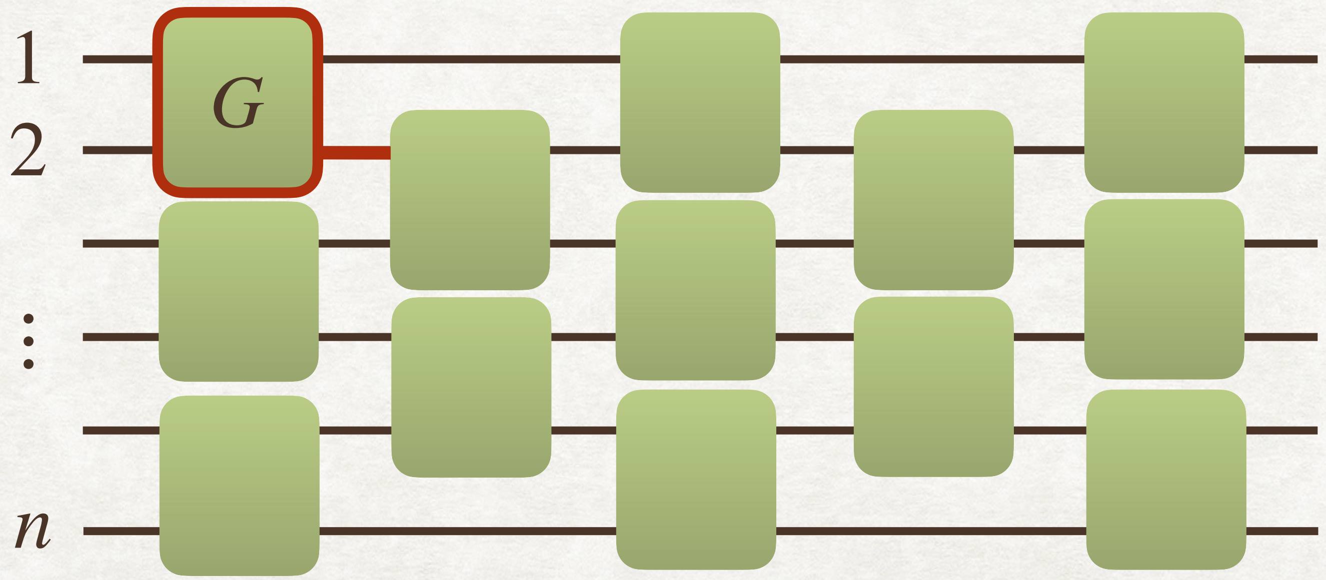
This set of parameters contains redundancies.



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

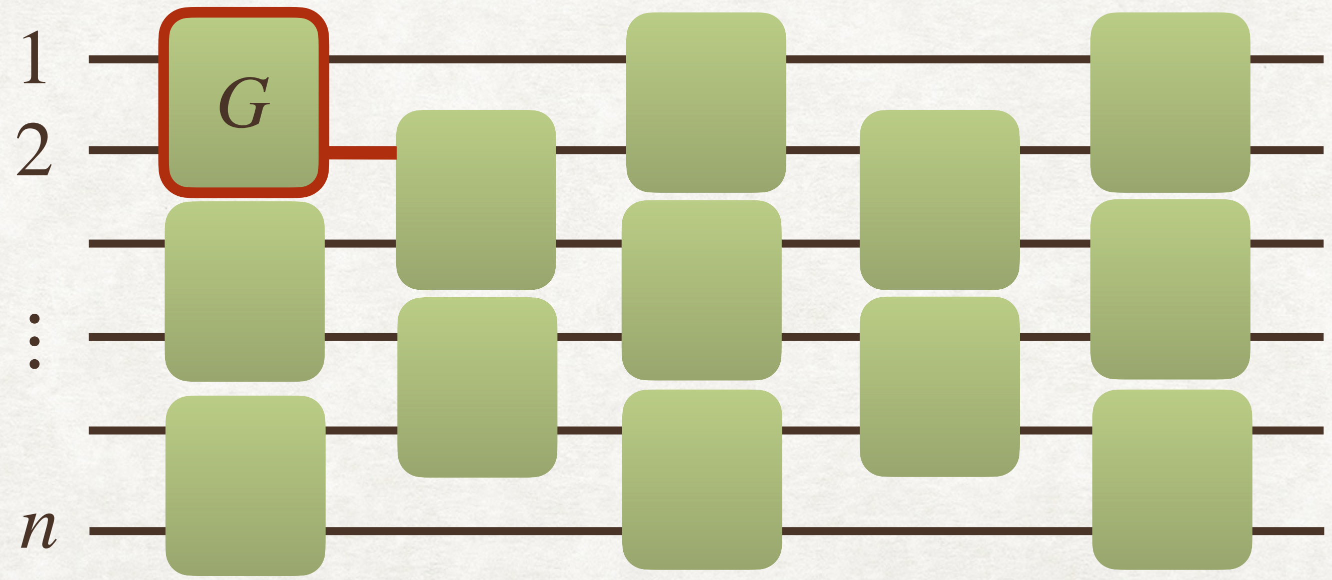
- This description of G includes a rotation of qubit 2



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

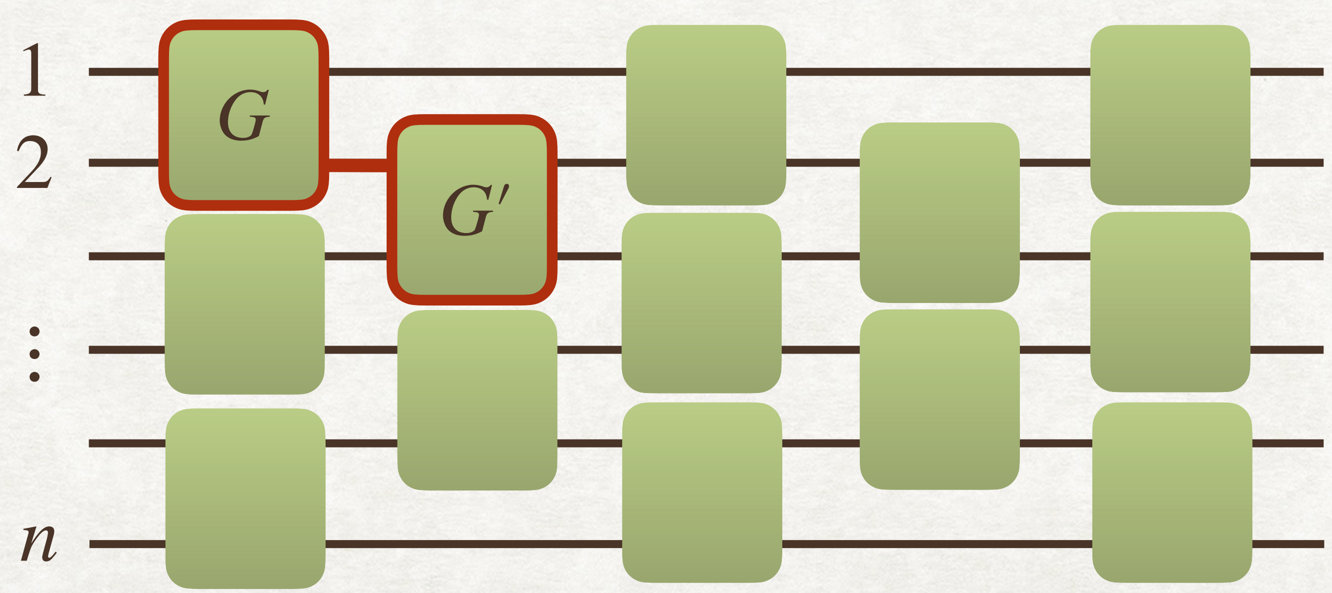
- This description of G includes a rotation of qubit 2 ⟶ 3 parameters



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

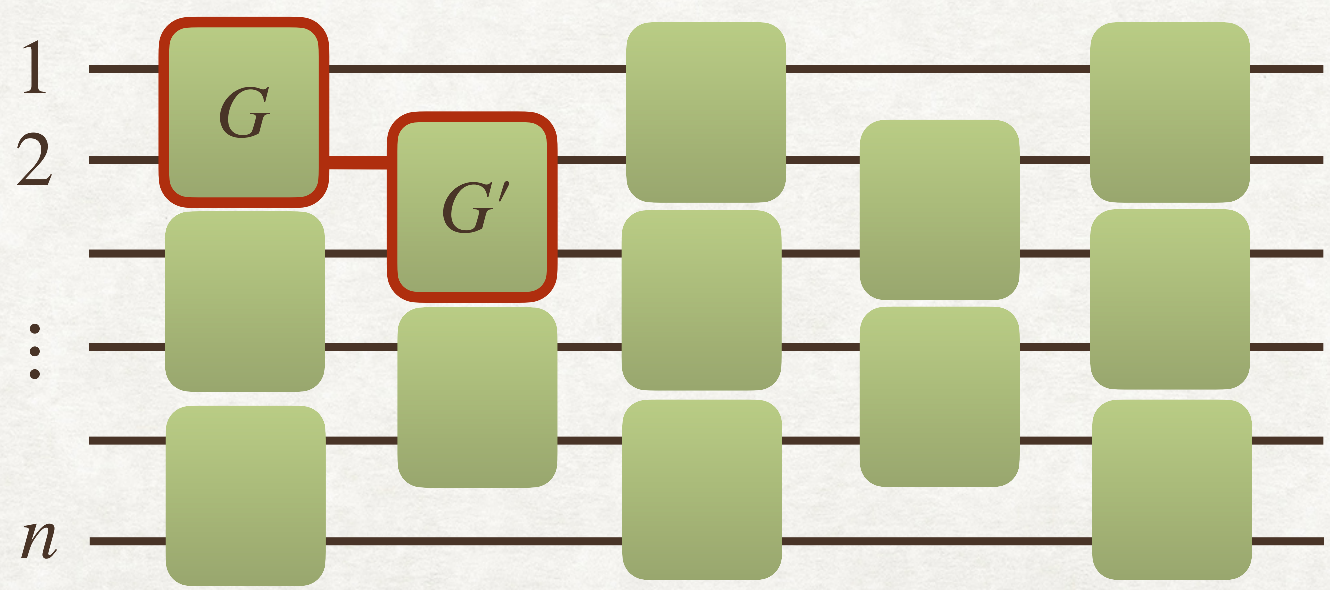
- This description of G includes a rotation of qubit 2 $\longrightarrow$ 3 parameters
- This description of G' includes a rotation of qubit 2



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

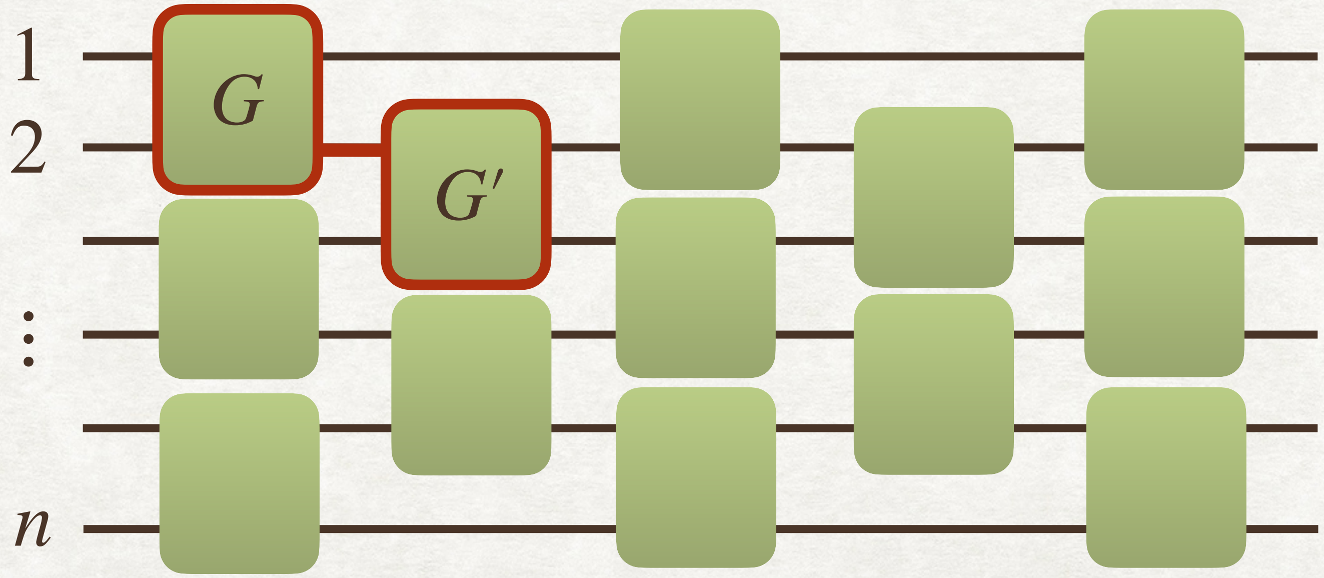
- This description of G includes a rotation of qubit 2 $\longrightarrow$ 3 parameters
- This description of G' includes a rotation of qubit 2 $\longrightarrow$ another 3 parameters



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- This description of G includes a rotation of qubit 2 ⟶ 3 parameters
- This description of G' includes a rotation of qubit 2 ⟶ another 3 parameters
- (1st rotation) * (2nd rotation) = just 1 rotation

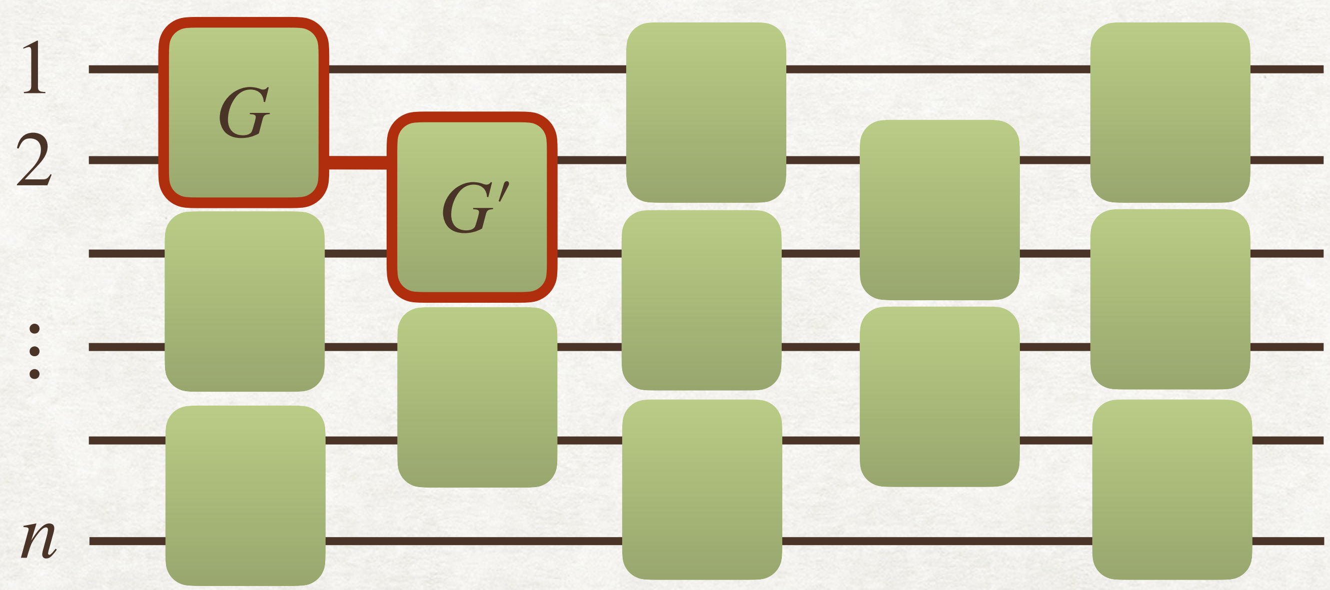


Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- This description of G includes a rotation of qubit 2 $\longrightarrow$ 3 parameters
- This description of G' includes a rotation of qubit 2 $\longrightarrow$ another 3 parameters
- (1st rotation) * (2nd rotation) = just 1 rotation

$\therefore$ We're describing just 1 rotation of qubit 2 with 6 parameters

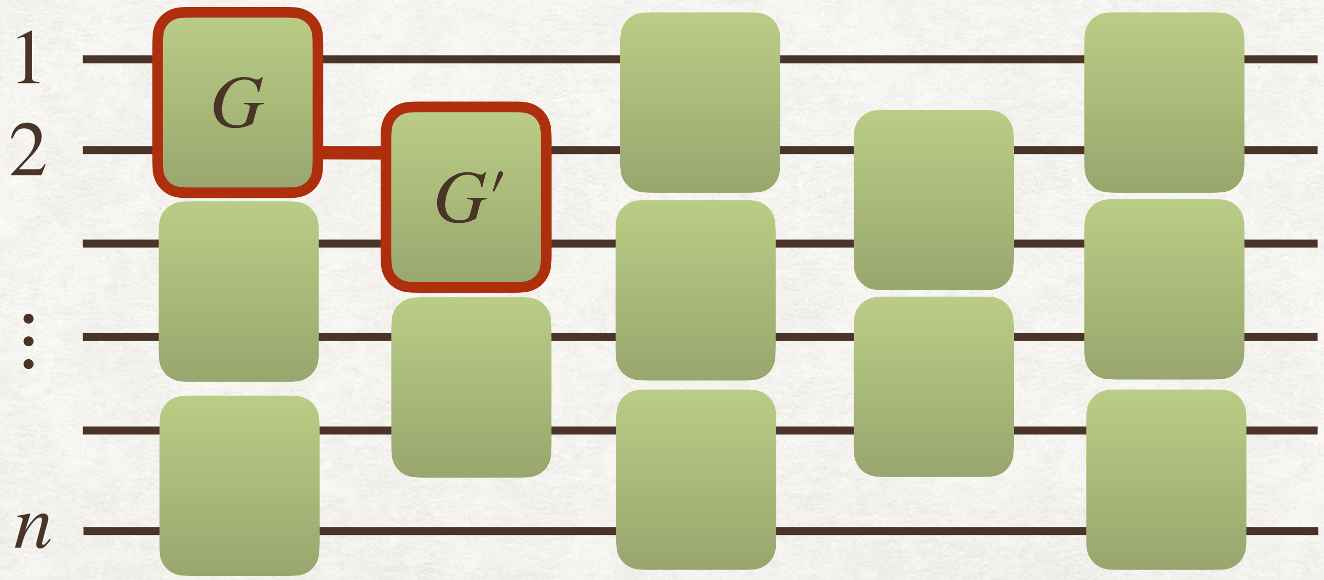


Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- This description of G includes a rotation of qubit 2 $\longrightarrow$ 3 parameters
- This description of G' includes a rotation of qubit 2 $\longrightarrow$ another 3 parameters
- (1st rotation) * (2nd rotation) = just 1 rotation

$\therefore$ We're describing just 1 rotation of qubit 2 with 6 parameters $\longrightarrow$ 3 parameters more than necessary



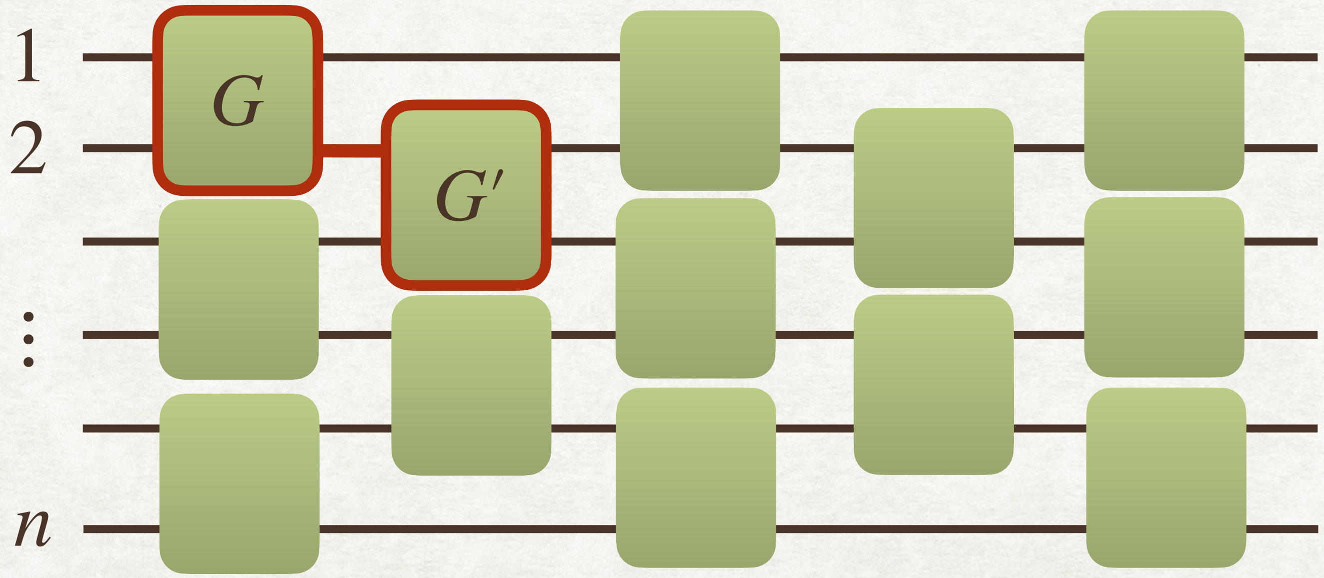
Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- This description of G includes a rotation of qubit 2 $\longrightarrow$ 3 parameters
- This description of G' includes a rotation of qubit 2 $\longrightarrow$ another 3 parameters
- (1st rotation) * (2nd rotation) = just 1 rotation

$\therefore$ We're describing just 1 rotation of qubit 2 with 6 parameters $\longrightarrow$ 3 parameters more than necessary

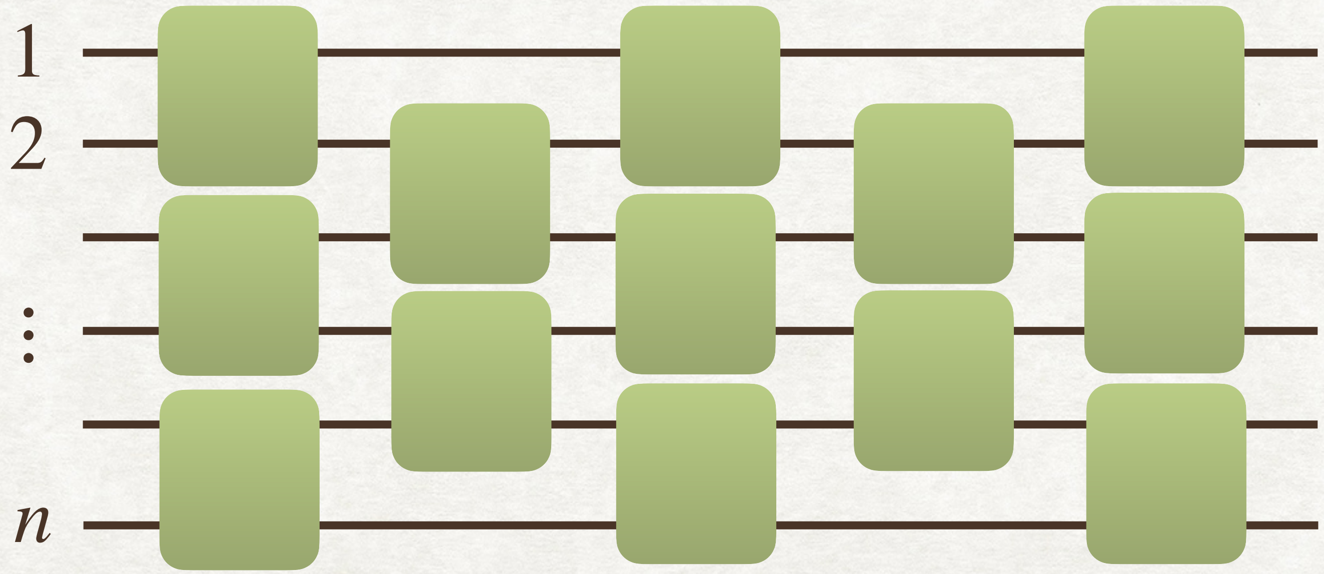
$\therefore$ Subtract off 3 parameters per shared qubit



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

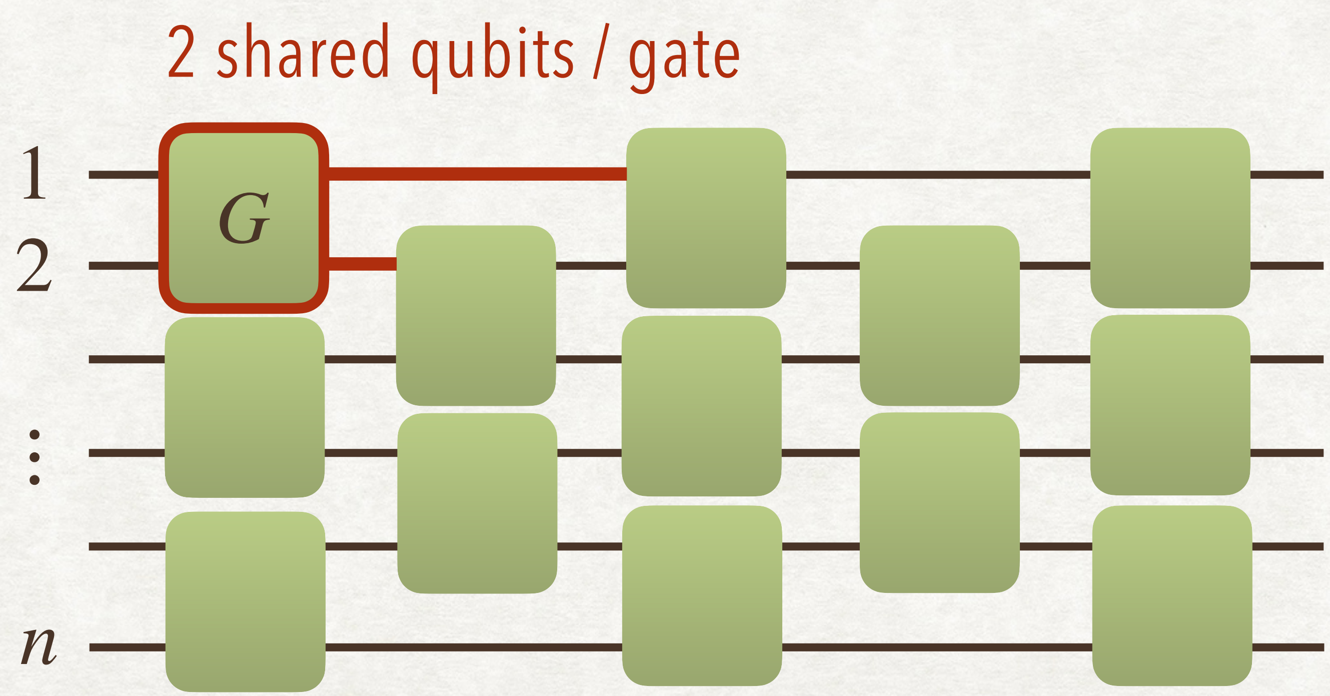
- # of shared qubits



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

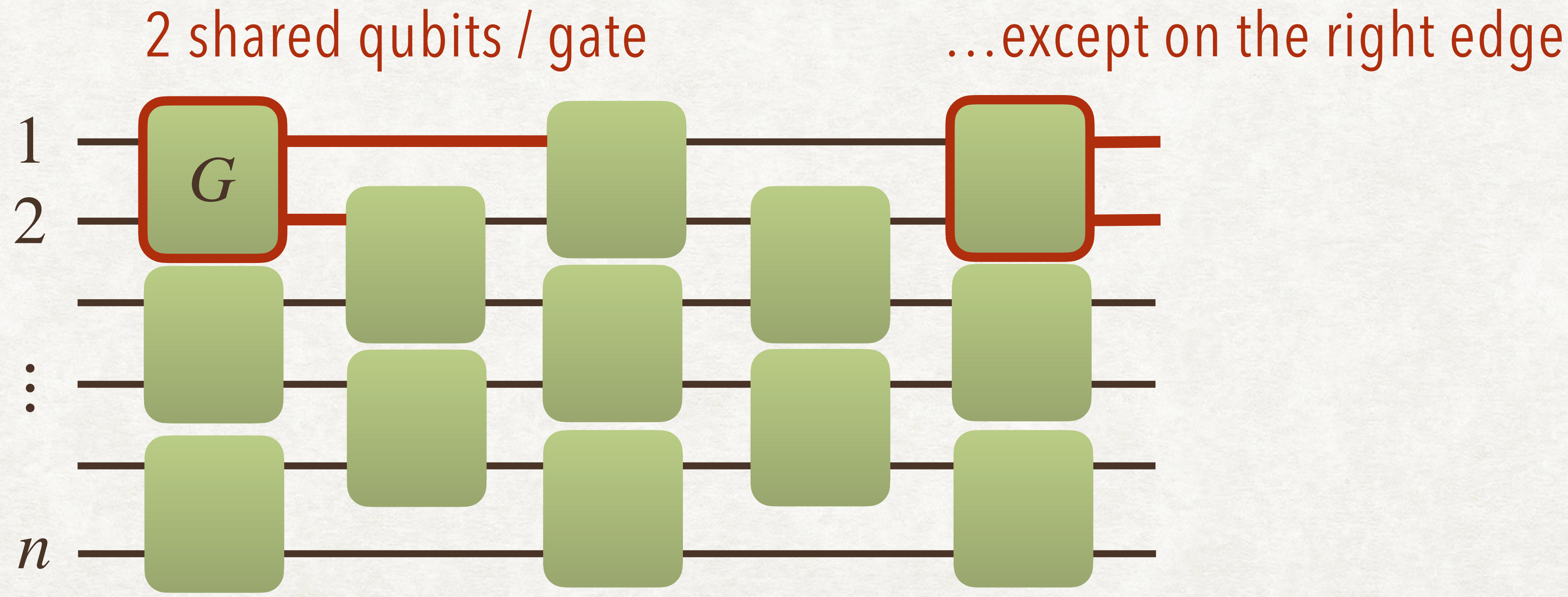
- # of shared qubits



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

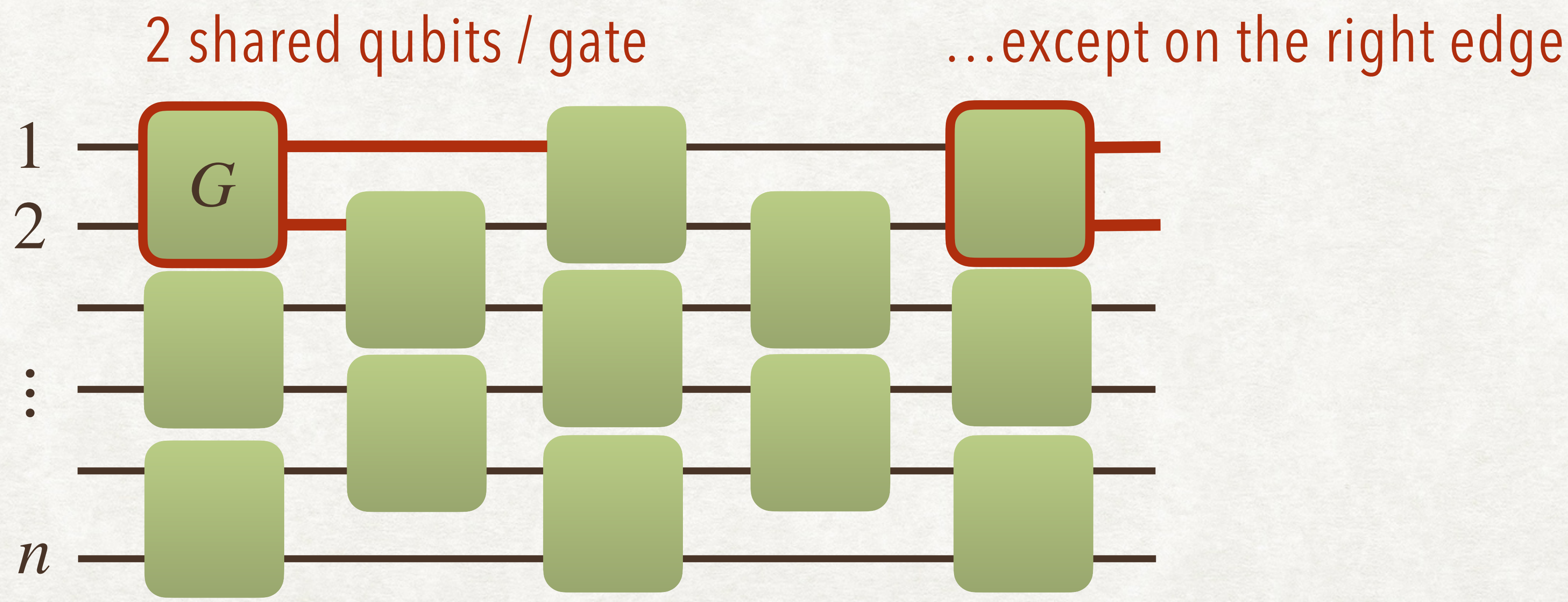
- # of shared qubits



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- # of shared qubits $= 2$ (# gates) -2 (# gates on right-hand boundary)

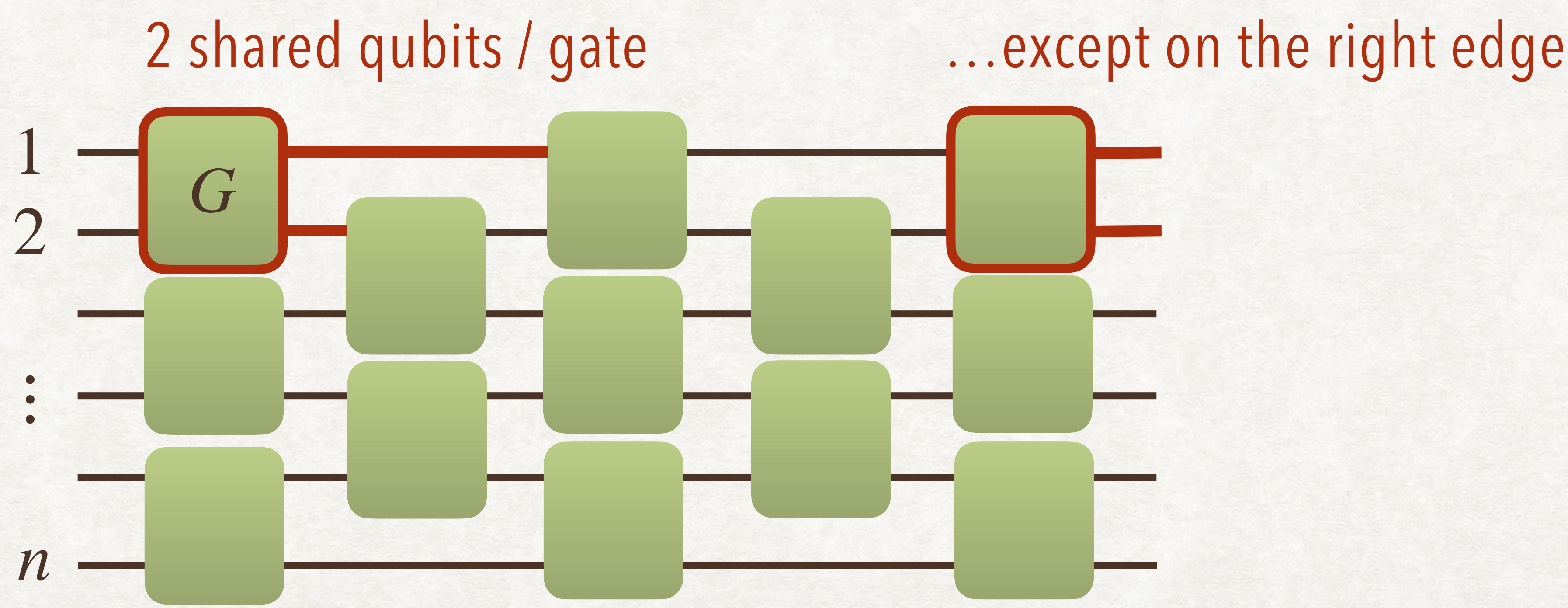


Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- \# of shared qubits $= 2$ (# gates) -2 (# gates on right-hand boundary)

$$= 2R - 2(n/2)$$



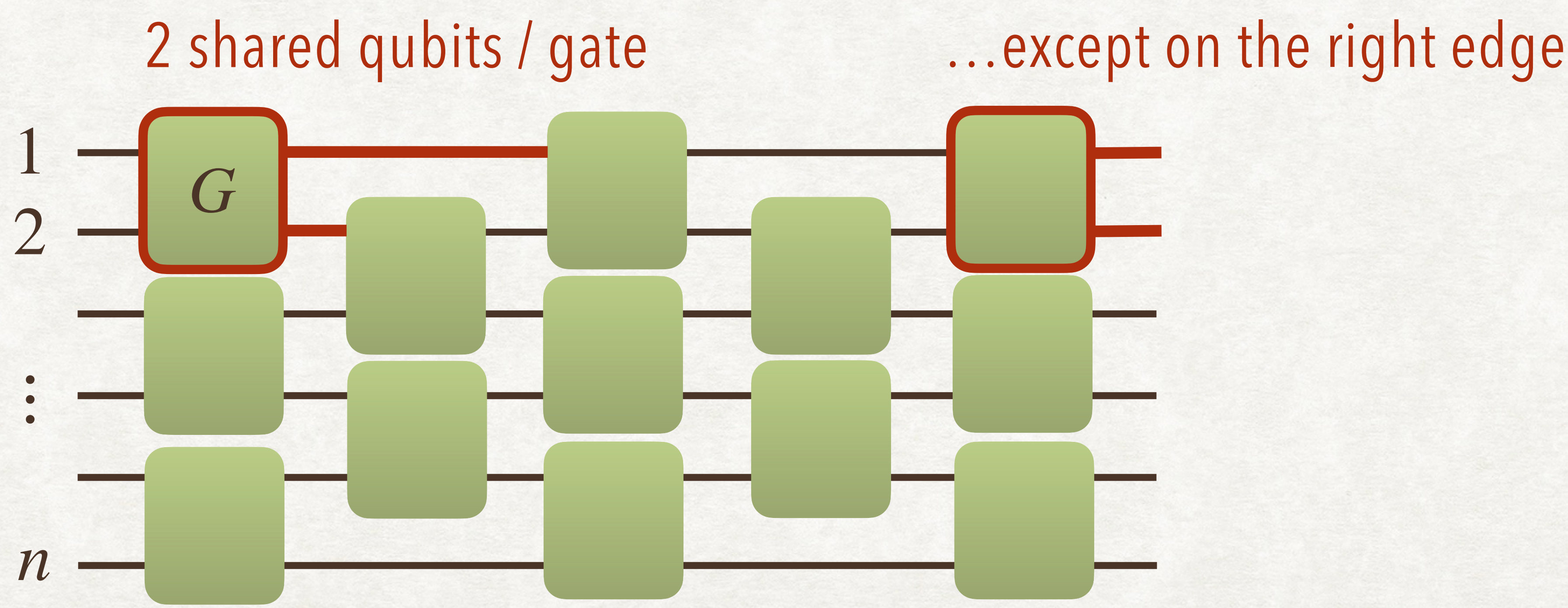
Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

This set of parameters contains redundancies.

- # of shared qubits $= 2$ (# gates) -2 (# gates on right-hand boundary)

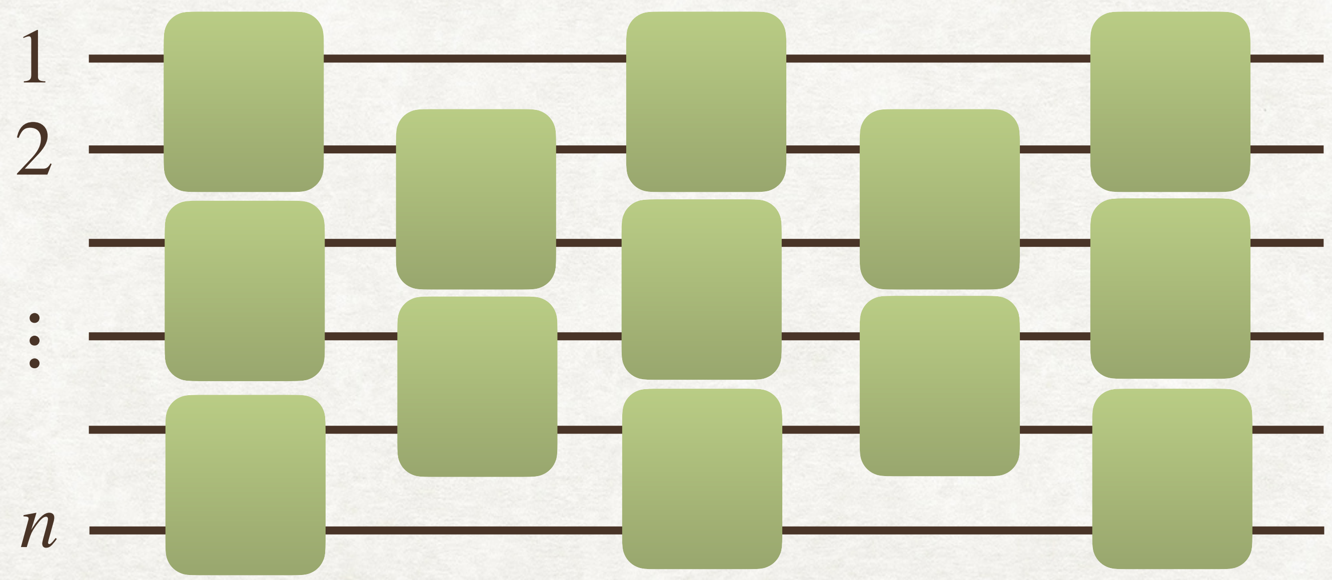
$$= 2R - 2(n/2)$$

$$= 2R - n$$



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

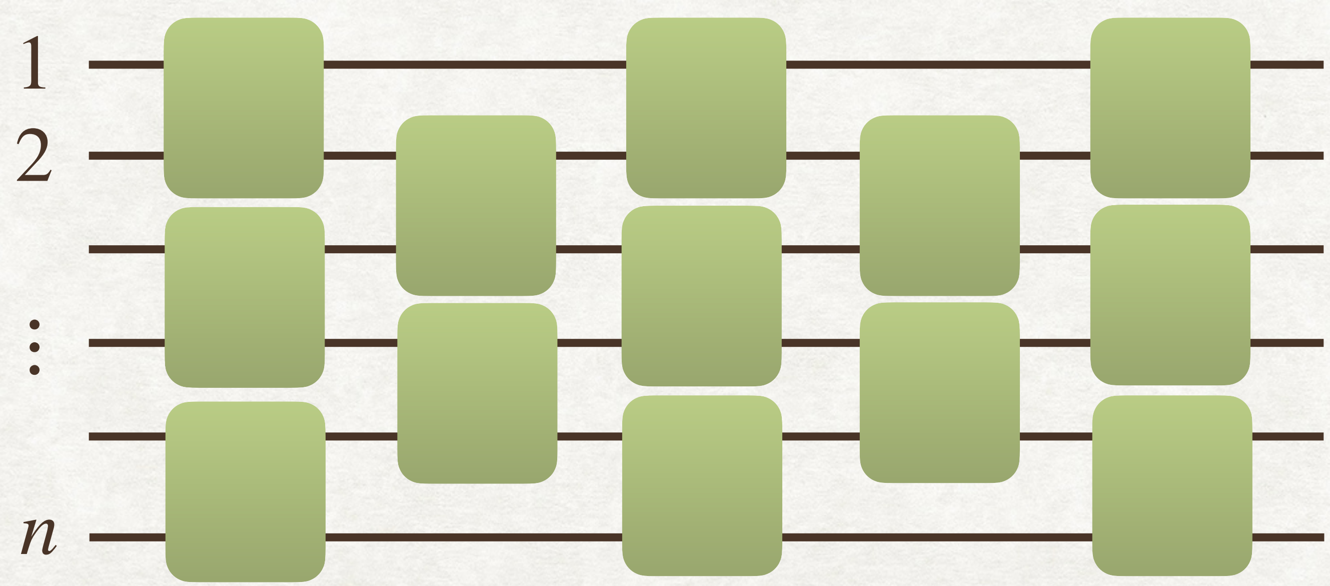
$\therefore$ # of parameters needed to specify circuit



Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

$\therefore$ # of parameters needed to specify circuit

$\leq$ (naïve guess) − 3 (# shared qubits)

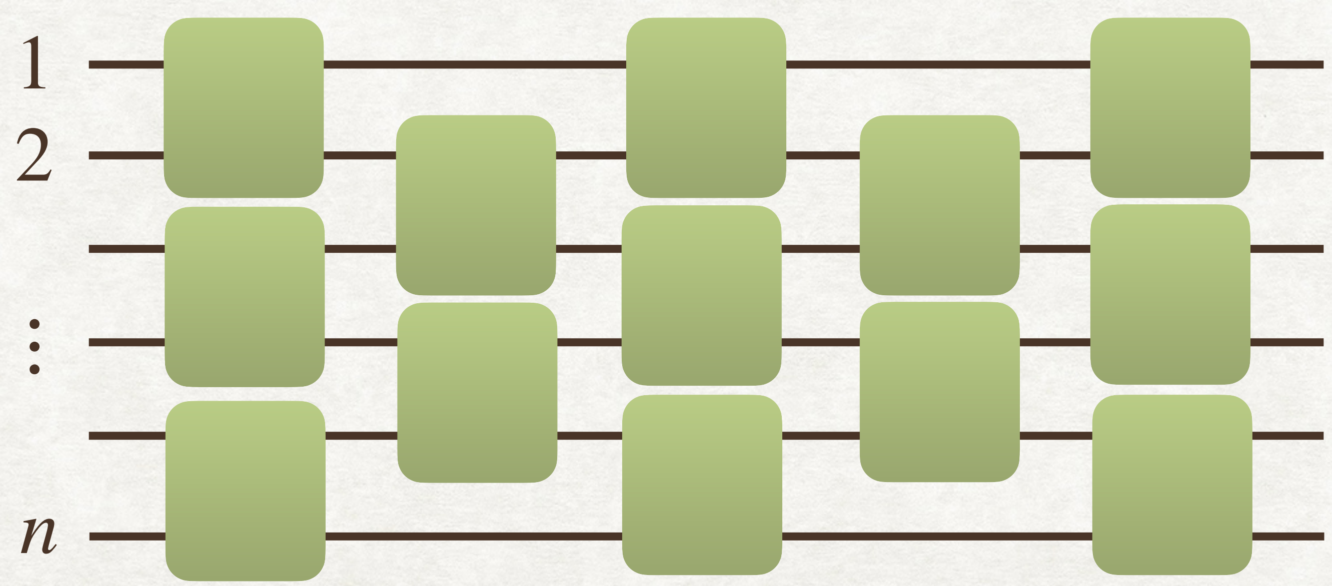


Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

$\therefore$ # of parameters needed to specify circuit

$\leq$ (naïve guess) − 3 (# shared qubits)

$= 15R - 3(2R - n)$



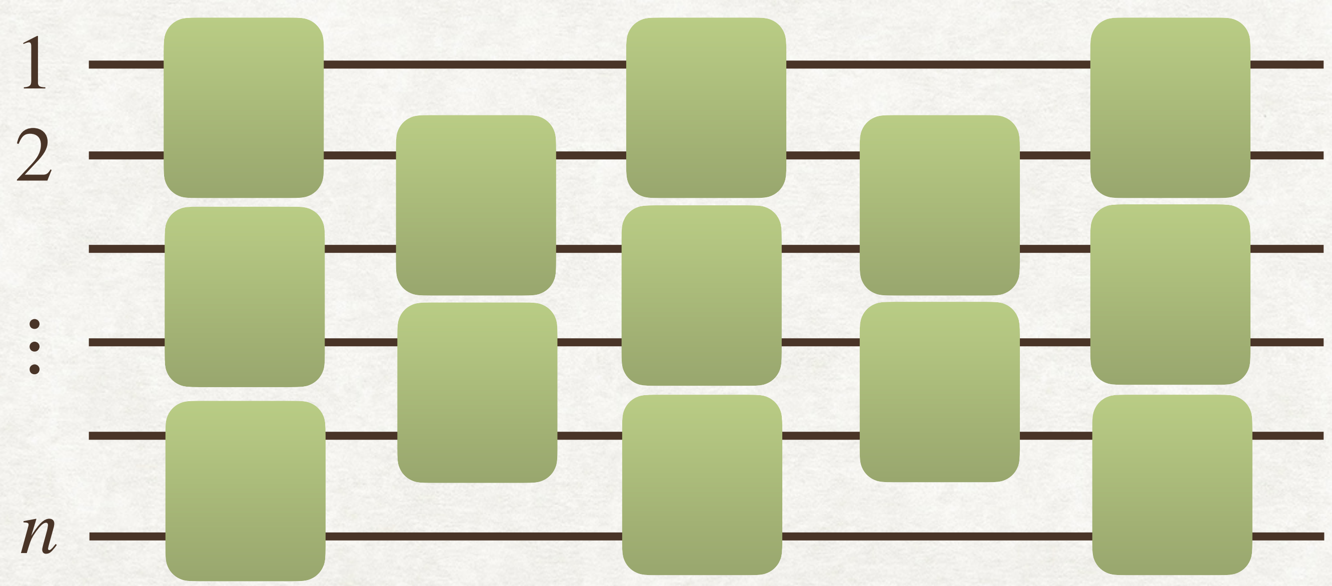
Proof of upper bound on accessible dimension, $d_A \leq 9R + 3n$

$\therefore$ # of parameters needed to specify circuit

$\leq$ (naïve guess) − 3 (# shared qubits)

$= 15R - 3(2R - n)$

$= 9R + 3n$



Proof of upper bound on accessible dimension, $d_A \le 9R + 3n$

$\therefore$ # of parameters needed to specify circuit

$\le$ (naïve guess) − 3 (# shared qubits)

$= 15R - 3(2R - n)$

$= 9R + 3n$ ✓

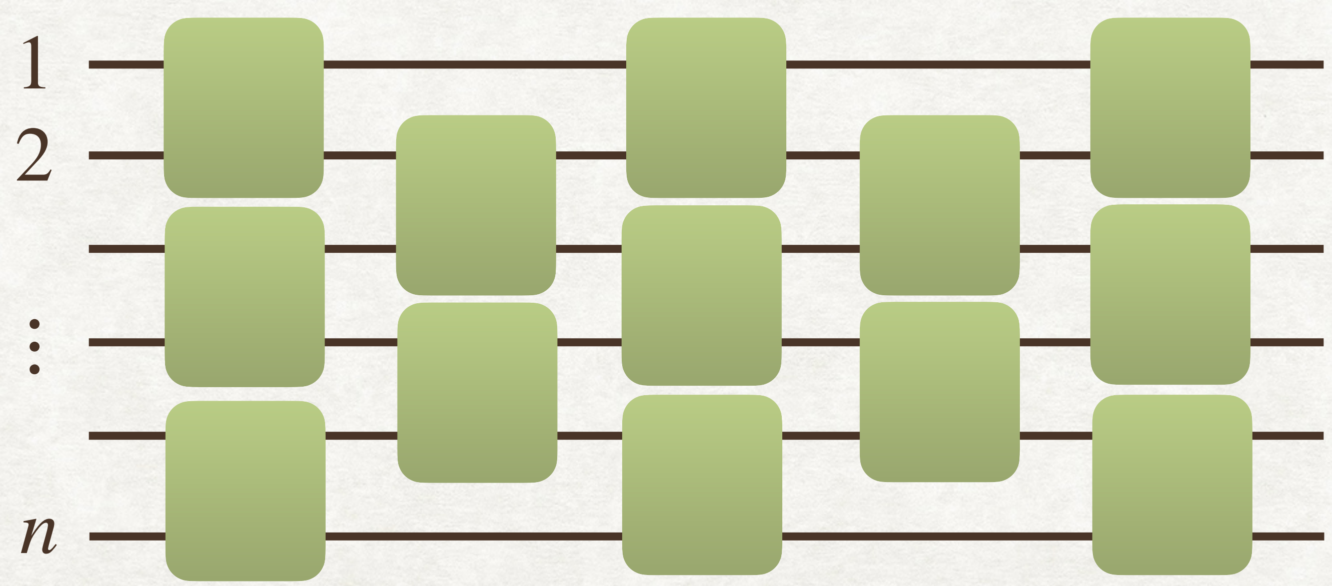


Image sources

- Knot: https://falkonry.com/blog/historical-data-the-gordian-knot-of-machine-learning/
- Mary, Mary: https://www.catsmeow.com/products/new-mother/mary-mary-quite-contrary
- Home: https://icon-icons.com/icon/house/99129
- *Hamilton* set: https://www.pinterest.de/pin/569072102906184687/
- Not-so-fast sloth: https://www.teepublic.com/sticker/2782891-not-so-fast
- Opportunity: https://www.moodyonthemarket.com/cornerstone-alliance-publishes-opportunity-zone-prospectus-for-potential-projects/
- Complexity (“Thanks” slide): https://www.facebook.com/complexandchaos/
- Emptying glass: https://www.istockphoto.com/photos/half-full-glass

Proof of lower bound on accessible dimension, $d_A \geq T$

- $r_{\max}$ = greatest rank achieved by F^A at any $x \in \mathrm{SU}(4)^{\times R}$
- $E_{r_{\max}}$ = locus of points x where F^A achieves rank $r_{\max}$
- $E_{<r_{\max}}$ = locus of points x where F^A achieves rank $< r_{\max}$
- Lemma: $E_{<r_{\max}}$ is an algebraic set of measure 0.

$\Leftrightarrow \quad E_{r_{\max}}$ is an open, measure-1 set.

$\Rightarrow \quad$ Accessible dimension = rank: $d_A = r_{\max}$.