Packing and covering rainbow spanning trees for small color classes

CCCIC 2021

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Joint work with Tomáš Kaiser and Matthias Kriesell

Factorization of matroids in rainbow bases

Problem setting

Given k matroids $M_1, \ldots, M_k$ on a common ground set S, can we partition S in subsets $S_1, \ldots, S_t$ such that S_j is a basis of M_i for $i = 1, \ldots, k$ and $j = 1, \ldots, t$?

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Included problems

- Arborescences in digraphs,
- Matchings in bipartite graphs,
- Rota's basis conjecture.

Recent developpements

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A direct sum of uniform matroids (of rank 1) is a (unitary) partition matroid.

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Theorem (Bérczi, Schwarcz, 2020)

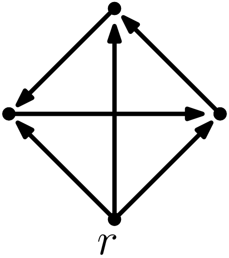
The above problem is difficult in two ways.

Factorization in arborescences

Theorem(Edmonds, 1975)

A digraph D can be factorized in k spanning r-arobrescences if and only if

- its underlying graph can be factorized in k spanning trees,
- $d_D^-(r) = 0$ and $d_D^-(v) = k$ for all $v \in V(D) - r$.



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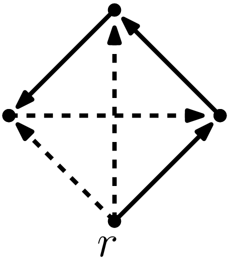

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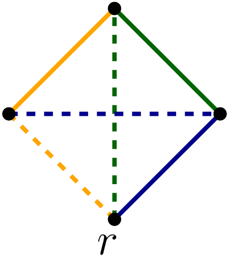
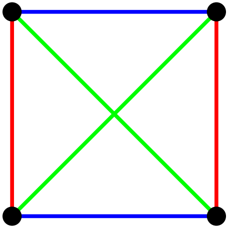


Connection

Special case of rainbow spanning tree factorization !

Complexity results



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- Deciding whether a digraph can be factorized in spanning trees of bounded in-degree is NP-hard.
- This answers a question of Frank.

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Observation

All negative instances occur when the size of some color classes equals the number of bases.

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Let G be a k-multiple tree with a $(k-1)$-bounded coloring for some positive integer k. Then G can be factorized in k rainbow spanning trees.

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There is an integer k such that every k-multiple tree with a 2-bounded coloring can be factorized in k spanning trees one of which is rainbow.

Approximative results

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Factorizing in rainbow spanning trees can be read in 3 different ways :

- packing rainbow spanning trees,
- covering by rainbow forests,
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Let G be a k-multiple tree for some $k \geq 4$ with some 2-bounded coloring. Then G can be covered by $4k$ rainbow spanning trees.

Proof preparation

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Let T be a tree with a 2-bounded coloring. Then T can be factorized in two rainbow forests.

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Lemma (Broersma, Li, 1997)

Let G be a 2-multiple tree with a 2-bounded coloring. Then G contains a rainbow spanning tree.

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- Let G be a k-multiple tree with a 2-bounded partition,

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- Let G be a k-multiple tree with a 2-bounded partition,
- let X be a rainbow forest in G,
- the partner of some $x \in X$ is the edge with the same color.

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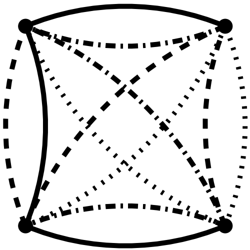

Proof steps

- Let $T_1, \ldots, T_4$ be 4 edge-disjoint spanning trees in G,

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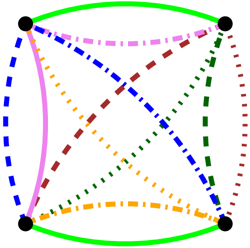

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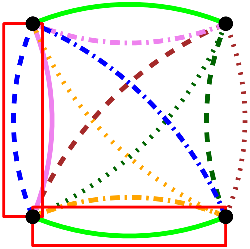

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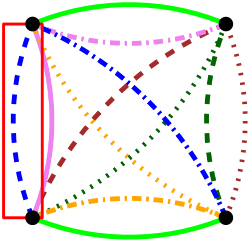

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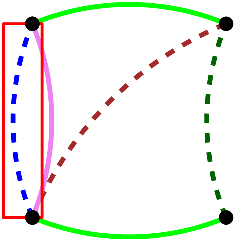

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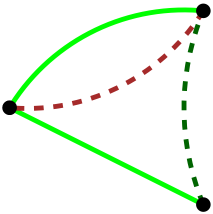

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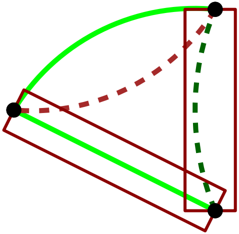

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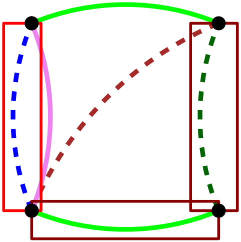

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- G'' contains a rainbow spanning tree T by the above lemma,
- $T \cup X_1$ forms a rainbow spanning tree in G.

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$k = 3$

A similar proof leads to a slightly weaker bound.

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Theorem

Every 2-multiple tree with a 2-bounded coloring can be covered by $O(\log(|V(G)|))$ rainbow spanning trees.

Thank You !