Linear and quasi-linear fractional operators in Lipschitz domains

regularity and approximation

Juan Pablo Borthagaray
Universidad de la República, Uruguay

Inverse Problems for Anomalous Diffusion Processes
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Fractional Laplacian

- **Definition:** the fractional Laplacian of order $s \in (0,1)$ of $u : \mathbb{R}^d \to \mathbb{R}$ is

$$(-\Delta)^s u(x) = C(d,s) \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+2s}}\, dy, \quad C(d,s) = \frac{2^{2s} s\Gamma(s+\frac{d}{2})}{\pi^{d/2}\Gamma(1-s)}.$$

- **Fourier transform:** definition above is equivalent to

$$\mathcal{F}\left((-\Delta)^s u\right)(\xi) = |\xi|^{2s}\mathcal{F}u(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

- **Problem:** let $\Omega \subset \mathbb{R}^d$ be open with Lipschitz boundary and $f \colon \Omega \to \mathbb{R}$,

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c = \mathbb{R}^d \setminus \overline{\Omega}. \end{cases}$$

- **'Boundary' conditions:** imposed in $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega}$.

  (This is the so-called *integral*, *Riesz* or *restricted* fractional Laplacian on Ω. There are other non-equivalent fractional Laplacians on bounded domains.)

- **Fractional Sobolev space:** $\widetilde{H}^s(\Omega) = \{v \in L^2(\mathbb{R}^d) \colon |v|_{H^s(\mathbb{R}^d)} < \infty, v|_{\Omega^c} = 0\}$,

$$(v, w)_s := \frac{C(d,s)}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x-y|^{d+2s}} \, dy dx,$$

$$|v|_{H^s(\mathbb{R}^d)} = (v, v)_s^{1/2}.$$

- **Variational formulation:** for any $f \in H^{-s}(\Omega) =$ dual of $\widetilde{H}^s(\Omega)$, consider

$$u \in \widetilde{H}^s(\Omega) : \quad (u, v)_s = \langle f, v \rangle \quad \forall v \in \widetilde{H}^s(\Omega),$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing.

Existence, uniqueness of weak solutions, and stability: Lax-Milgram Thm.

- Weak solution is the minimizer of the energy $\mathcal{F} \colon \widetilde{H}^s(\Omega) \to \mathbb{R}$,

$$\mathcal{F}(v) := \frac{1}{2} |v|^2_{H^s(\mathbb{R}^d)} - \langle f, v \rangle.$$

- In **finite element discretizations**[1], one typically finds a Galerkin projection: considers $\mathbb{V}_h \subset \widetilde{H}^s(\Omega)$ with $\dim(\mathbb{V}_h) < \infty$, and computes $u_h \in \mathbb{V}_h$ satisfying

$$|u - u_h|_{H^s(\mathbb{R}^d)} \simeq \inf_{v_h \in \mathbb{V}_h} |u - v_h|_{H^s(\mathbb{R}^d)}.$$

- Using **interpolation,** one can construct $v_h \in \mathbb{V}_h$ such that

$$|u - v_h|_{H^s(\mathbb{R}^d)} \le Ch^\alpha |u|_{H^{s+\alpha}(\mathbb{R}^d)} \quad \text{if } u \in H^{s+\alpha}(\mathbb{R}^d), \ 0 < \alpha \le 2 - s.$$

If f is smoother than $H^{-s}(\Omega)$, is necessarily u any smoother than $\widetilde{H}^s(\Omega)$?

- In FE applications, the domain Ω would typically be a polygon/polyhedron.

[1]For clarity, we consider conforming approximations with piecewise linear functions.

- **Sobolev regularity** (Vishik & Eskin (1965), Grubb (2015), Abels & Grubb (2020)): if $f \in H^r(\Omega)$ for some $r \geq 0$ and $\partial\Omega \in C^{1+\beta}$ ($\beta > 2s$), then

$$u \in \begin{cases} H^{2s+r}(\Omega) & \text{if } s+r < 1/2, \\ \bigcap_{\varepsilon>0} H^{s+1/2-\varepsilon}(\Omega) & \text{if } s+r \geq 1/2. \end{cases}$$

Generically, we cannot expect any regularity beyond $H^{s+1/2-\varepsilon}(\Omega)$.

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- **Hölder regularity** (Ros-Oton & Serra (2014)). If $\partial\Omega$ satisfies the exterior ball condition, $\beta > 0$ and $\delta(x, y) = \min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(y, \partial\Omega)\}$, then

$$\sup_{x,y\in\overline{\Omega}} \left\{ \delta(x, y)^{\beta+s} \frac{|u(x) - u(y)|}{|x - y|^{\beta+2s}} \right\} \leq C(f, u).$$

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- **Example:** If $\Omega = B(0,r)$ and $f \equiv 1$, then the solution u is given by

$$u(x) = C(r^2 - |x|^2)_+^s \Rightarrow u(x) \approx \mathrm{dist}(x,\partial\Omega)^s,$$

  which **does not belong to** $H^{s+1/2}(\Omega)$. The regularity above is sharp!

- **Definition of space $\widetilde{H}^t_\gamma(\Omega)$:** let $\gamma \geq 0$ and $t \in (0,1)$,

$$|v|^2_{H^t_\gamma(\Omega)} := \iint_{\Omega\times\Omega} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2t}}\, \delta(x,y)^{2\gamma} dx\, dy$$

where $\delta(x,y) = \min\{\operatorname{dist}(x,\partial\Omega), \operatorname{dist}(y,\partial\Omega)\}$. Then,

$$\widetilde{H}^t_\gamma(\Omega) = \{v \in H^t_\gamma(\mathbb{R}^d) \colon\ v|_{\Omega^c} = 0\},$$

and analogous definitions for spaces with differentiability order $t > 1$.

- **Weighted estimates:** let Ω satisfy the exterior ball condition.
If $s \leq \frac{d}{2(d-1)}$, let $\beta = \frac{d}{2(d-1)} - s$; otherwise, let $\beta > 0$. Let $f \in C^\beta(\overline{\Omega})$.
Then, the solution u of $(-\Delta)^s u = f$ that vanishes in Ω^c belongs to $\widetilde{H}^t_\gamma(\Omega)$ and satisfies the estimate

$$\|u\|_{\widetilde{H}^t_\gamma(\Omega)} \leq \frac{C(\Omega,s)}{\varepsilon} \|f\|_{C^\beta(\overline{\Omega})},$$

where $t = s + \frac{d}{2(d-1)} - d\varepsilon$, $\gamma = \frac{d}{2(d-1)} - \varepsilon$, $\varepsilon > 0$.
(This is based on boundary weighted Hölder estimates by Ros-Oton & Serra (2014).)

Regularity in $\widetilde{W}^t_p(\Omega)$ for $d \geq 2$ (JPB & Nochetto (2021))

- **Heuristics:** $v(x) = x_+^s$ for $x \in \mathbb{R}$ satisfies $\partial^t v(x) \approx x_+^{s-t}$. Then

$$v \in L^p(\mathbb{R}) \quad \Leftrightarrow \quad t < s + \frac{1}{p}.$$

  If $v(x) \simeq d(x, \partial\Omega)^s$, this regularity is valid for any $d \geq 2$.

- **Nonlinear approximation:** Sobolev embedding $W^t_p(\Omega) \subset H^s(\Omega)$ needs

$$t - \frac{d}{p} = \mathrm{Sob}(W^t_p) > \mathrm{Sob}(H^s) = s - \frac{d}{2} \quad \Rightarrow \quad t > s + d\left(\frac{1}{p} - \frac{1}{2}\right).$$

- **Optimal parameters:** These two lines intersect at $p = \frac{2(d-1)}{d}$, $t = s + \frac{d}{2(d-1)}$.

- **Theorem** (differentiability vs integrability) Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and satisfy the exterior ball condition. Let $f \in C^\beta(\overline{\Omega})$, with β as before. Then, the solution $u \in \widetilde{W}^{t-\varepsilon}_{p+\varepsilon}(\Omega)$ satisfies

$$|u|_{W^{t-\varepsilon}_{p+\varepsilon}(\mathbb{R}^d)} \leq \frac{C(\Omega, s)}{\varepsilon^2} \|f\|_{C^\beta(\overline{\Omega})} \quad \forall\, \varepsilon > 0.$$

- **Characterization by difference quotients:** given $1 \leq p < \infty$, $v \in L^p(\Omega)$, and $h \in \mathbb{R}^d$, we set $\delta_2(h)v(x) := v(x+h) - 2v(x) + v(x-h)$, and define

$$|v|_{B^\sigma_{p,q}(\Omega)} := \begin{cases} \left(q\sigma(2-\sigma) \int_D \frac{\|\delta_2(h)v\|^q_{L^p(\Omega_{|h|})}}{|h|^{d+q\sigma}} \right)^{1/q}, & 1 \leq q < \infty, \\ \sup_{h \in D} \frac{\|\delta_2(h)v\|_{L^p(\Omega_{|h|})}}{|h|^\sigma}, & q = \infty. \end{cases}$$

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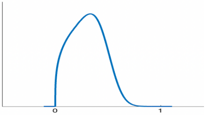
- For spaces of order $\sigma \in (0,1)$, we can also use first-order differences to characterize $B^{\sigma}_{p,q}(\Omega)$, with a norm equivalent to the one defined through second-order differences[2].
- **Zero-extension spaces:** $\dot{B}^{\sigma}_{p,q}(\Omega) := \{v \in B^{\sigma}_{p,q}(\mathbb{R}^d) \colon \text{supp } v \subset \overline{\Omega}\}$.
- **Relation with fractional Sobolev spaces:** $B^{\sigma}_{p,p}(\Omega) = W^{\sigma}_p(\Omega)$ for all $\sigma \in (0,2) \setminus \{1\}$, $1 \le p < \infty$.

[2]This is due to the so-called *Marchaud inequality*.

Recall the typical solution behavior $u(x) \approx \mathrm{dist}(x, \partial\Omega)^s$.

Let $s \in (0, 1/2)$, and $v(x) = x_+^s$ near 0 but smooth otherwise.



An example

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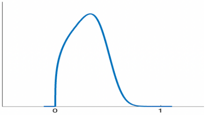

We compute

$$\|\delta_1(h)v\|^2_{L^2(\mathbb{R})} = \|v_h - v\|^2_{L^2(\mathbb{R})} \simeq \int_0^c [(x+h)^s - x^s]^2 \, dx \simeq h^{2s+1} \Rightarrow \|v_h - v\|_{L^2(\mathbb{R})} \simeq h^{s+1/2}.$$

Therefore, if $1 \le q < \infty$, we have

$$|v|_{B^{s+1/2}_{2,q}(\mathbb{R})} = \left(\int_D \frac{\|v_h - v\|^q_{L^2(\mathbb{R})}}{|h|^{1+q(s+1/2)}} \, dh \right)^{1/q} \simeq \int_D \frac{1}{h} \, dh = \infty,$$

while

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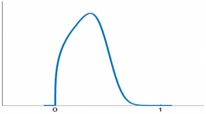

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In particular, $v \in B^{s+1/2}_{2,\infty}(\mathbb{R})$ but $v \notin B^{s+1/2}_{2,2}(\mathbb{R}) = H^{s+1/2}(\mathbb{R})$.

The following regularity is valid without a uniform exterior ball condition, thus allowing for reentrant corners[3].

- **Regularity assumptions:** Let $\Omega \subset \mathbb{R}^d$ be Lipschitz, $f \in B^{-s+1/2}_{2,1}(\Omega)$ and let $u \in \widetilde{H}^s(\Omega)$ solve:

$$(-\Delta)^s u = f \text{ in } \Omega, \qquad u = 0 \text{ in } \mathbb{R}^d \setminus \Omega.$$

- **Optimal shift property:** The solution u belongs to the Besov space $\dot{B}^{s+1/2}_{2,\infty}(\Omega)$ and satisfies

$$\|u\|_{\dot{B}^{s+1/2}_{2,\infty}(\Omega)} \leq C(\Omega, d, s)\|f\|_{B^{-s+1/2}_{2,1}(\Omega)}.$$

Therefore, $u \in \cap_{\varepsilon>0}\widetilde{H}^{s+1/2-\varepsilon}(\Omega)$ and $|u|_{H^{s+1/2-\varepsilon}(\mathbb{R}^d)} \lesssim \frac{1}{\sqrt{\varepsilon}}\|f\|_{B^{-s+1/2}_{2,1}(\Omega)}$.

[3] Related work by Gimperlein-Stephan-Stocek (2021) and Faustmann-Melenk-Marcati-Schwab (2021).

- **Besov norms and translations:** if $s, \sigma \in (0,1)$, $p \in (1,\infty)$, and $r \in [1,\infty]$, then (recall $\delta_1(h)v(x) = v(x+h) - v(x)$)

$$|v|_{B^{s+\sigma}_{p,\infty}(\mathbb{R}^d)} = \sup_{h \in D} \frac{\|\delta_2(h)v\|_{L^p(\mathbb{R}^d)}}{|h|^{s+\sigma}} \lesssim \sup_{h \in D} \frac{|\delta_1(h)v|_{B^s_{p,r}(\mathbb{R}^d)}}{|h|^{\sigma}}.$$

- **Functionals in $\widetilde{H}^s(\Omega)$:** $u \in \widetilde{H}^s(\Omega)$ minimizes $v \mapsto \mathcal{F}_2(v) - \mathcal{F}_1(v)$ where

$$\mathcal{F}_1(v) := \int_\Omega fv, \qquad \mathcal{F}_2(v) := \frac{1}{2}|v|^2_{H^s(\mathbb{R}^d)} \qquad \forall\, v \in \widetilde{H}^s(\Omega).$$

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- **Minimization problem:** Solution of $(-\Delta)^s u = f$ in Ω, $u = 0$ in Ω^c satisfies

$$\frac{1}{2}|u - v|^2_{H^s(\mathbb{R}^d)} = [\mathcal{F}_2(v) - \mathcal{F}_2(u)] - [\mathcal{F}_1(v) - \mathcal{F}_1(u)] \quad \forall\, v \in \widetilde{H}^s(\Omega).$$

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Idea: take $v = u_h$ and bound $\mathcal{F}(u_h) - \mathcal{F}(u)$... but u_h may not belong to $\widetilde{H}^s(\Omega)$!

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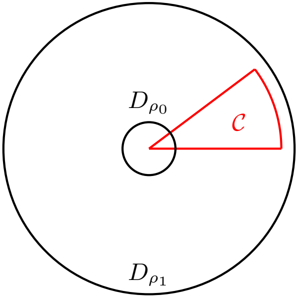
A seemingly harmless technical detail

In the definition of Besov seminorms, one can replace balls by cones.

Let $\mathcal{C}$ be a convex cone in $\mathbb{R}^d$ so that $\mathcal{C} \subset D_{\rho_1} = D_{\rho_1}(0)$.

Then, there exist ρ_0 and c such that for every $v \colon \mathbb{R}^d \to \mathbb{R}$

$$\frac{1}{c^\sigma(2^\sigma + 1)} |v|_{B^\sigma_{p,\infty}(\mathbb{R}^d; D_{\rho_0/2})} \leq |v|_{B^\sigma_{p,\infty}(\mathbb{R}^d; \mathcal{C})} \leq |v|_{B^\sigma_{p,\infty}(\mathbb{R}^d; D_{\rho_1})}.$$



- Because Ω is Lipschitz, it satisfies a **uniform cone property:** there exist $\rho > 0$, $\theta \in (0, \pi]$, and a map $\mathbf{n} : \Omega \to \mathbb{R}^d$ such that for all $x \in \Omega$, the cone $\mathcal{C}_\rho(\mathbf{n}(x), \theta)$ with height ρ, aperture θ, apex x and axis $\mathbf{n}(x)$ gives admissible outward vectors:

$$h \in \mathcal{C}_\rho(\mathbf{n}(x_0), \theta) \;\Rightarrow\; (D_{3\rho}(x_0) \setminus \Omega) + th \subset \Omega^c \quad \forall t \in [0, 1].$$

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- **Localized translations:** given a smooth cut-off function ϕ, such that $0 \le \phi \le 1$, $\phi = 1$ in $D_\rho(x_0)$, supp $\phi \subset D_{2\rho}(x_0)$, let

$$T_h v(x) = v(x + h\phi(x)).$$

The operator T_h translates v within $D_\rho(x_0)$ and is the identity in $D_{2\rho}(x_0)^c$.
By construction: $x_0 \in \Omega$, $h \in \mathcal{C}_\rho(\mathbf{n}(x_0), \theta)$, $v \in \widetilde{H}^s(\Omega) \Rightarrow T_h v \in \widetilde{H}^s(\Omega)$.

Admissible localized translations

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- We write $T_h v = v \circ S_h$, where $S_h = I + h\phi$; if $|h|$ is sufficiently small, it is one-to-one from $D_{2\rho}$ to $D_{2\rho}$. Moreover,

$$\det \nabla S_h \simeq 1 + \mathcal{O}(h),$$

$$|v - T_h v|_{\dot{B}^{1-\sigma}_{2,\infty}(D_{2\rho}(x_0))} \lesssim |h|^\sigma |v|_{B^1_{2,\infty}(D_{3\rho}(x_0))} \quad \forall v \in B^1_{2,\infty}(D_{3\rho}(x_0)).$$

- **Localization:** let $\{D_\rho(x_j)\}$ be a finite covering of Ω, then

$$|v|^p_{B^\sigma_{p,q}(\Omega)} \simeq \sum_{j=1}^{M} |v|^p_{B^\sigma_{p,q}(D_\rho(x_j))}.$$

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- Thus, if we can prove that

$$\mathcal{F}(T_h u) - \mathcal{F}(u) \leq C|h|^\sigma$$

for every ball $D_\rho(x_j)$ and $h \in \mathcal{C}_\rho(\mathbf{n}(x_j), \theta)$, then we can assure that $u \in B^{s+\sigma/2}_{2,\infty}(\Omega)$:

$$\begin{aligned} |u|^2_{B^{s+\sigma/2}_{2,\infty}(D_\rho(x_j))} &\lesssim \sup_{h \in D\setminus\{0\}} \frac{|\delta_1(h)u|^2_{B^s_{2,2}(D_\rho(x_j))}}{|h|^\sigma} \\ &= \sup_{h \in D\setminus\{0\}} \frac{|T_h u - u|^2_{B^s_{2,2}(D_\rho(x_j))}}{|h|^\sigma} \\ &\lesssim \sup_{h \in D\setminus\{0\}} \frac{\mathcal{F}(T_h u) - \mathcal{F}(u)}{|h|^\sigma} \leq C. \end{aligned}$$

[Note that we can argue with the functionals $\mathcal{F}_1$ and $\mathcal{F}_2$ separately.]

Regularity of functional $\mathcal{F}_1(v) = \langle f, v \rangle$

Given $\sigma \in (0,1]$, $t \in [\sigma - 1, 1]$, a fixed $x_j \in \Omega$, and a cone $\mathcal{C}_\rho(\mathbf{n}(x_j), \theta)$ we have, for all $v \in B^{\sigma-t}_{2,\infty}(D_{3\rho}(x_j))$,

$$\sup_{h \in \mathcal{C}_\rho(\mathbf{n}(x_j),\theta)} \frac{\mathcal{F}_1(T_h v) - \mathcal{F}_1(v)}{|h|^\sigma} \le C \, \|f\|_{B^t_{2,1}(\Omega \cap D_{2\rho}(x_j))} \, |v|_{B^{\sigma-t}_{2,\infty}(D_{3\rho}(x_j))}.$$

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Proof.
Note $\mathcal{F}_1(T_h v) - \mathcal{F}_1(v) = \int_\Omega f(T_h v - v)$, and the result follows if $t = \sigma - 1$.

If $t = 1$, note $\int_\Omega f T_h v = \int_{S_h(\Omega)} (f \circ S_h^{-1})\, v|J|$ with $J = \det \nabla S_h^{-1} \simeq 1 + \mathcal{O}(h)$, and then the result follows as well in that case.

Finally, the mapping $(f, v) \mapsto \mathcal{F}_1(T_h v) - \mathcal{F}_1(v)$ is bilinear and we interpolate.

Regularity of functional $\mathcal{F}_2(v) = \frac{1}{2}|v|^2_{H^s(\mathbb{R}^d)}$

Given a fixed $x_j \in \Omega$, and a cone $\mathcal{C}_\rho(\mathbf{n}(x_j), \theta)$ we have

$$\sup_{h \in \mathcal{C}_\rho(\mathbf{n}(x_j),\theta)} \frac{\mathcal{F}_2(T_h v) - \mathcal{F}_2(v)}{|h|} \leq C \iint_{Q_{D_{2\rho}(x_j)}} \frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}}\, dy dx$$

for all $v \in \widetilde{H}^s(\Omega)$, where $Q_{D_{2\rho}(x_j)} = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (D_{2\rho}(x_j)^c \times D_{2\rho}(x_j)^c)$.

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Proof: recall $T_h v = v \circ S_h$, write $Q = Q_{D_{2\rho}(x_j)}$, and split

$$\mathcal{F}_2(T_h v) - \mathcal{F}_2(v) = \iint_Q \frac{|v(x) - v(y)|^2}{|S_h^{-1}(x) - S_h^{-1}(y)|^d} \left(\frac{1}{|S_h^{-1}(x) - S_h^{-1}(y)|^{2s}} - \frac{1}{|x-y|^{2s}} \right) |J| \, dydx$$
$$+ \iint_Q \frac{|v(x) - v(y)|^2}{|x-y|^{2s}} \left(\frac{|J|}{|S_h^{-1}(x) - S_h^{-1}(y)|^d} - \frac{1}{|x-y|^d} \right) dydx.$$

Use that $\frac{|S_h^{-1}(x) - S_h^{-1}(y)|}{|x-y|} = 1 + \mathcal{O}(h)$ and that $J = \det \nabla S_h^{-1} \simeq 1 + \mathcal{O}(h)$ to prove that both integrals are $\mathcal{O}(h)$.

Regularity for $f \in B^{-s+1/2}_{2,1}(\Omega)$

- **Fundamental recursion formula:** if $f \in B^{t}_{2,1}(\Omega)$ with $t > -s$ and the minimizer u of the energy $\mathcal{F}$ belongs to $\dot{B}^{\sigma-t}_{2,\infty}(\Omega)$, then

$$\|u\|^2_{\dot{B}^{s+\sigma/2}_{2,\infty}(\Omega)} \leq \left(C_1 |u|^2_{H^s(\mathbb{R}^d)} + C_2 \|f\|_{B^t_{2,1}(\Omega)} \|u\|_{\dot{B}^{\sigma-t}_{2,\infty}(\Omega)} \right).$$

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- **Parameters:** set $t = -s + \frac{1}{2}$, $\sigma_{k+1} - t = s + \frac{\sigma_k}{2}$ $(\sigma_0 = 0)$

$$\sigma_{k+1} = t + s + \frac{\sigma_k}{2} = \frac{1}{2} + \frac{\sigma_k}{2} \quad \Rightarrow \quad \sigma_k = 1 - \frac{1}{2^{k-1}} \to 1 \text{ as } k \to \infty.$$

- **Master iteration:**

$$|u|^2_{\dot{B}^{s+\sigma_{k+1}/2}_{2,\infty}(\Omega)} \leq \left(C_1 \|f\|_{B^{-s+1/2}_{2,1}(\Omega)} + C_2 |u|_{\dot{B}^{s+\sigma_k/2}_{2,\infty}(\Omega)} \right) \|f\|_{B^{-s+1/2}_{2,1}(\Omega)},$$

- **Induction:** for $\{\Lambda_k\}$ uniformly bounded, $|u|_{\dot{B}^{s+\sigma_k/2}_{2,\infty}(\Omega)} \leq \Lambda_k \|f\|_{B^{-s+1/2}_{2,1}(\Omega)}$.
  - For $k = 0$: we have $\sigma_0 = 0$ and

$$|u|_{\dot{B}^s_{2,\infty}(\Omega)} \lesssim |u|_{H^s(\mathbb{R}^d)} \lesssim \|f\|_{H^{-s}(\Omega)} \lesssim \|f\|_{B^{-s+1/2}_{2,1}(\Omega)}.$$

  - For $k > 0$: $\Lambda^2_{k+1} = C_1 + C_2\Lambda_k$ is uniformly bounded depending on Λ_0, C_1, C_2,

$$|u|^2_{\dot{B}^{s+\sigma_{k+1}/2}_{2,\infty}(\Omega)} \leq \left(C_1 + C_2\Lambda_k \right) \|f\|^2_{B^{-s+1/2}_{2,1}(\Omega)}.$$

Regularity for $f \in L^2(\Omega)$

- **Case $s \neq 1/2$:** if $\alpha = \min\left\{s, \frac{1}{2}\right\}$, then $u \in \dot{B}^{s+\alpha}_{2,\infty}(\Omega)$ satisfies

$$|u|_{\dot{B}^{s+\alpha}_{2,\infty}(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

  with constant $C = C(\Omega, d, s)$ that blows up as $s \to 1/2$.

- **Case $s = 1/2$:** for all $0 < \varepsilon < 1$,

$$|u|_{\dot{B}^{1-\varepsilon}_{2,\infty}(\Omega)} \leq \frac{C}{\varepsilon^{1/2}}\|f\|_{L^2(\Omega)}.$$

- **Master iteration for $s \leq \frac{1}{2}$:**

$$|u|_{\dot{B}^{s+\sigma/2}_{2,\infty}(\Omega)} \leq \left(C_1\|f\|_{L^2(\Omega)} + \frac{C_2}{(1-\sigma)^{1/2}}|u|_{\dot{B}^{\sigma}_{2,\infty}(\Omega)}\right)\|f\|_{L^2(\Omega)}$$

- **Induction:** set $\sigma_0 = s$, $\sigma_k = s + \sigma_{k-1}/2$, then $\sigma_k = 2s\left(1 - \frac{1}{2^{(k+1)}}\right) \to 2s$ and

$$|u|_{\dot{B}^{\sigma_k}_{2,\infty}(\Omega)} \leq \Lambda_k\|f\|_{L^2(\Omega)},$$

  with a constant $\Lambda_k \leq \Lambda(\Omega, d, s)$ uniformly bounded for $s < 1/2$ that blows up for $s = 1/2$ precisely as $(1-\sigma_k)^{-1/2}$.

- Let $\mathcal{T}_h$ be a **shape-regular mesh** of Ω; h_T is the diameter of $T \in \mathcal{T}_h$ and $h = \max_T h_T$.
- **Conforming finite element space:**

$$\mathbb{V}_h := C^0(\overline{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_h) \subset \widetilde{H}^s(\Omega).$$

- **Discrete problem:** find $u_h \in \mathbb{V}_h$ such that, for all $v_h \in \mathbb{V}_h$,

$$\frac{C(d,s)}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u_h(x) - u_h(y))(v_h(x) - v_h(y))}{|x-y|^{d+2s}} \, dx \, dy = \langle f, v_h \rangle.$$

- **Best approximation:** since we project over $\mathbb{V}_h$ with respect to the energy norm $\| \cdot \|_{\widetilde{H}^s(\Omega)} = | \cdot |_{H^s(\mathbb{R}^d)}$, we get

$$|u - u_h|_{H^s(\mathbb{R}^d)} = \min_{v_h \in \mathbb{V}_h} |u - v_h|_{H^s(\mathbb{R}^d)}.$$

- **A priori error analysis:** must account for nonlocality and boundary behavior.

- **Local interpolation error:**

$$|v - \Pi_h v|_{H^s(T)} \le C h_T^{r-s} |v|_{H^r(S_T^1)},$$

where S_T^1 is a patch surrounding T.

- Faermann (2002) accounts for the **nonlocal** nature of the H^s-norm,

$$\|v\|^2_{\widetilde{H}^s(\Omega)} \le \left[\sum_{T \in \mathcal{T}_h} \int_T \int_{\widetilde{S}_T^1} \frac{|v(x)-v(y)|^2}{|x-y|^{d+2s}} \, dy dx + \frac{C(d,\sigma)}{s h_T^{2s}} \|v\|^2_{L^2(T)} \right],$$

so that in shape-regular meshes we have the **global approximation estimate**

$$\min_{v_h \in \mathbb{V}_h} \|v - v_h\|_{\widetilde{H}^s(\Omega)} \le C \left(\sum_{T \in \mathcal{T}_h} h_T^{2(r-s)} |v|^2_{H^r(\widetilde{S}_T^2)} \right)^{1/2}.$$

- **Quasi-uniform meshes:**

$$|u - u_h|_{H^s(\mathbb{R}^d)} \lesssim \begin{cases} h^{\frac{1}{2}} |\log h| \, \|f\|_{H^{-s+1/2}(\Omega)}, & \Omega \text{ smooth,} \\ h^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \, \|f\|_{\dot{B}^{-s+1/2}_{2,1}(\Omega)}, & \Omega \text{ Lipschitz.} \end{cases}$$

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- **Graded meshes** $(d \geq 2)$**:** if $h_T \approx h \, \mathrm{dist}(T, \partial\Omega)^{1/d}$ then

$$|u - u_h|_{\widetilde{H}^s(\Omega)} \lesssim h^{\frac{d}{2(d-1)}} |\log h| \, \|f\|_{C^\beta(\overline{\Omega})} \approx N^{-\frac{1}{2(d-1)}} \log N \, \|f\|_{C^\beta(\overline{\Omega})},$$

where $N = \#\mathcal{T}_h \approx h^{-d}|\log h|$ is the number of degrees of freedom of $\mathcal{T}_h$.

- **Example:** $u(x) = C(r^2 - |x|^2)^s_+$ with $\Omega = B(0,1) \subset \mathbb{R}^2$, $f = 1$

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Uniform $\mathcal{T}_h$	0.49	0.49	0.49	0.50	0.50	0.50	0.50	0.50	0.53
Graded $\mathcal{T}_h$	1.06	1.04	1.02	1.00	1.06	1.05	0.99	0.98	0.97

CONSTRUCTIVE APPROXIMATION IN BISECTION GRIDS $(d = 2)$

- **Interpolation error:** $|u - \Pi_h u|^2_{H^s(\mathbb{R}^d)} \lesssim \sum_{T \in \mathcal{T}_h} |u - \Pi_h u|^2_{H^s(\widetilde{S}^1_T)}$ and

$$|u - \Pi_h u|_{H^s(\widetilde{S}^1_T)} \lesssim h_T^t |u|_{W^{s+1-\varepsilon}_{1+\varepsilon}(\widetilde{S}^2_T)} \lesssim |T| \operatorname{dist}(x_T, \partial\Omega)^{-1} := E_T,$$

with $\widetilde{S}^1_T, \widetilde{S}^2_T$ first and second extended patch of T and $t = 2 - \varepsilon - \frac{2}{1+\varepsilon} > 0.$

- **Greedy algorithm:** given a tolerance $\delta > 0$, iterate

GREEDY $(\mathcal{T}, \delta)$
  while $\mathcal{M} := \{T \in \mathcal{T} : E_T > \delta\} \neq \emptyset$
    $\mathcal{T}$ = REFINE $(\mathcal{T}, \mathcal{M})$
  end while
  return

- REFINE is a bisection algorithm acting on the marked elements $\mathcal{M}$.

- **Optimal mesh:** GREEDY terminates in finite steps, the number of elements N satisfies $N \approx \delta^{-1} |\log \delta|$ and the error of the interpolant $u_N = \Pi_h u$ obeys

$$|u - u_N|_{H^s(\mathbb{R}^d)} \lesssim N^{-1/2} |\log N|^2.$$

[Constructive proof. Rate consistent with a priori graded meshes.]

L^2-ERROR ESTIMATES $(d \geq 2)$

- **Sobolev regularity:** lift theorem for Ω Lipschitz and $\alpha = \min\{s, \frac{1}{2}\}$

$$|u_g|_{H^{s+\alpha-\varepsilon}(\mathbb{R}^d)} \leq \frac{C(\Omega, d, s)}{\varepsilon^{\kappa/2}} \|g\|_{L^2(\Omega)},$$

  where $\kappa = 1$ if $s \neq 1/2$ and $\kappa = 2$ if $s = 1/2$.

- **Quasi-uniform meshes:** If Ω is Lipschitz and $f \in B^{s+1/2}_{2,1}(\Omega)$, then

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{\frac{1}{2}+\min\{s,\frac{1}{2}\}} |\log h|^{\kappa} \|f\|_{B^{-s+1/2}_{2,1}(\Omega)},$$

  where $\kappa = 1$ if $s \neq 1/2$ and $\kappa = 2$ if $s = 1/2$.

- **Graded meshes:** if Ω satisfies the exterior ball condition, $f \in C^{\beta}(\overline{\Omega})$ $(\beta = \max\{\frac{d}{2(d-1)} - s, 0\})$, and $h_T \approx h \operatorname{dist}(T, \partial\Omega)^{1/d}$, then

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{\frac{d}{2(d-1)}+\min\{s,\frac{1}{2}\}} |\log h|^{1+\frac{\kappa}{2}} \|f\|_{C^{\beta}(\overline{\Omega})},$$

  where $\kappa = 1$ if $s \neq \frac{1}{2}$ and $\kappa = 2$ if $s = \frac{1}{2}$.

- **Caccioppoli inequality** (Cozzi (2017)): Let $B_R \subset \mathbb{R}^d$ be a ball of radius R centered at x_0. Let u satisfy
  - u is s-harmonic in B_R, namely $(u,v)_s = 0$ for all $v \in \widetilde{H}^s(B_R)$
  - $\int_{B_R^c} \frac{|u(x)|}{|x-x_0|^{d+2s}} dx < \infty$, where $B_R^c = \mathbb{R}^d \setminus B_R$.

  Then u satisfies

$$|u|^2_{H^s(B_{R/2})} \leq \frac{C}{R^{2s}} \|u\|^2_{L^2(B_R)} + CR^{d+2s} \left(\int_{B_R^c} \frac{|u(x)|}{|x-x_0|^{d+2s}} dx \right)^2 .$$

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- A pair $(u, u_h) \in \widetilde{H}^s(\Omega) \times \mathbb{V}_h$ satisfies the **local Galerkin orthogonality (LGO)** relation in B_R if

$$(u - u_h, v_h)_s = 0 \quad \forall\, v_h \in \mathbb{V}_h(B_R), \quad \text{where } v_h \text{ vanishes in } B_R^c.$$

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- **Theorem** Let $(u, u_h) \in \widetilde{H}^s(\Omega) \times \mathbb{V}_h$ satisfy LGO in B_R. If $\mathcal{T}_h$ is a shape-regular mesh with $16h_T \leq R$ for all $T \in \mathcal{T}_h$, $T \subset B_R$, then

$$|u-u_h|_{H^s(B_{R/2})} \leq C \inf_{v_h \in \mathbb{V}_h} \left(|u-v_h|_{H^s(B_R)} + \frac{1}{R^s} \|u-v_h\|_{L^2(\Omega)} \right) + \frac{C}{R^s} \|u-u_h\|_{L^2(\Omega)}.$$

  [Similar results by Faustmann, Karkulik, & Melenk (2020)]

Global vs interior error estimates $(d = 2)$

Comparison of convergence rates (up to logarithmic factors) between interior $|u - u_h|_{H^s(B_{R/2})}$ and global $|u - u_h|_{H^s(\mathbb{R}^d)}$ error estimates.

Quasi-uniform meshes: Let $f \in B^{-s+1/2}_{2,1}(\Omega)$ or smoother. The interior estimates exhibit an improvement rate $h^{\min\{s,1/2\}}$ regardless of the regularity of Ω.

	Interior rates		Global rates	
	Ω-smooth	Ω-Lipschitz	Ω-smooth	Ω-Lipschitz
$s \leq \frac{1}{2}$	$h^{s+\frac{1}{2}}$	$h^{s+\frac{1}{2}}$	$h^{\frac{1}{2}}$	$h^{\frac{1}{2}}$
$s > \frac{1}{2}$	h	h	$h^{\frac{1}{2}}$	$h^{\frac{1}{2}}$

Graded meshes: Let $f \in H^{2-2s}(\Omega) \cap C^{1-s}(\overline{\Omega})$ and local meshsize satisfy $h_T \approx h\, \mathrm{dist}(T, \partial\Omega)^{1/2}$. The interior estimates exhibit an improvement rate $h^{\min\{s,1-s\}}$ for Ω either smooth or Lipschitz with an exterior ball condition (e.b.c.).

	Ω-smooth or Lipschitz e.b.c.	
	Interior rates	Global rates
$s \leq \frac{1}{2}$	h^{s+1}	h
$s > \frac{1}{2}$	h^{2-s}	h

- **Fractional Sobolev spaces:** let $1 < p < \infty$,

$$\widetilde{W}_p^s(\Omega) = \{v \in L^p(\mathbb{R}^d) \colon |v|_{W_p^s(\mathbb{R}^d)} < \infty, v|_{\Omega^c} = 0\},$$

$$|v|_{W_p^s(\mathbb{R}^d)} = \left(C_{d,s,p} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|v(x) - v(y)|^p}{|x-y|^{d+sp}} \, dydx \right)^{1/p}.$$

- The minimizer of the energy $\mathcal{F} \colon \widetilde{W}_p^s(\Omega) \to \mathbb{R}$,

$$\mathcal{F}(v) := \frac{1}{p} |v|^p_{W_p^s(\mathbb{R}^d)} - \langle f, v \rangle$$

is the **unique weak solution** to the problem

$$\begin{cases} (-\Delta)_p^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where

$$(-\Delta)_p^s u(x) := \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{d+sp}} \, dy.$$

- **Variational formulation:** weak solution is the unique $u \in \widetilde{W}^{s,p}(\Omega)$ such that for every $v \in \widetilde{W}^{s,p}(\Omega)$,

$$\langle (-\Delta)_p^s u, v \rangle := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+sp}} \, dx \, dy = \langle f, v \rangle.$$

- **Hölder regularity:** (Iannizzotto, Mosconi, & Squassina (2016), Brasco, Lindgren, & Schikorra (2018)) if $\partial\Omega$ is $C^{1,1}$

$$\|u\|_{C^\alpha(\overline{\Omega})} \lesssim \|f\|_{L^\infty(\Omega)}^{1/(p-1)},$$

with $\alpha \in (0, s]$ and $\alpha = s$ if $p \geq 2$.

- **Interior Sobolev regularity:** (Brasco & Lindgren (2017)) in the $p \geq 2$ case.

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- **Interior Sobolev regularity:** (Brasco & Lindgren (2017)) in the $p \geq 2$ case.
- **Monotonicity:** the operator $(-\Delta)_p^s$ satisfies

$$\langle (-\Delta)_p^s u - (-\Delta)_p^s v, u - v \rangle \geq \alpha_p |u - v|_{W_p^s(\mathbb{R}^d)}^p \quad \forall u, v \in \widetilde{W}_p^s(\Omega).$$

and therefore the **minimizer** $u \in \widetilde{W}_p^s(\Omega)$ of $\mathcal{F}$ satisfies

$$\frac{\alpha_p}{p} |u - v|_{W_p^s(\mathbb{R}^d)}^p \leq \mathcal{F}(v) - \mathcal{F}(u) \quad \forall v \in \widetilde{W}_p^s(\Omega).$$

Regularity estimate (JPB, Li & Nochetto (2022))

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $s \in (0,1)$, $p \in (1,\infty)$, $q = \max\{p,2\}$, and $f \in B^{-s+\frac{1}{q'}}_{p',1}(\Omega)$. Let $u \in \widetilde{W}^s_p(\Omega)$ be the minimizer of the energy $\mathcal{F}(v) = \frac{1}{p}|v|^p_{\widetilde{W}^s_p(\Omega)} - \langle f, v\rangle$.

- *If $p \geq 2$, then $u \in \dot{B}^{s+\frac{1}{p}}_{p,\infty}(\Omega)$, with $\|u\|_{\dot{B}^{s+\frac{1}{p}}_{p,\infty}(\Omega)} \leq C\|f\|^{\frac{1}{p-1}}_{B^{-s+\frac{1}{p'}}_{p',1}(\Omega)}$.*
- *If $p < 2$, then $u \in \dot{B}^{s+\frac{1}{2}}_{p,\infty}(\Omega)$, with $\|u\|_{\dot{B}^{s+\frac{1}{2}}_{p,\infty}(\Omega)} \lesssim \|f\|^{\frac{1}{p-1}}_{B^{-s+\frac{1}{2}}_{p',1}(\Omega)}$.*

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Remarks:

- One can **interpolate** between the maximal regularity and the stability $|u|_{\widetilde{W}^s_p(\Omega)} \leq C\|f\|_{W^{-s}_{p'}(\Omega)}$ to prove intermediate regularity estimates.
- By **embedding,** we obtain estimates in the **Sobolev scale:**

$$|u|_{W^{s+\frac{1}{q}-\varepsilon}_p(\mathbb{R}^d)} \leq \frac{C}{\varepsilon^{1/p}} \|f\|_{B^{-s+\frac{1}{q'}}_{p',1}(\Omega)}, \quad \varepsilon > 0.$$

- **Discretization:** let $\mathbb{V}_h := C^0(\overline{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_h) \subset \widetilde{W}^{s,p}(\Omega)$, $s \in (0,1)$, and $p \in (1,\infty)$. We seek $u_h \in \mathbb{V}_h$ such that for all $v_h \in \mathbb{V}_h$

$$\int_{\mathbb{R}^d} \frac{\big|u_h(x) - u_h(y)\big|^{p-2}\big(u_h(x) - u_h(y)\big)\big(v_h(x) - v_h(y)\big)}{|x-y|^{d+sp}}\,dy = \int_\Omega f(x) v_h(x).$$

- **Error estimates:** Let $p \in (1,\infty)$, $s \in (0,1)$ and $f \in B^{-s+\gamma'}_{p',1}(\Omega)$. Then

$$p \in (1,2] \quad \Rightarrow \quad |u - u_h|_{W^s_p(\mathbb{R}^d)} \le C h^{\frac{p}{4}} |\log h|^{\frac{1}{2}} \|f\|^{\frac{p'}{2}}_{B^{-s+\gamma'}_{p',1}(\Omega)}$$

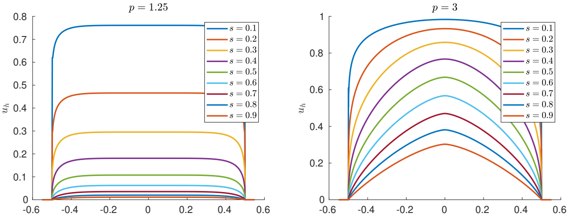
$$p \in [2,\infty) \quad \Rightarrow \quad |u - u_h|_{W^s_p(\mathbb{R}^d)} \le C h^{\frac{2}{p^2}} |\log h|^{\frac{2}{p^2}} \|f\|^{\frac{2}{p(p-1)}}_{B^{-s+\gamma'}_{p',1}(\Omega)},$$

over quasi-uniform meshes $\mathcal{T}_h$ (recall $\gamma' = \max\{1/p', 1/2\}$).

- Error analysis inspired by Chow (1989) for **classical p-Laplacian.**
- Damped Newton's method to solve the nonlinear system of equations[5].

[5] In case $p \in (1,2)$, we add a regularization following Bartels, Diening, & Nochetto (2018).

Example: $\Omega = (-0.5, 0.5) \subset \mathbb{R}$ and $f = 1$.



Uniform meshes:

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p = 1.25$	0.80	0.78	0.78	0.79	0.80	0.81	0.82	0.85	0.90
$p = 3$	0.34	0.33	0.33	0.33	0.33	0.33	0.33	0.33	0.34

Remarks:

- Solution is $\widetilde{W}_p^{s+\min\{1/p,\,1/2\}-\varepsilon}(\Omega)$: interpolation error is $h^{\min\{1/p,\,1/2\}}$.
- Theoretical rates $h^{\min\{p/4,\,2/p^2\}}$ seem suboptimal (unless $p = 2$).
- Regularity (convergence rates) affected by boundary behavior
  $\Rightarrow$ graded meshes.

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If there exists some smooth φ such that

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We fix $\mu > 1$ and set

$$h_T \approx \begin{cases} h\,\mathrm{dist}(T, \partial\Omega)^{\frac{\mu-1}{\mu}}, & \text{if } \mathrm{dist}(T, \partial\Omega) > 0, \\ h^\mu, & \text{if } \mathrm{dist}(T, \partial\Omega) = 0. \end{cases}$$

To fully exploit a weighted W^2_p-regularity, we require $\mu \geq p(2-s)$, and we expect the interpolation error to be of order $2-s$ in the energy norm.

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p = 1.25$	1.95	1.85	1.73	1.62	1.51	1.40	1.30	1.20	1.10
$p = 3$	1.98	1.80	1.70	1.60	1.46	1.28	1.12	0.99	0.89

[Recall the rates $4/5$ ($p = 1.25$) and $1/3$ ($p = 3$) we obtained on uniform meshes.]

An experiment about interior regularity

- For $p = 3$ and $s \geq 1/2$, the **interior regularity** limits the convergence rates.
- In the local ($s = 1$) case, we have

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega = (-1/2, 1/2), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \Rightarrow \quad u(x) = C\left(1 - |2x|^{\frac{p}{p-1}}\right)_+,$$

so the solution is locally W^2_p only if $\frac{p}{p-1} + \frac{1}{p} \geq 2$, ie, $p \leq \frac{3+\sqrt{5}}{2}$.

- We test with modified meshes: for $\mu = p(2-s)$, we set

$$h_T \approx \begin{cases} h \, \text{dist}(T, \partial\Omega)^{\frac{\mu-1}{\mu}}, & \text{if } \text{dist}(T, \partial\Omega \cup \{0\}) > 0, \\ h^\mu, & \text{if } \text{dist}(T, \partial\Omega \cup \{0\}) = 0. \end{cases}$$

Value of s	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p = 1.25$	1.85	1.74	1.62	1.51	1.41	1.31	1.21	1.13	1.05
$p = 3$	1.90	1.76	1.65	1.55	1.45	1.35	1.25	1.16	1.09

- Linear problems
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  - Boundary behavior of solutions $\Rightarrow$ graded meshes.
  - Finite element error analysis in $\widetilde{H}^s(\Omega)$, $L^2(\Omega)$, $H^s(B_R)$, $B_R \Subset \Omega$.
- Quasi-linear problems
  - Besov regularity.
  - Finite element error analysis in $\widetilde{W}^s_p(\Omega)$.
  - Suboptimal regularity estimates in the case $p < 2$;
  - Suboptimal convergence rates (w.r.t. interpolation and experiments).
- Technique to prove Besov regularity is **variational** and can be extended to operators with variable diffusivity, or finite horizon.

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Thank you!