# Subconvexity in twisted mean values of exponential sums


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# 1. Introduction: cubic Vinogradov systems

Consider $s > 0$ and $\mathbf{h} = (h_1, h_2, h_3) \in \mathbb{Z}^3$. When $X$ is large, write

$$f(\boldsymbol{\alpha}; X) = \sum_{1 \le x \le X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3),$$

where $e(z)$ denotes $e^{2\pi i z}$. We consider the twisted mean value

$$B_s(X; \mathbf{h}) = \int_{[0,1)^3} |f(\boldsymbol{\alpha}; X)|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, \mathrm{d}\boldsymbol{\alpha},$$

in which $\boldsymbol{\alpha} \cdot \mathbf{h} = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$.

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in which $\boldsymbol{\alpha} \cdot \mathbf{h} = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$.
Note: when $s \in \mathbb{N}$, it follows via orthogonality that $B_s(X; \mathbf{h})$ counts the number of integral solutions of the system

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = h_j \quad (1 \le j \le 3),$$

with $1 \le x_i, y_i \le X$ $(1 \le i \le s)$.

Observe that by the triangle inequality, one has

$$\begin{aligned} B_s(X;\mathbf{h}) &= \int_{[0,1)^3} |f(\boldsymbol{\alpha};X)|^{2s} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha} \\ &\le \int_{[0,1)^3} |f(\boldsymbol{\alpha};X)|^{2s}\,\mathrm{d}\boldsymbol{\alpha} = B_s(X;\mathbf{0}) \\ &\ll X^{s+\varepsilon} + X^{2s-6}, \end{aligned}$$

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This is the (classical) convexity bound, and in particular, for any $\varepsilon > 0$, one has

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Analogous statements and conclusions for degree exceeding $3$.

## 2. Subconvexity: an asymptotic formula

We require some notation. Write

$$I(\boldsymbol{\beta}) = \int_0^1 e(\beta_1\gamma + \beta_2\gamma^2 + \beta_3\gamma^3)\,\mathrm{d}\gamma$$

$$S(q,\mathbf{a}) = \sum_{r=1}^{q} e((a_1 r + a_2 r^2 + a_3 r^3)/q).$$

Next, put $n_j = h_j X^{-j}$ $(1 \le j \le 3)$, and define

$$\mathfrak{J}(\mathbf{h}) = \int_{\mathbb{R}^3} |I(\boldsymbol{\beta})|^{12} e(-\boldsymbol{\beta}\cdot\mathbf{n})\,\mathrm{d}\boldsymbol{\beta}$$

$$\mathfrak{S}(\mathbf{h}) = \sum_{q=1}^{\infty} \sum_{\substack{1\le a_1,a_2,a_3\le q\\ (q,a_1,a_2,a_3)=1}} \left|q^{-1}S(q,\mathbf{a})\right|^{12} e(-\mathbf{a}\cdot\mathbf{h}/q).$$

We note that the *singular integral* $\mathfrak{J}(\mathbf{h})$, and *singular series* $\mathfrak{S}(\mathbf{h})$, are known to converge absolutely (see Arkhipov, Chubarikov and Karatusuba (2004)).

## Theorem (W., 2022; arxiv:2202.05804)

*Suppose that* $\mathbf{h} \in \mathbb{Z}^3$ *and* $h_1 \neq 0$. *Then whenever* $X$ *is sufficiently large, one has*

$$B_6(X;\mathbf{h}) = \mathfrak{J}(\mathbf{h})\mathfrak{S}(\mathbf{h})X^6 + o(X^6),$$

*in which* $0 \leq \mathfrak{J}(\mathbf{h}) \ll 1$ *and* $0 \leq \mathfrak{S}(\mathbf{h}) \ll 1$. *If the system below possesses a non-singular real solution with positive coordinates, moreover, then* $\mathfrak{J}(\mathbf{h}) \gg 1$. *Likewise, if the system below possesses primitive non-singular* $p$*-adic solutions for each prime* $p$, *then* $\mathfrak{S}(\mathbf{h}) \gg 1$.

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$$\begin{aligned} x_1^3 - x_2^3 + \ldots + x_{11}^3 - x_{12}^3 &= h_3 \\ x_1^2 - x_2^2 + \ldots + x_{11}^2 - x_{12}^2 &= h_2 \\ x_1 - x_2 + \ldots + x_{11} - x_{12} &= h_1 \end{aligned}$$

with $1 \le x_i \le X$.

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Define a Hardy-Littlewood dissection of the unit cube $[0,1)^3$ into major and minor arcs. Put $L = X^{1/72}$, and define the set of major arcs $\mathfrak{M}$ to be the union of the arcs

$$\mathfrak{M}(q,\mathbf{a}) = \{\boldsymbol{\alpha} \in [0,1)^3 : |\alpha_j - a_j/q| \le LX^{-j}\ (1 \le j \le 3)\},$$

with $1 \le q \le L$, $0 \le a_j \le q$ $(1 \le j \le 3)$ and $(q, a_1, a_2, a_3) = 1$. We then define the complementary set of minor arcs $\mathfrak{m} = [0,1)^3 \setminus \mathfrak{M}(L)$.

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It is straightforward to show that

$$\int_{\mathfrak{M}} |f(\boldsymbol{\alpha}; X)|^{12} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, \mathrm{d}\boldsymbol{\alpha} = \mathfrak{J}(\mathbf{h}) \mathfrak{S}(\mathbf{h}) X^6 + o(X^6).$$

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Thus, since the theorem shows that $B_6(X;\mathbf{h}) = \mathfrak{J}(\mathbf{h})\mathfrak{S}(\mathbf{h})X^6 + o(X^6)$, we deduce that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\boldsymbol{\alpha}; X)|^{12} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha} &= B_6(X;\mathbf{h}) - \int_{\mathfrak{M}} |f(\boldsymbol{\alpha}; X)|^{12} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha} \\ &= o(B^6).\end{aligned}$$

### Theorem (W., 2022; arxiv:2202.05804)

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What about the situation when $h_1 = 0$?

### Theorem (W., 2022; arxiv:2202.05804)

*Let* $s$ *be a natural number with* $s \geq 6$. *Then the asymptotic formula*

$$B_6(X; \mathbf{h}) = \mathfrak{J}(\mathbf{h})\mathfrak{S}(\mathbf{h})X^6 + o(X^6),$$

*holds when* $h_2 \neq 0$, *and* $X$ *is sufficiently large in terms of* $h_2$.

(Uses work on small cap decouplings by Demeter, Guth and Wang (2020)).

# 3. Related results

Work of Brandes and Hughes (2021) shows that when one of $(h_1, h_2) \neq (0,0)$, then

$$B_s(X;\mathbf{h}) = o(X^s) \quad \text{for} \quad 1 \leq s \leq 5.$$

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Note that by applying the circle method, when $s > 6$ and appropriate real and $p$-adic solubility conditions are satisfied, one has

$$B_s(X; \mathbf{h}) \gg X^{2s-6},$$

and in such circumstances, subconvex estimates willl not be possible.

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and in such circumstances, subconvex estimates willl not be possible.

Also, when $s > 1$, there are values of $\mathbf{h}$ for which one has the lower bound

$$B_s(X; \mathbf{h}) \gg X^{s-1}.$$

Just take $h_i = a^i - b^i$ for integers $a \neq b$. (Note: have $\mathbf{h} \neq 0$).

Let $J_{s,k}(X;\mathbf{h})$ denote the number of integral solutions of the system

$$\sum_{i=1}^{s}(x_i^j - y_i^j) = h_j \quad (1 \le j \le k),$$

with $1 \le \mathbf{x}, \mathbf{y} \le X$. Improving on Brandes and Hughes (2022) we obtain:

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**Theorem (W., 2022; arxiv:2202.14003)**

*Suppose that $k \ge 3$ and $\mathbf{h} \in \mathbb{Z}^k \setminus \{\mathbf{0}\}$. Let $l$ be the smallest index with $h_l \ne 0$. Then, whenever $l < k$ and $s$ is an integer with*

$$1 \le s \le \tfrac{1}{2}k(k+1) - \frac{1}{2} - \frac{l}{k-l+1},$$

*one has*

$$J_{s,k}(X;\mathbf{h}) \ll X^{s-1/2+\varepsilon}.$$

*In particular, this holds when $1 \le l \le (k+1)/3$ and $s < k(k+1)/2$. Moreover, when $1 \le s \le l(l+1)/2$, one has*

$$J_{s,k}(X;\mathbf{h}) \ll X^{s-1+\varepsilon}$$

# 4. Ideas in the proof

Recall

$$f(\boldsymbol{\alpha}; X) = \sum_{1 \le x \le X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3).$$

When $\mathbf{h} \in \mathbb{Z}^3$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we put

$$I_s(\mathfrak{B}; X; \mathbf{h}) = \int_{\mathfrak{B}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; X)|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, \mathrm{d}\boldsymbol{\alpha},$$

where $\mathrm{d}\boldsymbol{\alpha} = \mathrm{d}\alpha_1 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3$. Thus, in particular, $I_s([0,1); X; \mathbf{h}) = B_s(X; \mathbf{h})$.

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We also make use of the auxiliary generating function

$$g(\boldsymbol{\alpha},\theta;X)=\sum_{1\le y\le X}e\left(y\theta+2h_1y\alpha_2+(3h_2y+3h_1y^2)\alpha_3\right).$$

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**Lemma**

*Suppose that $s\in\mathbb{N}$, $\mathbf{h}\in\mathbb{Z}^3$ and $\mathfrak{B}\subseteq\mathbb{R}$ is measurable. Then*

$$I_s(\mathfrak{B};X;\mathbf{h})\ll X^{-1}(\log X)^{2s}\sup_{\Gamma\in[0,1)}\int_{\mathfrak{B}}\int_0^1\int_0^1 |f(\boldsymbol{\alpha};2X)^{2s}g(\boldsymbol{\alpha},\Gamma;X)|\,\mathrm{d}\boldsymbol{\alpha}.$$

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This has essentially introduced a new variable underlying the exponential sum $g$ with an accompanying factor $X^{\varepsilon-1}$, and so generates extra cancellation. The idea originates with earlier work (W., 2012) on the asymptotic formula in Waring's problem.

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To get an idea of how the proof here works, simplify to the situation where $s = 6$, $\mathfrak{B} = [0, 1)$ and apply orthogonality. The left hand side counts the number of solutions with $1 \leq x_i \leq X$ of

$$x_1^3 - x_2^3 + \ldots + x_{11}^3 - x_{12}^3 = h_3$$
$$x_1^2 - x_2^2 + \ldots + x_{11}^2 - x_{12}^2 = h_2$$
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Shift by any $y$ with $1 \le y \le X$ to obtain

$$(x_1 + y)^3 - (x_2 + y)^3 + \ldots + (x_{11} + y)^3 - (x_{12} + y)^3 = h_3 + 3h_2 y + 3h_1 y^2$$
$$(x_1 + y)^2 - (x_2 + y)^2 + \ldots + (x_{11} + y)^2 - (x_{12} + y)^2 = h_2 + 2h_1 y$$
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with $1 \le y + x_i \le 2X$.

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with $1 \le y + x_i \le 2X$. Average over $y$ and apply orthogonality to get

$$B_6(X;\mathbf{h}) \ll X^{-1} \int_{[0,1)^3} |f(\boldsymbol{\alpha};2X)|^{12} g(\boldsymbol{\alpha},0;X)\, \mathrm{d}\boldsymbol{\alpha}.$$

(The conclusion of the lemma requires some standard harmonic analysis to work over the set $\mathfrak{B}$ in place of $[0,1)$.)

$$g(\boldsymbol{\alpha},\theta;X)=\sum_{1\le y\le X} e\left(y\theta+2h_1y\alpha_2+(3h_2y+3h_1y^2)\alpha_3\right).$$

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Now seek to bound this mean value in terms of

$$\Theta_m(X;\mathbf{h})=\int_{[0,1)^3}|f(\boldsymbol{\alpha};2X)^{2m}g(\boldsymbol{\alpha},0;X)^6|\,\mathrm{d}\boldsymbol{\alpha}\quad (m\in\mathbb{N}).$$

Key difficulty here: the exponential sum $g$ is only quadratic, and so less efficient at saving powers of $X$ than the cubic exponential sum $f$.

## Lemma (essentially optimal)

*When $\mathbf{h}\in\mathbb{Z}^3$ and $h_1\neq 0$, one has $\Theta_5(X;\mathbf{h})\ll X^{10+\varepsilon}$.*

## Lemma

*When* $\mathbf{h} \in \mathbb{Z}^3$ *and* $h_1 \neq 0$, *one has* $\Theta_1(X;\mathbf{h}) \ll X^4 \log(2X)$.

To see why this is true, observe that by applying orthogonality,

$$\int_0^1 |f(\boldsymbol{\alpha};2X)|^2 \, \mathrm{d}\alpha_1 \leq 2X.$$

Since $g(\boldsymbol{\alpha};X)$ is independent of $\alpha_1$,

$$\Theta_1(X;\mathbf{h}) \leq 2X \int_{[0,1)^2} |g(0,\alpha_2,\alpha_3;X)|^6 \, \mathrm{d}\alpha_2 \, \mathrm{d}\alpha_3.$$

The integral here counts the number of integral solutions $T_0(X)$ of

$$3h_1 \sum_{i=1}^3 (x_i^2 - y_i^2) + 3h_2 \sum_{i=1}^3 (x_i - y_i) = 0,$$

$$2h_1 \sum_{i=1}^3 (x_i - y_i) = 0,$$

with $1 \leq x_i, y_i \leq X$ $(1 \leq i \leq 3)$.

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with $1 \leq x_i, y_i \leq X$ $(1 \leq i \leq 3)$.
Since, by hypothesis, one has $h_1 \neq 0$, we see that $T_0(X)$ counts the integral solutions of the Vinogradov system of equations

$$\sum_{i=1}^{3} (x_i^j - y_i^j) = 0 \quad (j = 1, 2),$$

with the same conditions on $\mathbf{x}$ and $\mathbf{y}$. Thus $T_0(X) \ll X^3 \log(2X)$, whence

$$\Theta_1(X; \mathbf{h}) \ll X \cdot X^3 \log X \ll X^4 \log X.$$

## Lemma

*When* $\mathbf{h} \in \mathbb{Z}^3$ *and* $h_1 \neq 0$*, one has* $\Theta_1(X;\mathbf{h}) \ll X^4 \log(2X)$*.*

We have to estimate

$$\Theta_5(X;\mathbf{h}) = \int_{[0,1)^3} |f(\boldsymbol{\alpha};2X)^{10} g(0,\alpha_2,\alpha_3;X)^6| \, \mathrm{d}\boldsymbol{\alpha}.$$

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Here, as a guide, we can observe that a version of Weyl's inequality shows that $f(\boldsymbol{\alpha};2X) \ll X^{3/4+\varepsilon}$ on a set of minor arcs. By adapting a pruning argument to the present situation, one may show that this guideline applies on average for all $\boldsymbol{\alpha}$, yielding

$$\begin{aligned}\Theta_5(X;\mathbf{h}) &\ll (X^{3/4+\varepsilon})^8 \int_{[0,1)^3} |f(\boldsymbol{\alpha};2X)^2 g(0,\alpha_2,\alpha_3;X)^6| \,\mathrm{d}\boldsymbol{\alpha} \\ &\ll X^{6+8\varepsilon}\Theta_1(X;\mathbf{h}) \ll X^{10+9\varepsilon}.\end{aligned}$$

(Optimal estimate – saves $X^{6-\varepsilon}$).

When $Q$ is a real parameter with $1 \le Q \le X$, we define the set of major arcs $\mathfrak{M}(Q)$ to be the union of the arcs

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le QX^{-3}\},$$

with $0 \le a \le q \le Q$ and $(a,q) = 1$. Also, put $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$.

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From our earlier lemma (simplifying slightly),

$$I_6(\mathfrak{m}(Q); X; \mathbf{h}) \ll X^{\varepsilon - 1} \int_{\mathfrak{m}(Q)} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)^{12} g(\boldsymbol{\alpha}, 0; X)| \, \mathrm{d}\boldsymbol{\alpha}.$$

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Apply Hölder's inequality to obtain

$$I_6(\mathfrak{m}(Q); X; \mathbf{h}) \ll \Bigl( \sup_{\alpha_3 \in \mathfrak{m}(Q)} \sup_{(\alpha_1, \alpha_2) \in [0,1)^2} |f(\boldsymbol{\alpha}; 2X)| \Bigr)^{1/3} U_1^{5/6} U_2^{1/6},$$

where

$$U_1 = \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)|^{12} \, \mathrm{d}\boldsymbol{\alpha} \quad \text{and} \quad U_2 = \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, 0; X)^6| \, \mathrm{d}\boldsymbol{\alpha}.$$

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Since Weyl's inequality yields

$$\sup_{\alpha_3\in\mathfrak{m}(Q)} \sup_{(\alpha_1,\alpha_2)\in[0,1)^2} |f(\boldsymbol{\alpha};2X)| \ll X^{1+\varepsilon} Q^{-1/4},$$

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This is what provides our subconvex minor arc estimate. For the major arcs, use technical pruning arguments and standard major arc technique.

# 5. Further results

Define

$$f_k(\boldsymbol{\alpha}; X) = \sum_{1 \le x \le X} e(\alpha_1 x + \ldots + \alpha_k x^k).$$

We have formulated an extension to the main conjecture in Vinogradov's mean value theorem as follows.

**Conjecture (W., 2022; arxiv:2202.14003)**

*When $k \in \mathbb{N}$, $\mathfrak{B} \subseteq [0,1)^k$ is measurable and $s \ge \frac{1}{4}k(k+1) + 1$,*

$$\int_{\mathfrak{B}} |f_k(\boldsymbol{\alpha}; X)|^{2s} \, \mathrm{d}\boldsymbol{\alpha} \ll X^{\varepsilon} \left( X^s \mathit{mes}(\mathfrak{B}) + X^{2s - k(k+1)/2} \right).$$

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Notice that this is **not** a subconvex estimate.

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We now consider the twisted mean value

$$B_k(X;\mathbf{h}) = \int_{[0,1)^k} |f_k(\boldsymbol{\alpha};X)|^{k(k+1)} e(-\boldsymbol{\alpha}\cdot\mathbf{h})\,\mathrm{d}\boldsymbol{\alpha},$$

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## Theorem (W., 2022; arxiv:2202.14003)

*Assume the above conjecture. Suppose that* $\mathbf{h} \in \mathbb{Z}^k$ *and* $h_l \neq 0$ *for some* $1 \leq l < k$. *Then when* $X$ *is sufficiently large in terms of* $\mathbf{h}$,

$$B_k(X;\mathbf{h}) = \mathfrak{J}_k(\mathbf{h})\mathfrak{S}_k(\mathbf{h})X^{k(k+1)/2} + o(X^{k(k+1)/2}),$$

*in which* $0 \leq \mathfrak{J}_k(\mathbf{h}) \ll 1$ *and* $0 \leq \mathfrak{S}_k(\mathbf{h}) \ll 1$.

THE END