Field counting and arboreal degrees
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  Today: Progress on these two questions and an unexpected link between them!

Key players: nilpotent groups

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A finite group is nilpotent if and only if it is a product of groups of sizes powers of primes.

Malle's conjecture

Let G be a finite group, K be a number field. Define

$$N(K, G, X) := \#\{L \subseteq K^{\text{sep}} : \text{Gal}(L/K) \simeq G, |N_{K/\mathbb{Q}}(\text{Disc}(L/K))| \leq X\}.$$

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Malle conjectured that there are $a(G), b_{\text{Malle}}(G, K)$ and $c > 0$ such that

$$N(K, G, X) \sim c \cdot X^{a(G)} \cdot \log(X)^{b_{\text{Malle}}(G,K)-1}.$$

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For G *nilpotent* the last was excluded by a celebrated theorem of Shafarevich, recently reproved by Harpaz and Wittenberg.

Previous results: asymptotics

Established for:

- All G abelian over any K number field (Wright, 1989).
- For $G := S_3$ standard action and $K := \mathbb{Q}$ (Davenport–Heilbronn, 1971).
- For $G := S_4, S_5$ standard action and $K := \mathbb{Q}$ (Bhargava, 2005, 2010).
- For generalized quaternions (Klüners, 2005).
- For $G := S_3$ regular action and $K := \mathbb{Q}$ (Bhargava–Wood, 2008).
- For $G := S_n \times A$, with $n \in \{3, 4, 5\}$ and A abelian, $K := \mathbb{Q}$ (Wang, 2017).
- For D_4 by conductor (Shankar–Varma–Wilson, 2017).
- For nonic Heisenberg (Koymans–Fouvry, 2021).

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- All nilpotent groups G and all number fields K (Klüners–Malle, 2004).
- The upper bound $N(K, G, X) = O(X^{a(G)} \log(X)^{b_{\mathrm{Kl}}(G,K)-1})$ with $b_{\mathrm{Kl}}(G, K) \geq b_{\mathrm{Malle}}(G, K)$ established for all nilpotent G and all number fields K (Klüners, 2020).

Main results: a general upper bound

We improve $b(G,K) \leq b'(G,K) \leq b_{\mathrm{Kl}}(G,K)$, with $b'(G,K) + 4 \leq b_{\mathrm{Kl}}(G,K)$ for some G's. Having the following

Theorem 1, Koymans–P., 2021

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- Once the parametrization is set-up the proof is extremely easy!
- It makes possible making the estimate *effective*.

Main results: asymptotics

Basically: Whenever $b_{\mathrm{Malle}}(G, K) = b'(G, K)$ we promote Theorem 1 to an asymptotic:

Theorem 2, Koymans–P., 2021

Let G a nilpotent group where all the elements of minimal non-trivial order are *central*. Then Malle's conjecture holds, i.e. there is $c > 0$ such that

$$N(K, G, X) \sim c \cdot X^{a(G)} \cdot \log(X)^{b_{\mathrm{Malle}}(G,K)-1}.$$

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- There are 2-groups G of arbitrarily large nilpotency class for which this theorem applies.
- We have a sharp upper bound in case all elements of minimal order are pairwise commuting, a yet even larger class of groups.
- Our parametrization yields a heuristic understanding of $b_{\mathrm{Malle}}(G, K)$ and of Malle's conjecture when ordering fields by discriminants.

The growth of arboreal degrees

Let K be a number field, $\phi \in K(x)$ a map of degree at least 2 and α in K and focus on the fields $K(\phi^{-N}(\alpha))/K$.

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Wide open in general!

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- As soon as $\{\phi^{-N}(\alpha)\}_{N\geq 1}$ is *infinite*, then we have a positive constant $c(\phi,\alpha)$ such that

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An application of nilpotent Malle: linear lower bounds for quadratic ϕ

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- If ϕ has degree 2 then leveraging on the new results on nilpotent Malle we can promote it to:

$$[K(\phi^{-N}(\alpha)):K] \geq c(\phi,\alpha,\epsilon)\cdot N^{1-\epsilon}.$$

Methods

For the first up to constants:

- One has about d^N algebraic numbers of uniformly bounded height. Let D be the largest of their degrees.
- A theorem of Schmidt permits no more than $\exp(cD^2)$ numbers of degree D and of unformly bounded height. Hence D must be at least $\sqrt{N}$.

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- This requires running Theorem 1 with a moving G, which is handled by the effectivity of the upper bound: the wonders of the parametrization.
- Controlling fibers: $\alpha \mapsto K(\alpha)$ (Lemke-Olivier–Thorne) and number of 2-groups of given size (Highman–Sims).

Exponential lower bounds: PCF polynomials

Let K be a number field. We have the following.

Theorem 3, P., 2021

Assume GRH. Suppose that f is a PCF polynomials of degree $d \geq 2$. Let α be outside the critical orbits of f. Then there is $c(f, \alpha) > 0$ such that

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Idea: If ramification insists to be a finite set: smallest splitting prime is no less d^N. Hence (GRH) huge ramification at these primes. Hence (finite set of prime once again) huge degrees.

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- Apply the magic modulo a suitably chosen prime.

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This is part of a more general conjecture of Andrews–Petsche.

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If a quadratic polynomial $x^2 + c$ over any number field K, gives abelian arboreal Galois group for some α, then the orbit of 0 is preperiodic.

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- This condition (essentially) translates into making the critical orbit modulo squares unidimensional;
- If the orbit were infinite one would get curves of very high genus having infinitely many rational points.

Progress on Andrews–Petsche

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This follows from the magic of period critical orbit (and, for example, Amoroso–Zannier lower bounds in K^{ab}).

Andrews–Petsche over $\mathbb{Q}$

We have the following:

Theorem 6, Ferraguti–P., 2020

Andrews–Petsche conjecture holds for any quadratic polynomial over $\mathbb{Q}$.

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- Superexponential lower bounds would settle this conjecture.
- Recently, Ferraguti–Ostafe–Zannier explored the case of rational functions and Lattes maps.

Thanks for the attention!